Publ. Math. Debrecen
53 / 1-2 (1998), 217–223

On quasi-slant submanifolds of an almost Hermitian manifold

By FERNANDO ETAYO (Santander)

Abstract. Quasi-slant submanifolds of an almost Hermitian manifold are defined in this paper as a generalization of slant submanifolds introduced by B. Y. Chen. We prove their basic properties, obtain sufficient conditions for such a manifold to be a slant submanifold and show examples.

B. Y. Chen [C] defined a slant submanifold M of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$ as a real submanifold verifying that the Wirtinger angle, i.e., the angle between $\overline{J}(X)$ and T_xM, is constant for every vector $X \in T_xM$ and $x \in M$. Geometry of slant submanifolds has been studied in several papers after [C]. In [P], N. Papaghiuc gave a first generalization of this notion defining semi-slant submanifolds, obtaining slant and Cauchy-Riemann submanifolds as particular cases.

In this work we study submanifolds such that the Wirtinger angle is pointwise constant, but it is not globally constant. We call them *quasi-slant* submanifolds. The organization of the paper is as follows: in Section 1 we give the basic definitions and in Section 2 we show the results. In particular, we prove that surfaces are always quasi-slant and that odd-dimensional quasi-slant submanifolds are totally real. Finally, we prove that quasi-slant totally geodesic submanifolds of a Kähler manifold are slant submanifolds.

All the manifolds considered are real, connected and of class C^∞. The author whishes to thank to his wife Ujué R. Trías for their valuable suggestions.

Mathematics Subject Classification: 53C40, 53C55.
Partially supported by Spanish Grant PB95-0124

1. Basic definitions

Let $(\overline{M}, \overline{J}, \overline{g})$ be an almost Hermitian manifold and let M be a real submanifold of $\overline{M}$. Then M is said *holomorphic* (resp. *totally real*) if $\overline{J}(T_xM) \subset T_xM$, $\forall x \in M$ (resp. $\overline{J}(T_xM) \subset T_x^{\perp}M$, $\forall x \in M$), where T_xM and $T_x^{\perp}M$ denote the tangent and the normal space to M at the point x. Many results have been obtained for these kinds of submanifolds. The above definitions have been generalized in several ways:

(1) The submanifold M is said a *Cauchy-Riemann submanifold* (cfr. [B]) if there exists a differentiable distribution $D : x \to D_x \subset T_xM$ such that D is $\overline{J}$-invariant and the complementary orthogonal distribution $D^{\perp}$ is anti-invariant, i.e., $\overline{J}(D_x^{\perp}) \subset T_x^{\perp}M$, $\forall x \in M$.

(2) The submanifold M is said *slant* (cfr. [C]) if for all non-zero vector X tangent to M the angle $\vartheta(X)$ between $\overline{J}(X)$ and T_xM is a constant, i.e., it does not depend on the choice of $x \in M$ and $X \in T_xM$.

(3) The submanifold M is said *semi-slant* (cfr. [P]) if it is endowed with two orthogonal distributions D and $D^{\perp}$, where D is $\overline{J}$-invariant and $D^{\perp}$ is slant, i.e., the angle $\vartheta(X)$ between $\overline{J}(X)$ and $D_x^{\perp}$ is a constant.

(4) The submanifold M is said *generic* (cfr. [C]) if the maximal holomorphic subspace $H_x = T_xM \cap \overline{J}(T_xM)$ is of constant dimension.

One can easily check that holomorphic and totally real submanifolds are Cauchy-Riemann and slant submanifolds; these are semi-slant submanifolds and all of them are generic submanifolds.

In this work we give the following natural definition:

Definition 1.1. A submanifold M of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$ is said *quasi-slant* if, for each $x \in M$, the angle $\vartheta(X)$ between $\overline{J}(X)$ and T_xM is a constant, for all non-zero vector $X \in T_xM$, i.e., it does not depend on the choice of $X \in T_xM$, but it depends on the choice of the point $x \in M$.

Then, the angle function can be defined on the submanifold M and it will be denoted as $\vartheta : M \to [0, \frac{\pi}{2}]$, $\vartheta(x) = \vartheta_x$.

