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Normal contact CR-submanifolds of a quasi-Sasakian manifold

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Abstract. We obtain geometric characterisations of a normal contact CR-submanifolds of a quasi-Sasakian manifold by means of a new tensor S, called torsion tensor (see (2.5)). Finally an example of normal contact CR-submanifold of a quasi-Sasakian manifold is given.

0. Introduction

The concept of CR-submanifold of a Kahlerian, manifold has been defined by A. Bejancu [3] and it is studied by, many authors [10], [13]. Later A. Bejancu and N. Papaghiuc [5] introduced and studied the notion of semi-invariant submanfold of a Sasakian manifold. These submanifolds are closely related to CR-submanifolds in a Kahlerian manifold. However, the existence of the structure vector field implies some important changes. Extensions of CR-submanifolds of a Sasakian manifold have also been studied by M. Kon and K. Yano [13].

The purpose of the present paper is to define what we call normal CR-submanifolds of a Quasi-Sasakian manifold and to obtain fundamental results on their geometry. In the first section we recall some results and formulae for later use. In the second section, we prove some important properties of a normal contact CR-submanifolds of a Quasi-Sasakian manifold and we close this section with an example of a normal contact CR-submanifolds of a Quasi-Sasakian manifold. In the last section we give some results of a cosymplectic CR-submanifolds of a quasi-Sasakian manifold.

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1. Preliminaries

Let $\widetilde{M}$ be a real $2n+1$-dimensional differentiable manifold and f, ξ and η be a tensor field of type (1,1), a vector field, a 1-form, respectively, and a Riemannian metric g, on $\widetilde{M}$ satisfying:

$$(1.1)\qquad \begin{gathered} f^2 = -I + \eta \oplus \xi, \quad \eta(\xi) = 1, \quad f(\xi) = 0, \quad \eta \oplus f = 0, \\ g(X,Y) = g(fX, fY) + \eta(X)\eta(Y), \end{gathered}$$

where I is the identity on the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ and X, Y the vector fields tangent to M.

We say that $\widetilde{M}$ is an almost contact metric manifold and (f, ξ, η) is an almost contact metric structure on M (c.f. D. E. Blair [6]). Throughout the paper, all manifolds and maps are differentiable of, class C^∞. We denote by $\mathcal{F}(M)$ the algebra of the differentiable function on M and by $\Gamma(E)$ the $\mathcal{F}(M)$-module of sections of a vector, bundle E over $\widetilde{M}$. We use the same notations for any other manifold involved in the study.

Next, we define the fundamental 2-form Ω of $\widetilde{M}$, by,

$$\Omega(X,Y) = g(X, fY), \quad \forall X, Y \in \Gamma(T\widetilde{M}).$$

The Nijenhuis tensor N_f of f is defined by

$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[fX,Y] - f[X,fY], \forall X, Y \in \Gamma(T\widetilde{M}).$$

We say that the almost contact structure (f, ξ, η) is normal if the, following condition is satisfied,

$$N_f(X,Y) + 2d\eta(X,Y)\xi = 0, \quad \forall X, Y \in \Gamma(T\widetilde{M}).$$

Finally we say that M is a quasi-Sasakian manifold, if it is , endowed with a normal almost contact metric structure (f, ξ, η, g) and, the fundamental 2-form Ω is closed. Quasi-Sasakian manifold has been, introduced by D. E. Blair [6]. A characterisation of a Quasi-Sasakian, manifold by means of covariant derivative of f has been given by, S. Kanemaki [9] as follows: *M is a quasi-Sasakian manifold if and, only if it is endowed with* *an almost contact metric structure,* (f, ξ, η, g) *and a tensor field* F *of type* $(1,1)$ *such as,*

$$(1.2)\qquad \begin{gathered}(\widetilde{\nabla}_X f)Y = \eta(Y)FX - g(FX, Y)\xi; fFX = FfX;\\ g(FX, Y) = g(X, FY), \forall X, Y\Gamma(T\widetilde{M}),\end{gathered}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to the metric, tensor g. From (1.2) we deduce (see [7])

$$(1.3)\qquad \widetilde{\nabla}_X \xi = fFX, \forall X, Y \in \Gamma(T\widetilde{M}).$$

Now, let M be an m-dimensional Riemannian manifold isometrically, immersed in M, and suppose that the structure vector field ξ of $\widetilde{M}$ be tangent to M. We denote by TM and $TM^\perp$ the tangent bundle to M and, respectively, the normal bundle to M. Also we denote by $\{\xi\}$ the, 1-dimensional distribution spanned by ξ on M.

