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Solutions of linear recursive systems

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PETER R. J. ASVELD [1], [2], furthermore MARJORIE BICKNELL–JOHNSON and GERALD BERGUM [3] investigated sequences determined by initial values and linear recursive systems. The Problem H–351 of the Fibonacci Quarterly, proposed by V. E. HOGGATT, Jr. [4], considers a similar question: Determine the sequences $U = \{U_n\}_{n=1}^{\infty}$, $V = \{V_n\}_{n=1}^{\infty}$ with $U_1 = V_1 = F_1 = F_2 = 1$ and

$$\begin{gathered} U_{n+1} - U_n - V_n - F_{n+1} = 0, \qquad -U_{n+1} + V_{n+1} - V_n = 0, \\ F_{n+2} - F_{n+1} - F_n = 0, \quad \text{for any } \; n \geq 1. \end{gathered} \tag{1}$$

The purpose of this paper is to investigate a generalization of these problems.

For a given integer $r \geq 1$ let $X^{(j)} = \left\{x_n^{(j)}\right\}_{n=0}^{\infty}$ $(1 \leq j \leq r)$ be sequences of real numbers with initial terms $x_0^{(j)}, x_1^{(j)}, \dots, x_{m_j-1}^{(j)}$ $(1 \leq j \leq r)$ and let $c_{i,j,t}$ $(1 \leq i, j \leq r;\ 0 \leq t \leq m_j)$ be fixed real numbers. Suppose that the sequences satisfy the equation system

$$\begin{aligned} &\sum_{t=0}^{m_1} c_{1,1,t} x_{n+t}^{(1)} + \sum_{t=0}^{m_2} c_{1,2,t} x_{n+t}^{(2)} + \dots + \sum_{t=0}^{m_r} c_{1,r,t} x_{n+t}^{(r)} = 0 \\ &\sum_{t=0}^{m_1} c_{2,1,t} x_{n+t}^{(1)} + \sum_{t=0}^{m_2} c_{2,2,t} x_{n+t}^{(2)} + \dots + \sum_{t=0}^{m_r} c_{2,r,t} x_{n+t}^{(r)} = 0 \\ &\qquad \vdots \\ &\sum_{t=0}^{m_1} c_{r,1,t} x_{n+t}^{(1)} + \sum_{t=0}^{m_2} c_{r,2,t} x_{n+t}^{(2)} + \dots + \sum_{t=0}^{m_r} c_{r,r,t} x_{n+t}^{(r)} = 0 \end{aligned} \tag{2}$$

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for any natural number n. We assume that

$$\det(c_{i,j,m_j}) \neq 0, \tag{3}$$

where $\det(c_{i,j,m_j})$ is the determinant of the $r \times r$ matrix with entries c_{i,j,m_j} $(0 \leq i,j \leq r)$. By (2) and (3) the sequences $X^{(j)}$ are uniquely determined since the initial terms are given. We shall show that these sequences also satisfy a linear recurrence relation.

Let M be the set of operators $A = A(a_0, a_1, \ldots, a_m)$, $B = B(b_0, b_1, \ldots, b_k), \ldots,$ defined on the set of sequences of real numbers $X = \{X_n\}_{n=0}^{\infty}$, $Y = \{Y_n\}_{n=0}^{\infty}, \ldots,$ so that

$$\begin{aligned} A(X) &= X' = \{x_n'\}_{n=0}^{\infty} = \left\{\sum_{i=0}^{m} a_i x_{n+i}\right\}_{n=0}^{\infty} \\ B(X) &= X'' = \{x_n''\}_{n=0}^{\infty} = \left\{\sum_{i=0}^{k} b_i x_{n+i}\right\}_{n=0}^{\infty} \\ &\vdots \end{aligned}$$

where $a_0, a_1, \ldots, a_m; b_0, b_1, \ldots, b_k, \ldots$ are fixed real numbers. Let $X+Y = \{x_n + y_n\}_{n=0}^{\infty}$ and $aX = \{ax_n\}_{n=0}^{\infty}$ for a real number a.

It can easily be checked that

$$A(aX + bY) = aA(X) + bA(Y) \tag{4}$$

for any operator A of M and for any real numbers a and b, that is each element of M is a linear operator. We define the addition and multiplication of operators by

$$(A + B)(X) = A(X) + B(X) \tag{5}$$

and

$$(A \cdot B)(X) = A(B(X)) \tag{6}$$

for any sequence X of real numbers.