Obviously, a slant submanifold is a quasi-slant submanifold. Observe that, for each $x \in M$, M being a quasi-slant manifold, the maximal holomorphic subspace $H_x = T_xM \cap \overline{J}(T_xM)$ is $H_x = T_xM$ or $H_x = \{0\}$, thus proving that quasi-slant submanifolds may be no generic submanifolds.

On the other hand, one can consider the unit tangent bundle of M, $T^1M \to M$, and define the angle function $\vartheta : T^1M \to [0, \frac{\pi}{2}]$, where $\vartheta(X)$ is the angle between $\overline{J}(X)$ and T_xM, when $X \in T_xM$. If M is a quasi-slant submanifold the function ϑ is constant along the fibres of the unit tangent bundle of M.

Definition 1.2. Let M be a quasi-slant submanifold of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$. The tensor field F of type $(1,1)$ on M given by $F_x = \pi \circ \overline{\pi}|_{T_xM} : T_xM \to T_xM$, where π (resp. $\overline{\pi}$) denotes the orthogonal projection over T_xM (resp. $\overline{J}(T_xM)$) is said the *canonical tensor field* of M.

If M is a holomorphic (resp. totally real) submanifold, its canonical tensor field is the identity (resp. the null tensor field). We shall show the important rôle of the canonical tensor field in the following section.

2. The results

First of all, we show that surfaces are quasi-slant submanifolds and we obtain a restricion to the embedding of a non-orientable surface into an almost Hermitian manifold.

Proposition 2.1. *Let M be a submanifold of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$. If M is a surface, then M is a quasi-slant submanifold. Moreover, if M is non- orientable, then there exists a point $x \in M$ such that $\vartheta_x = \frac{\pi}{2}$.*

Proof. Let $x \in M$. If $H_x = T_xM$, then F_x is the identity and $\vartheta_x = 0$, for all $X \in T_xM$. Let us assume that $H_x = \{0\}$ and let $\{X, Y\}$ be an orthonormal basis of T_x. Then $\{\overline{J}X, \overline{J}Y\}$ is an orthonormal basis of $\overline{J}(T_xM)$ and $\{X, Y, \overline{J}X, \overline{J}Y\}$ is a basis of $E_x = T_xM \oplus \overline{J}(T_xM)$. Then, the matrix of $\pi_x|_{E_x}$ is

$$M(\pi|_{E_x}) = \begin{pmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Taking into account that $(\overline{J}X - \pi(\overline{J}X)) \perp T_xM$ and $(\overline{J}Y - \pi(\overline{J}Y)) \perp T_xM$, one obtains:

$$M(\pi|_{E_x}) = \begin{pmatrix} 1 & 0 & 0 & -\alpha_x \\ 0 & 1 & \alpha_x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\alpha_x = \overline{g}(\overline{J}X, Y) = -\overline{g}(X, \overline{J}Y)$. Then, $|\cos\vartheta(X)| = |\overline{g}(\overline{J}X, \pi(\overline{J}X))| = |\alpha_x|$ and $|\cos\vartheta(Y)| = |\overline{g}(\overline{J}Y, \pi(\overline{J}Y))| = |-\alpha_x| = |\alpha_x|$, and one easily check that for all $Z \in T_xM$, $|\cos\vartheta(Z)| = |\alpha_x|$ thus proving that M is quasi-slant.

For the last part of the theorem, let us consider that M is a surface verifying $\vartheta_x < \frac{\pi}{2}$, for all $x \in M$. Given two orthonormal basis $\{X, Y\}$ and $\{Z, W\}$ of T_xM, one easily checks that $\overline{g}(\overline{J}X, Y) = \overline{g}(\overline{J}Z, W)$ if both basis define the same orientation and $\overline{g}(\overline{J}X, Y) = -\overline{g}(\overline{J}Z, W)$ if their orientations are mutually inverse. Then, taking into account that $\cos\vartheta_x \neq 0$, for all $x \in M$, and choosing an orthonormal basis on a point $x \in M$, one can define a global orientation on M, thus showing that such a surface is orientable. $\square$

Remarks. (1) We have obtained the matrix expression of $\pi|_{E_x}$. In a similar way one obtains:

$$M(\overline{\pi}|_{E_x}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_x & 1 & 0 \\ -\alpha_x & 0 & 0 & 1 \end{pmatrix}$$

and then F_x is a homothety of ratio α_x^2. This result will be generalized in Theorem 2.3.