The submanifold M is called contact CR-submanifold if it is endowed with the pair of distributions $(D, D^\perp)$ satisfying (cf. [3])

i) $TM = D \perp D^\perp \oplus \{\xi\}$ and D, $D^\perp$ and $\{\xi\}$ are orthogonal on each other,

ii) the distribution D is invariant by f, i.e. we have $fD \subseteq D$,

iii) the distribution $D^\perp$ is anti-invariant by f, i.e. $fD \subseteq TM^\perp$

Throughout the paper we denote by $2p$ (resp. q) the dimension of D (resp. $D^\perp$). Thus, if $p = 0$ (resp. $q = 0$) then CR-submanifold is an anti-invariant submanifold tangent to ξ (resp. an invariant submanifold). An anti-holomorphic submanifold is a CR-submanifold which satisfyies: $\dim T\widetilde{M} = q$ (see [4]). The CR-submanifold M is called a proper CR-submanifold if it is neither an invariant submanifold nor an anti-invariant submanifold.

The projection morphisms of TM to D and $D^\perp$ are denoted by P and Q, respectively, and we have,

$$(1.4)\qquad X = PX + QX + \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$

The vector fields fX, fN and FZ are decomposed into tanget parts, tX, BN, αZ and normal parts ωX, CN, βZ, as follows

$$fX = tX + \omega X \qquad \forall X \in \Gamma(TM), \tag{1.5}$$

$$fN = BN + CN, \quad \forall N \in \Gamma(TM^{\perp}), \tag{1.6}$$

$$FZ = \alpha Z + \beta Z, \qquad \forall Z \in \Gamma(TM). \tag{1.7}$$

It is easy to say that t and C are f-structures in the sense of K. Yano [12] on TM and $TM^{\perp}$, respectively.

Next, from general theory of Riemannian submanifolds recall the Gauss and Weingarten formulae

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \forall X, Y \in \Gamma(TM), \tag{1.8}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \forall X \in \Gamma(TM), N \in \Gamma(TM^{\perp}), \tag{1.9}$$

where h is the fundamental form, A_N is the shape operator with respect to the normal section N, ∇ and $\nabla^{\perp}$ are the connections induced by $\widetilde{\nabla}$ on TM and $TM^{\perp}$, respectively.

The Nijenhuis tensor of t is given by

$$\begin{gathered} N_t(X,Y) = [tX,tY] + t^2[X,Y] - t[X,tY] - t[tX,Y], \\ \forall X, Y \in \Gamma(TM). \end{gathered} \tag{1.10}$$

We say that a contact CR-submanifold M of a quasi-Sasakian manifold is mixed geodesic if $h(X,Z) = 0$ for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$

Because ξ is a Killing vector field (see [9]) and by using (1.3), (1.5) and (1.7) we deduce

$$\begin{gathered} d\eta(X,Y) = g(Y, \widetilde{\nabla}_X \xi) = g(X, fFY) = g(\beta Y, \omega X) + \Omega(\alpha X, Y), \\ \forall X, Y \in \Gamma(TM). \end{gathered}$$

Finally, we recall some fundamental results from [7] for later use.

Lemma 1.1 [7]. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then we have*

$$\begin{gathered} (\nabla_X t)Y = A_{\omega Y} X + Bh(X,Y) + \eta(Y)\alpha X - g(FX,Y)\xi, \\ \forall X, Y \in \Gamma(TM), \end{gathered} \tag{1.11}$$

and

$$(1.12)\qquad \begin{gathered}(\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) + \eta(Y)\beta X,\\ \forall X, Y \in \Gamma(TM),\end{gathered}$$

where $(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y$ *and* $(\nabla_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega(\nabla_X Y)$.