Let T be the mapping of the set M of operators onto the set $R[x]$ of polynomials with real coefficients defined by

$$T(A) = \sum_{i=0}^{m} a_i x^i$$

where the operator A is determined by the real numbers $a_0, a_1, \ldots, a_m$. The following auxiliary result will be proved at the and of the paper.

Lemma. *The mapping T is an isomorphism between the structures $(M,+,\cdot)$ and $(R[x],+,\cdot)$.*

Using the notation of operators, equation system (2) can be written in the form

$$\begin{aligned} &C_{1,1}\left(x^{(1)}\right)+C_{1,2}\left(x^{(2)}\right)+\cdots+C_{1,r}\left(x^{(r)}\right)=0^* \\ &C_{2,1}\left(x^{(1)}\right)+C_{2,2}\left(x^{(2)}\right)+\cdots+C_{2,r}\left(x^{(r)}\right)=0^* \\ &\vdots \\ &C_{r,1}\left(x^{(1)}\right)+C_{r,2}\left(x^{(2)}\right)+\cdots+C_{r,r}\left(x^{(r)}\right)=0^* \end{aligned} \tag{7}$$

where $C_{i,j}\in M$ $(1\le i,j\le r)$ is an operator determined by the constants $c_{i,j,1}$, $c_{i,j,2},\ldots,$ c_{i,j,m_j} and 0^* is the sequence of zeros.

Let Z be the zero element of the ring $(M,+,\cdot)$, i.e. $Z(X)=0^*$ for any sequence X (Z is determined by zeros).

Using the above notation our main result is as follows:

Theorem. *Let $X^{(j)}$ $(1\le j\le r)$ be sequences determined uniquely by their initial terms and by (2) and (3). Then these sequences satisfy the recursive relation*

$$(\det(C_{i,k}))\left(X^{(j)}\right)=0^* \quad (1\le j\le r), \tag{8}$$

furthermore

$$(\det(C_{i,k}))\ne Z, \tag{9}$$

where $\det(C_{i,k})\in M$ is the determinant of the $r\times r$ matrix with entries $C_{i,k}\in M$, $(1\le i,k\le r)$.

Before proving the Theorem we show some consequences and applications of our result.

If $A=A(a_0,\ldots,a_m)$ is an operator and $A(X)=0^*$ for a sequence X of real numbers, then X is a linear recursive sequence of order m since

$$a_mX_{n+m}+a_{m-1}X_{n+m-1}+\cdots+a_0X_n=0$$

for any $n\ge 0$, furthermore the characteristic polynomial of this sequence is

$$a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0=T(A),$$

where T is the isomorphism defined in the Lemma. So, as a consequence of our theorem we have

Corollary. *Let $X^{(j)}$ $(1 \le j \le r)$ be sequences such as in the Theorem. Then these are linear recursive sequences of order maximum*

$$m = m_1 + m_2 + \cdots + m_r$$

and their common characteristic polynomial is

$$T\left(\det\left(C_{i,j}\right)\right) = \det\left(T\left(C_{i,j}\right)\right).$$

As an application of the theorem we show the way of solving the system (1). From $U_1 = V_1 = F_1 = F_2 = 1$, by (1), the initial terms of the sequences are $U_0 = V_0 = F_0 = 0$, $U_1 = 1$, $U_2 = 3$, $U_3 = 9$; $V_1 = 1$, $V_2 = 4$, $V_3 = 13$ and $F_1 = F_2 = 1$, $F_3 = 2$. In our case $r = 3$, $X^{(1)} = U$, $X^{(2)} = V$, $X^{(3)} = F$; $m_1 = m_2 = 1$, $m_3 = 2$ and

$$\det\left(c_{i,j,m_j}\right) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The operators $C_{i,j}(1 \le i, j \le 3)$ are $C_{1,1}(-1,1)$, $C_{1,2}(-1,0)$, $C_{1,3}(0,-1,0)$, $C_{2,1}(0,-1)$, $C_{2,2}(-1,-1)$, $C_{2,3}(0,0,0) = C_{3,1}(0,0) = C_{3,2}(0,0) = 0^*$, $C_{3,3}(-1,-1,1)$ and so, by the Corollary, U,V,F are linear recursive sequences of order maximum 4 with characteristic polynomial

$$f(x) = \det\left(T\left(C_{i,j}\right)\right) = \begin{vmatrix} x-1 & -1 & -1 \\ -x & x-1 & 0 \\ 0 & 0 & x^2-x-1 \end{vmatrix} =$$
$$= \left(x^2 - 3x + 1\right) \cdot \left(x^2 - x - 1\right)$$