(2) In the case $\vartheta_x < \frac{\pi}{2}$, for all $x \in M$, one can choose a continuous determination of α_x and define an almost complex structure J on M given by $J_x(Z) = \frac{1}{\alpha_x}\pi_x(\overline{J}Z)$, when $Z \in T_xM$.

(3) Generalizing the proof of the above proposition one has the following: let us assume that M is a quasi-slant submanifold of dimension r, $x \in M$ is a point such that $H_x = \{0\}$, $\{X_1, \dots, X_r\}$ is an orthonormal basis of T_xM, $\{\overline{J}(X_1), \dots, \overline{J}(X_r)\}$ is the induced orthonormal basis of $\overline{J}(T_xM)$, and $\{X_1, \dots, X_r, \overline{J}(X_1), \dots, \overline{J}(X_r)\}$ is the induced basis of $E_x = T_xM \oplus \overline{J}(T_xM)$. Then,

$$M(\pi|_{E_x}) = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}; \quad M(\overline{\pi}|_{E_x}) = \begin{pmatrix} 0 & 0 \\ -B & I \end{pmatrix}$$

where I is the identity matrix and $B = (b_{ij})$, where $b_{ij} = \overline{g}(\overline{J}X_i, X_j)$; thus B is an anti- symmetric matrix, i.e., $B^t = -B$. Then, the matrix of the canonical tensor field F_x respect to the orthonormal basis $\{X_1, \dots, X_r\}$ is $-B^2 = -BB = B^tB$, which is symmetric, thus proving that the canonical tensor field is a self-adjoint endomorphism.

Moreover, the matrix of $\overline{g}|_{E_x}$ is

$$M(\overline{g}|_{E_x}) = \begin{pmatrix} I & B \\ -B & I \end{pmatrix}$$

The following result generalizes one obtained for slant submanifolds:

Proposition 2.2. *Let M be a quasi-slant submanifold of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$. If M has odd dimension, then M is a totally real submanifold.*

Proof. First of all, observe that 0 is an eigenvalue of F_x, for all $x \in M$, because dimension of M odd implies $H_x = \{0\}$ and, using the above Remark (3), $\det(F_x) = \det(-B^2) = \det(-B)\det B = \det(B^t)\det B = \det(B^2)$, which is null, rank of B being odd.

Let $x \in M$. As 0 is an eigenvalue of F_x, there exists a non-zero vector $X \in T_xM$ such that $0 = F_x(X) = \pi(\overline{\pi}(X))$ and then two cases appear:

(a) If $\overline{\pi}(X) = 0$, then $X \perp \overline{J}(T_xM)$ and for all $Y \in T_xM$, $0 = \overline{g}(X, \overline{J}Y) = -\overline{g}(\overline{J}X, Y) \Rightarrow \overline{J}X \perp T_xM \Rightarrow \vartheta_x(X) = \frac{\pi}{2} \Rightarrow \vartheta_x = \frac{\pi}{2}$, as we wanted.

(b) If $\overline{\pi}(X) \neq 0$, then there exists a non-zero vector $Y \in T_xM$ such that $\overline{\pi}(X) = \overline{J}Y$ and then one obtains: $0 = F_x(X) = \pi(\overline{\pi}(X)) = \pi(\overline{J}Y) \Rightarrow \overline{J}Y \perp T_xM \Rightarrow \vartheta_x(Y) = \frac{\pi}{2} \Rightarrow \vartheta_x = \frac{\pi}{2}$. □

Remarks. (1) Observe that in the last theorem we have proved the following most general result: "If M is an odd-dimensional submanifold of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$ and $x \in M$ is a point such that $H_x = \{0\}$, then there exists a non-zero vector $X \in T_xM$ such that $\overline{J}X \perp T_xM$".

(2) The above two results allow us to think that a Schur-type theorem may be true: "If $\dim M > 2$ and M is quasi-slant, then M is slant". Nevertheless, this is not the case, as the following example shows: for any $k > 0$, the map $\mathbb{R}^4 \to \mathbb{C}^4$ given by $(u, v, w, z) \to (u, v, k\sin w, k\sin z, kw, kz, k\cos w, k\cos z)$ defines a slant submanifold in $\mathbb{C}^4$ with slant angle $\cos^{-1} k$ (cfr. [C, Example 8.7]). Changing k by a non-constant positive function (e.g., by $1+u^2$) one obtains a 4-dimensional quasi-slant submanifold which is not slant.