Theorem 1.1 [7]. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. The distribution $D^{\perp}$ is integrable if and only if*

$$(1.13)\qquad FD^{\perp} \perp fD^{\perp}.$$

Theorem 1.2. [7]. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold M. The distribution $D^{\perp} \oplus \{\xi\}$ is integrable.*

2. Normal contact CR-submanifold of a quasi-Sasakian manifold

The purpose of this section is to study the fundamental properties of a normal contact CR-submanifold. The triplet (t, ω, η) is called *contact CR-structure* on M. First by using (1.1) and (1.5) we infer

$$(2.1)\quad g(X,Y) = g(tX,tY) + g(\omega X, \omega Y) + \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM).$$

By using (2.1) we deduce

Lemma 2.1. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold M. Then we have*

$$(2.2)\qquad \Omega(tX,tY) = \Omega(X,Y), \quad \forall X, Y \in \Gamma(TM).$$

By using the same method as in [10], we derive

Theorem 2.1. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then the covariant derivative of t is given by*

$$(2.3)\qquad \begin{aligned}2g((\nabla_X t)Y, Z) = {} & 3d\Omega(X,tY,tZ) - 3d\Omega(X,Y,Z) + g(N_t(Y,Z),tX)\\ & + 2g(d\omega(tY,Z),\omega X) - 2g(d\omega(tZ,Y),\omega X)\\ & - 2g(d\omega(tZ,X),\omega Y) + 2g(d\omega(tY,X),\omega Z)\\ & + 2\eta(Y)d\eta(X,tZ) - 2\eta(Z)d\eta(X,tY),\\ & \forall X, Y, Z \in \Gamma(TM).\end{aligned}$$

By means of a contact CR-structure (t, ω, η), we define the tensor field S on M, by

$$S(X,Y) = N_t(X,Y) - 2Bd\omega(X,Y) + \{\Omega(\alpha Y, X) - \Omega(\alpha X, Y)\}\xi, \quad \forall X, Y \in \Gamma(TM). \tag{2.4}$$

The tensor field S is called the *torsion tensor* of a contact CR-structure. We say that the contact CR-submanifold M is *normal contact CR-submanifold* if the tensor S vanishes identically on M. By using (1.10) and by taking into account that ∇ is a torsion-free linear connection, (2.4) becomes

$$\begin{aligned} S(X,Y) = (\nabla_{tX}t)Y - (\nabla_{tY}t)X + t\{(\nabla_Y t)X - (\nabla_X t)Y\} + \{\Omega(\alpha Y, X) \\ - \Omega(\alpha X, Y)\} + B\{h(X, tY) - h(tX, Y) \\ + \eta(X)\beta Y - \eta(Y)\beta X\}, \quad \forall X, Y \in \Gamma(TM). \end{aligned} \tag{2.5}$$

By straightforward calculation and using (1.5), (1.6), (1.7), (1.11), (1.12) and (2.5) we infer

Lemma 2.2. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then we have*

$$\begin{aligned} S(X,Y) = (A_{\omega Y} \circ t - t \circ A_{\omega Y} - \eta(Y)\alpha \circ \omega)\, X \\ - (A_{\omega X} \circ t - t \circ A_{\omega X} - \eta(X)\alpha \circ \omega)\, Y, \quad \forall X, Y \in \Gamma(TM). \end{aligned} \tag{2.6}$$

Theorem 2.2. *The contact CR-submanifold M of a quasi-Sasakian manifold $\widetilde{M}$ is normal if and only if the distribution $D^\perp$ is integrable and*

$$A_{\omega Y} tX = tA_{\omega Y}X, \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp). \tag{2.7}$$

Proof. It is easy to see that if $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D^\perp)$ or $X = \xi$ and $Y \in \Gamma(D)$, we obtain $S(X,Y) = 0$. Next, for $X = \xi$ and $Y \in \Gamma(D^\perp)$, by using (2.6) we deduce

$$S(\xi, Y) = \alpha\omega Y - tA_{\omega Y}\xi. \tag{2.8}$$

Thus by using (1.5) and (2.8) we infer

$$g(S(\xi,Y), Z) = g(fY, FZ), \quad \forall Y, Z \in \Gamma(D^\perp). \tag{2.9}$$

Now let $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$. By using (1.5) and (2.6) we deduce

$$S(X,Y) = A_{\omega Y} tX - tA_{\omega Y} X. \tag{2.10}$$

Finally, our assertion follow form (2.9) and (2.10).