The roots of $f(x)$ are

$$\alpha_1 = \frac{1+\sqrt{5}}{2},\ \alpha_2 = \frac{1-\sqrt{5}}{2}, \alpha_3 = \frac{3+\sqrt{5}}{2},\ \text{and } \alpha_4 = \frac{3-\sqrt{5}}{2}$$

and so, as it is well-known, the terms of the sequences can be expressed as

$$x_n^{(j)} = a_j\alpha_1^n + b_j\alpha_2^n + c_j\alpha_3^n + d_j\alpha_4^n \quad (j = 1,2,3)$$

where a_j, b_j, c_j, d_j are fixed real numbers, depending on the initial terms, and they can be calculated by solving a linear equation system generated for $n = 0,1,2$ and 3. This way for the sequence V we get

$$a_2 = -\frac{5+2\sqrt{5}}{10},\ b_2 = \frac{2\sqrt{5}-5}{10},\ c_2 = \frac{2\sqrt{5}+5}{10},\ d_2 = \frac{5-2\sqrt{5}}{10}$$

and for the sequence F we obtain

$$a_3 = \frac{\sqrt{5}}{5}, \quad b_3 = -\frac{\sqrt{5}}{5}, \quad c_3 = d_3 = 0.$$

Thus F satisfies also a second order linear recursive relation with characteristic polynomial

$$(x-\alpha_1)(x-\alpha_2)=x^2-x-1,$$

hence F is really the Fibonacci sequence.

Another example shows a common generalization of the problems investigated in [1], [2] and [3].

Let $X^{(1)}$ be a sequence of real numbers defined by the initial terms $X_0^{(1)}, X_1^{(1)}, \dots, X_{m-1}^{(1)}$ and by the formula

$$\sum_{i=0}^{m} a_i x_{n+i}^{(1)} + \sum_{i=1}^{k} q_i(n)\alpha_i^n = 0 \quad (n \geq 0), \tag{10}$$

where $a_0, a_1, \dots, a_m$ $(a_m \neq 0)$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are fixed real numbers, and $q_i(x)$ are given polynomials with real coefficients of degree $(r_i-1) \geq 0$ for $i=1,2,\dots,k$. It is known that the sequence

$$X^{(2)} = \left\{x_n^{(2)}\right\}_{n=0}^{\infty} = \left\{\sum_{i=1}^{k} q_i(n)\alpha_i^n\right\}_{n=0}^{\infty}$$

is a linear recursive sequence of order $r_1+r_2+\dots+r_k$ with characteristic polynomial

$$q(x)=\prod_{i=1}^{k}(x-\alpha_i)^{r_i}.$$

So there is an operator $B \in M$ such that $B\left(X^{(2)}\right)=0^*$ and $T(B)=q(x)$. Let $A=A(a_0,a_1,\dots,a_m)$ be an operator of M and let E and Z be the unit operator, i.e. $E=E(1)$, and the zero operator, respectively. Then (10) can be written in the form

$$A\left(X^{(1)}\right)+E\left(X^{(2)}\right)=0^*$$
$$Z\left(X^{(1)}\right)+B\left(X^{(2)}\right)=0^*$$

From this, by the Theorem and the Corollary, it follows that $X^{(1)}$ is a linear recursive sequence with characteristic polynomial

$$T\left(\begin{vmatrix} A & E \\ Z & B \end{vmatrix}\right) = T(A\cdot B) = T(A)\cdot T(B) = \left(\sum_{i=0}^{m} a_i x^i\right)\cdot \prod_{j=1}^{k}(x-\alpha_j)^{r_j}$$

Now we prove the Lemma and the Theorem.

Proof of the Lemma. Let X be a sequence of real numbers and let $A = A(a_0, a_1, \dots, a_m)$ and $B = B(b_0, b_1, \dots, b_k)$ be operators of the set M. Then

$$A(X) = \left\{ \sum_{i=0}^{m} a_i x_{n+i} \right\}_{n=0}^{\infty} \text{ and } B(X) = \left\{ \sum_{i=0}^{k} b_i x_{n+i} \right\}_{n=0}^{\infty}$$