The following result shows that quasi-slant submanifolds are characterized by the canonical tensor field.

Theorem 2.3. *Let M be a submanifold of an almost Hermitian manifold $(\overline{M}, \overline{J}, \overline{g})$. Then M is a quasi-slant submanifold iff F_x is a homothety, for all $x \in M$.*

Proof. We use the notation and results of the Remark (3) below Proposition 2.1. Let $x \in M$. If $H_x = T_xM$, then F_x is the identity. Let us assume that $H_x = \{0\}$. Then, one easily checks:

(a) $\cos^2\vartheta_x = \|\overline{J}X_i\| \, \|\pi(\overline{J}X_i)\| \cos\vartheta_x = \overline{g}(\overline{J}X_i, \pi(\overline{J}X_i)) = \sum_{j=1}^{r}(b_{ij})^2$, thus proving that the sume of the squares of the elements of a column of B is constant.

(b) As the matrix of F_x is $-B^2 = B^tB$, and using the above result (a), the elements of the first diagonal of the matrix of F_x are equal to $-\cos^2\vartheta_x$.

(c) M being quasi-slant, the angle between Y and $\overline{J}(T_xM)$ is ϑ_x, for every non-zero vector $Y \in T_xM$.

(d) Let X_i a vector of the basis choosen in T_xM. As $\|X_i\| = 1$, then $\|\overline{\pi}(X_i)\| = |\cos\vartheta_x|$ and $\|F(X_i\| = \|\pi(\overline{\pi}X_i)\| = \cos^2\vartheta_x$. Taking account that the basis $\{X_1, \dots, X_r\}$ is orthonormal and the result (b) of the present proof, one concludes that the elements of the matrix of F_x are all of them null, except those of the diagonal.

(e) The inverse is trivial. □

Finally, we show that the slant condition and the extrinsic curvature of the submanifold are related geometric properties:

Theorem 2.4. *Let M be a quasi-slant complete and totally geodesic submanifold of a Kähler manifold $(\overline{M}, \overline{J}, \overline{g})$. Then M is a slant submanifold.*

Proof. Let $x, y \in M$. We have to prove that $\vartheta_x = \vartheta_y$. As M is complete, there exists a geodesic γ in M joining x and y. Let X be a non-zero vector, $X \in T_xM$ and let $Y = \tau(X) \in T_yM$ be the vector obtained from X by parallel transport along γ. M being totally geodesic, the parallel transport in M along γ is the restriction of the parallel transport in $\overline{M}$ along γ and then, $\tau(T_xM) = T_yM$ and $\tau(T_x\overline{M}) = T_y\overline{M}$. As $(\overline{M}, \overline{J}, \overline{g})$ is a Kähler manifold, parallel transport commutes with $\overline{J}$ and then $\tau(\overline{J}X) = \overline{J}(\tau X) = \overline{J}Y$. Finally, parallel transport is an isometry, thus proving that the angle between $\overline{J}X$ and T_xM coincides with the angle between $\overline{J}Y$ and T_yM, i.e., $\vartheta_x = \vartheta_y$. □

One can deduce from the above Theorem 2.4 and Proposition 2.1, the following result:

Corollary 2.5. *Let M be a complete totally geodesic surface embedded into a Kähler manifold $(\overline{M}, \overline{J}, \overline{g})$. Then M is a slant surface.*

References

[B] A. Bejancu, Geometry of CR-Submanifolds, *D. Reidel Publ. Comp.*, *Dordrecht*, 1986.

[C] B. Y. Chen, Slant immersions, *Bull. Austral. Math. Soc* **41** (1990), 135–147.

[P] N. Papaghiuc, Semi-slant submanifolds of a Kaehlerian manifold, *Ann. Stiint. Univ. Al. I Cuza Iaşi Secţ. I a Mat* **40** (1994), 55–61.

FERNANDO ETAYO
DEPARTAMENTO DE MATEMÁTICAS
ESTADÍSTICA Y COMPUTACIÓN
FACULTAD DE CIENCIAS UNIVERSIDAD DE CANTABRIA
AVDA. DE LOS CASTROS, S/N, 39071 SANTANDER
SPAIN

E-mail: etayof@matesco.unican.es

(Received July 29, 1997)