Remark 2.1. From the proof of the Theorem 2.2 we see that S(X,Y) $= 0$ $\forall X,\ Y \in \Gamma(TM)$ if and only if $S(X,Y) = 0$, $\forall X \in \Gamma(D \oplus \{\xi\})$, $Y \in \Gamma(D^{\perp})$.

Corollary 2.1. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then M is normal contact CR-submanifold if and only if $F(TM) \subseteq \mu \oplus D^{\perp}$, and*

$$h(X,tY) + h(tX,Y) \subseteq \Gamma(\mu), \quad \forall X,Y \in \Gamma(D), \tag{2.11}$$

$$h(tX,W) \in \Gamma(\mu), \quad \forall X \in \Gamma(D),\ W \in \Gamma(D^{\perp}). \tag{2.12}$$

where μ is the complement orthogonal of $f(D^{\perp})$ in $TM^{\perp}$.

Proof. Let $X,Y \in \Gamma(D)$ and $Z,W \in \Gamma(D^{\perp})$. By using (1.3) and (2.10) we infer

$$g(A_{\omega Z} tX - tA_{\omega Z} X, Y) = g(h(tX,Y) + h(X,tY), \omega Z), \tag{2.13}$$

$$g(A_{\omega Z} tX - tA_{\omega Z} X, W) = g(h(tX,W), \omega Z), \tag{2.14}$$

$$\begin{aligned} g(A_{\omega Z} tX - tA_{\omega Z} X, \xi) &= g(\widetilde{\nabla}_{tX}\xi, fZ) = g(fX, FZ) \\ &= -g(FfX, Z). \end{aligned} \tag{2.15}$$

By using Theorem 2.2, our assertion follow from (2.13)–(2.15). From Corollary 2.1 we deduce

Corollary 2.2. *Let M be an anti-holomorphic submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then M is normal contact CR-submanifold if and only if $F(TM) \subseteq D$ and*

$$h(X,tY) + h(tX,Y) = 0, \quad \forall X,Y \in \Gamma(D),$$

and

$$h(tX,W) = 0, \quad \forall X \in \Gamma(D),\ W \in \Gamma(D^{\perp}).$$

By using Corollary 2.2 we deduce

Corollary 2.3. *Each normal contact anti-holomorphic CR-submanifold M is mixed geodesic.*

We say that a CR-submanifold M of a quasi-Sasakian manifold $\widetilde{M}$ of a quasi-Sasakian manifold $\widetilde{M}$ is $(D \oplus \{\xi\}, D^\perp)$ contact CR-product (resp. $(D^\perp \oplus \{\xi\}, D)$ contact CR-prodoct if the distributions $D \oplus \{\xi\}$ and $D^\perp$, (resp. $D^\perp \oplus \{\xi\}$ and D) are integrable and their leaves are totally geodesic in M. From [8] we recall the following result

Lemma 2.3. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then M is $(D \oplus \{\xi\}, D^\perp)$ contact CR-product if and only if*

$$fD^\perp \perp FD^\perp, \text{ and } Bh(X,U) = 0, \quad \forall X \in \Gamma(D),\ U \in \Gamma(TM).$$

Lemma 2.4 [8]. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then M is $(D^\perp \oplus \{\xi\}, D)$ contact CR-product if and only if*

$$h(X,U) \in \Gamma(\mu), \text{ and } D \perp FD, \quad \forall X \in \Gamma(D),\ U \in \Gamma(TM).$$

From Corollary 2.1 and Lemma 2.3 it follows

Corollary 2.4. *Each $(D \oplus \{\xi\}, D)$ contact CR-product of a quasi-Sasakian manifold $\widetilde{M}$ so that $fD^\perp \perp F(TM)$ is a normal contact CR-submanifold.*