We can suppose that $m \geq k$ and $b_i = 0$ if $k < i \leq m$. By (5) and (6) we get

$$(A+B)(X) = \left\{ \sum_{i=0}^{m} a_i x_{n+i} \right\}_{n=0}^{\infty} + \left\{ \sum_{i=0}^{k} b_i x_{n+i} \right\}_{n=0}^{\infty} = \\ = \left\{ \sum_{i=0}^{m} (a_i + b_i) x_{n+i} \right\}_{n=0}^{\infty}$$

and

$$(A \cdot B)(X) = A\left(\left\{ \sum_{t=0}^{k} b_t x_{n+t} \right\}_{n=0}^{\infty} \right) = \left\{ \sum_{j=0}^{m} a_j \sum_{t=0}^{k} b_t x_{(n+t)+j} \right\}_{n=0}^{\infty} = \\ = \left\{ \sum_{i=0}^{m+k} \sum_{j+t=i} a_j \cdot b_t \cdot x_{n+i} \right\}_{n=0}^{\infty}$$

Combining the above equations with the definition of the mapping T, we obtain

$$T(A+B) = \sum_{i=0}^{m} (a_i + b_i) x^i = \sum_{i=0}^{m} a_i x^i + \sum_{i=0}^{k} b_i x^i = T(A) + T(B)$$

and

$$T(A \cdot B) = \sum_{i=0}^{m+k} \sum_{j+t=i} a_j b_t x^i = \left(\sum_{j=0}^{m} a_j x^j \right) \cdot \left(\sum_{t=0}^{k} b_j x^t \right) = T(A) \cdot T(B)$$

follow which proves the Lemma since T is obviously a bijective mapping.

Proof of the Theorem. The Lemma implies that $(M, +, \cdot)$ is an Euclidean ring and the usual properties of determinants are valid if the entries are operators of M.

Let $A^{i,j}$ be the determinant of the $(r-1) \times (r-1)$ matrix that we get from $C_{i,k} (1 \leq i, k \leq r)$ by omitting the i^{th} row and the j^{th} column. Further, let

$$A_{i,j} = (-E)^{i+j} A^{i,j} \quad (1 \leq i, j \leq r),$$

where E is the unit element of M. Similarly as in the proof of Cramer's rule, from (7) with some j $(1 \le j \le r)$

$$\begin{aligned}
&A_{1,j}\left(C_{1,1}\left(X^{(1)}\right)+C_{1,2}\left(X^{(2)}\right)+\cdots+C_{1,r}\left(X^{(r)}\right)\right)=A_{1,j}(0^*)\\
&A_{2,j}\left(C_{2,1}\left(X^{(1)}\right)+C_{2,2}\left(X^{(2)}\right)+\cdots+C_{2,r}\left(X^{(r)}\right)\right)=A_{2,j}(0^*)\\
&\vdots\\
&A_{r,j}\left(C_{r,1}\left(X^{(1)}\right)+C_{r,2}\left(X^{(2)}\right)+\cdots+C_{r,r}\left(X^{(r)}\right)\right)=A_{r,j}(0^*)
\end{aligned}$$

follows. From this system, using (4) and the fact that the multiplication in the ring of operators is commutative, we get

$$\begin{aligned}
&C_{1,1}A_{1,j}\left(X^{(1)}\right)+C_{1,2}A_{1,j}\left(X^{(2)}\right)+\cdots+C_{1,r}A_{1,j}\left(X^{(r)}\right)=0^*\\
&C_{2,1}A_{2,j}\left(X^{(1)}\right)+C_{2,2}A_{2,j}\left(X^{(2)}\right)+\cdots+C_{2,r}A_{2,j}\left(X^{(r)}\right)=0^*\\
&\vdots\\
&C_{r,1}A_{r,j}\left(X^{(1)}\right)+C_{r,2}A_{2,j}\left(X^{(2)}\right)+\cdots+C_{r,r}A_{r,j}\left(X^{(r)}\right)=0^*
\end{aligned}$$

since $A(0^*) = 0^*$ for any $A \in M$. Adding the equations of this system by (5) we obtain the equation

$$\sum_{t=1}^{r}\left(C_{1,t}A_{1,j}+C_{2,t}A_{2,j}+\cdots+C_{r,t}A_{r,j}\right)\left(X^{(t)}\right)=0^*. \tag{11}$$

But

$$\sum_{i=1}^{r} C_{i,t}A_{i,j} = \begin{cases} Z & \text{if } t \ne j \\ \det(C_{i,j}) & \text{if } t = j \end{cases}$$

and so (11) implies (8).

By the Lemma (9) is equivalent to the inequality

$$\det\left(T(C_{i,j})\right) \ne 0\ , \tag{12}$$

where 0 is the identically zero polynomial. But the leading coefficient of the polynomial $\det\left(T(C_{i,j})\right)$ is equal to $\det\left(c_{i,j,m_j}\right)$ and so (12) follows from (3). This completes the proof of the theorem.

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