By using Corollary 2.1 and Lemma 2.4 we obtain

Corollary 2.5. *Each $(D^\perp \oplus \{\xi\}, D)$ contact CR-product of a quasi-Sasakian manifold $\widetilde{M}$, so that $D \perp F(TM)$, is a normal contact CR submanifold.*

We say that M is totally contact-umbilical submanifold if there exists a normal vector field H such that the second fundamental form of M is given by

$$h(X,Y) = g(fX,fY)H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \quad \forall X,Y \in \Gamma(TM).$$

Then by using Corollary 2.1 we deduce

Corollary 2.6. *Each totally contact umbilical CR-submanifold M of a quasi-Sasakian manifold $\widetilde{M}$ with $fD \perp F(TM)$ is a normal contact CR-sumanifold.*

Next, suppose $\{E_1, \dots, E_q\}$ is a local field of orthonormal frames, for the anti-invariant distribution $D^\perp$. Denote by A_i the shape operator with respect to $V_i = fE_i$, $i = 1, \dots, q$. Then from Theorem 2.2 we have

Corollary 2.7. *The contact CR-submanifold M of a quasi-Sasakian manifold $\widetilde{M}$ is normal if and only if the distribution $D^\perp$ is integrable and*

$$A_i tX = tA_i X, \quad \forall X \in \Gamma(D). \tag{2.16}$$

By using (1.2), (1.5) and (1.6), for $X \in \Gamma(TM)$, we deduce

$$\nabla_X E_i = tA_i X - B\nabla^\perp_X V_i + g(FX, V_i)\xi, \quad \forall X \in \Gamma(TM), \tag{2.17}$$

$$\nabla^\perp_X V_i = \omega(\nabla_X E_i) + Ch(X, E_i), \quad \forall X \in \Gamma(TM). \tag{2.18}$$

It is well known that X is a Killing vector field if and only if

$$g(\nabla_Z X, Y) + g(\nabla_Y X, Z) = 0, \quad \forall Y, Z \in \Gamma(TM). \tag{2.19}$$

If $Y, Z \in \Gamma(D)$, so that (2.19) hold then we say that X is a D-Killing vector field.

Theorem 2.3. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then M is normal contact CR-submanifold if and only if E_i, $i = 1, \dots, q$, are D-Killing vector fields and distribution $D^\perp$ is integrable.*

Proof. By using (2.17) we deduce

$$g(\nabla_X E_i, Y) + g(\nabla_Y E_i, X) = g(tA_i X - A_i tX, Y), \quad \forall X, Y \in \Gamma(D). \tag{2.20}$$

Now our assertion follow form (2.20) and Corollary (2.7).

The Lie derivative of t with respect to $Y \in \Gamma(TM)$ is given by

$$(\mathcal{L}_Y t)X = [Y, tX] - t[Y, X], \quad \forall X \in \Gamma(TM). \tag{2.21}$$

Now we define a new tensor field $S^\star$ by

$$S^\star(Y, X) = (\mathcal{L}_Y t)X, \quad \forall X \in \Gamma(TM). \tag{2.22}$$

By using (2.4) we deduce

$$S(X,Y) = t^2[X,Y] - t[tX,Y] - Bh(tX,Y) - g(FY,tX)\xi, \quad \forall X \in \Gamma(D), Y \in \Gamma(D^\perp). \tag{2.23}$$

By using (2.21) and Theorem 1.2 we infer

$$S^\star(\xi, Y) = t[\xi, Y] = 0, \quad \forall Y \in \Gamma(D^\perp). \tag{2.24}$$

Next from (1.12), we deduce $h(tX,Y) = Ch(X,Y) + \omega(\nabla_Y X)$, $\forall X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Thus we obtain $Bh(tX,Y) = -Q(\nabla_Y X)$ and by using (2.24) we deduce

$$S(X,Y) = tS^\star(Y,X) + Q(\nabla_Y X) - g(FY, fX)\xi. \tag{2.25}$$

Theorem 2.4. *Suppose that M is a contact CR submanifold of a quasi-Saskian manifold $\widetilde{M}$ so that*

$$Q(\nabla_X Y) = 0, \quad \forall X \in \Gamma(D),\ Y \in \Gamma(D^\perp\{\xi\}). \tag{2.26}$$

M is normal contact CR-submanifold of a Quasi-Sasakian manifold $\widetilde{M}$ if and only if $F(TM) \perp fD^\perp$, and

$$S^\star(Y,X) = 0, \quad \forall X \in \Gamma(D \oplus \{\xi\}),\ Y \in \Gamma(D^\perp). \tag{2.27}$$

Proof. Suppose M be normal. By using Theorem 2.2 and relation (2.25) we obtain

$$Q(\nabla_Y X) = 0, \tag{2.28}$$

$$tS^\star(Y,X) = 0, \tag{2.29}$$

$$g(fX, FY) = 0, \tag{2.30}$$

for any $X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$. By using (2.25) and (2.27) we obtain $Q([X,Y]) = 0$ which is equivalent with

$$Q([tX,Y]) = 0, \quad \forall X \in \Gamma(D),\ Y \in \Gamma(D^\perp). \tag{2.31}$$

Now from (2.20), (2.21), (2.23), (2.28) and (2.31) we deduce (2.26). Because M is normal by using (2.9) and (2.31) we deduce that $F(TM) \perp$

$fD^\perp$ is true. Conversely suppose that (2.26) and $F(TM) \perp fD^\perp$ holds good. By using (2.9) we deduce that $S(\xi, Y) = 0$, $\forall Y \in \Gamma(D)$. By straightforward calculation it is easy to see that $\eta([X,Y]) = 0$, for $X \in \Gamma(D \oplus \{\xi\})$, $Y \in \Gamma(D^\perp)$. By using (1.4), (2.20) and (2.26) we infer

$$Q([X,Y]) = 0, \quad \forall X \in \Gamma(D),\ Y \in \Gamma(D^\perp). \tag{2.32}$$

Next from (2.25) and (2.32) it follows

$$Q(\nabla_Y X) = 0. \tag{2.33}$$

Finally from (2.9), (2.24), (2.33) and $F(TM) \perp fD^\perp$, we deduce that $S(X,Y) = 0$ for any $X \in \Gamma(D \oplus \{\xi\})$, $Y \in \Gamma(D^\perp)$. The proof is complete.

Now, we give an example of normal contact CR-submanifold of a Quasi-Sasakian manifold.

Let (f, ξ, η, g) the almost contact structure defined on R^5 as

$$f = f_i^j dx^i \otimes \frac{\partial}{\partial x^j}; \quad [f_i^j] = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2y^1 & 0 & 0 \end{bmatrix},$$

$$g = g_{ij} dx^i \otimes dx^j; \quad [g_{ij}] = \begin{bmatrix} 1+4(y^1)^2 & 0 & 0 & 0 & -2y^1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2y^1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$\xi = \frac{\partial}{\partial x^5} = \frac{\partial}{\partial z} = (0,0,0,0,1)^t$; $\eta = dz - 2y^1 dx = (-2y^1, 0, 0, 0, 1)^t$, where $(x^1, x^2, x^3, x^4, x^5) = (x^1, x^2, y^1, y^2, z)$, are the Cartesian coordinates on R^5. S. Kanemaki in [9] proved that this is a Quasi-Sasaki structure of rank 3. With respect to this cartesian coordinate let $e_i = \frac{\partial}{\partial x^i}$ be a local field of frames for the tangent bundle to R^5. Next let M be a hypersurface of R^5 defined by

$$x^i = f^i(u^1, u^2, u^3, u^4), \quad i \in \{1,2,3,4,5\}, \tag{2.34}$$

where f^i, are smooth functions on a domain $D \subseteq R^4$. If the functions f^i satisfy

$$\text{(2.35)} \qquad \begin{cases} a^1 f^1 + a^2 f^2 + a^3 f^3 + a^4 f^4 = a \\ -a^1 f^1 - a^2 f^2 + a^3 f^3 + a^4 f^4 = \phi_1(u^2, u^4), \\ a^1 f^1 + a^3 f^3 = \phi_2(u^2, u^4) \\ a^3 f^5 + a^1 (f^1)^2 = \phi_3(u^2, u^4). \end{cases}$$

where a^1, a^2, a^3, a^4, a, are real numbers not all null, ϕ_1, ϕ_2, ϕ_3 are smooth functions defined on a domain $D_1 \subseteq R^2$, then M defined by (2.34) so that (2.35) hold, is normal contact CR-submanifod. Indeed by straightforward calculation we deduce that $\Gamma^3_{11} = -4y^1$, $\Gamma^1_{13} = 2y^1$, $\Gamma^5_{13} = -1 + 4(y^1)^2$, $\Gamma^3_{15} = 1$, $\Gamma^1_{35} = -1$, $\Gamma^5_{35} = -2y^1$, and all other $\Gamma^k_{ij} = 0$. By using (1.3) we infer

$$F = F^j_i dx^i \otimes \frac{\partial}{\partial x^j}; \quad [F^j_i] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2y^1 & 0 & 0 & 0 & \alpha \end{bmatrix}$$

where $\alpha = \eta(F\xi)$.

On the other hand, a local field of frames for the tangent bundle TM is

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial f^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}.$$

It is easy to see that:

a) the normal vector bundle $TM^\perp$ is generated by the normal vector field N of M defined by: $N = (a^1, a^2, a^3, a^4, 2y^1 a^1)$,

b) the anti-invariant distribution $D^\perp = \text{span}\{fN\}$,

c) the distribution $D = \text{span}\{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^3}\}$,

d) $\xi = \frac{\partial}{\partial x^5}$ is tangent to M.

Therefore M is a contact CR-submanifold of M. More, by straightforward calculation we deduce

$$\widetilde{\nabla}_{f\frac{\partial}{\partial u^i}} N = f \widetilde{\nabla}_{\frac{\partial}{\partial u^i}} N = 0, \quad i \in \{1, 3\}$$

and cosequently $A_N X = 0, \forall X \in \Gamma(D)$, which prove that the Corollary 2.7 is verified and our assertion is proved. Next we prove that the set of functions and numbers which satisfy (2.35) is not null. Ineed if $a^1 = a^2 = a^3 = a^5 = a = 0$, $a^4 = 1$, $f^4 = \phi_2 = \phi_3 = 0$, $\phi_1 = -u^2$, $f^1 = u^1$, $f^2 = u^2$, $f^3 = u^3$, $f^5 = 2u^1u^3$, then (2.35) is verified.

3. Cosymplectic CR-submanifolds of a quasi-Sasakian manifold

Let $\widetilde{M}$ be a CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then we say that M is cosymplectic CR-submanifold if M is normal CR-submanifold and the differentiable form is closed, i.e. $d\omega = 0$.

The purpose of this section is to prove

Theorem 3.1. *Let M be a cosymplectic CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$ so that the distribution D is integrable. Then M is $(D, D^\perp \oplus \xi)$ CR-product.*

Proof. By using (2.5) we deduce that $N_t(X,Y) = [\Omega(FX,Y) - \Omega(FY,X)]\xi$, and from (2.3) we infer

$$g((\nabla_X t)Y, Z) = \eta(Y)d\eta(X,tZ) - \eta(Z)d\eta(X,tY), \quad \forall X,Y,Z \in \Gamma(TM). \tag{3.1}$$

Next, by using (1.13) we deduce

$$d\eta(X,tY) = -\Omega(\alpha X, tY) = -g(\alpha X, t^2Y) = g(FX, PY), \quad \forall X,Y \in \Gamma(TM). \tag{3.2}$$

and

$$d\eta(X,tZ) = -\Omega(\alpha X, tZ) = -g(\alpha X, t^2Z) = g(FX, PZ), \quad \forall X,Z \in \Gamma(TM). \tag{3.3}$$

Now, from (3.1), (3.2) and (3.3) we infer

$$(\nabla_X t)Y = -g(FX,PY)\xi + \eta(Y)P\alpha X, \quad \forall X,Y \in \Gamma(TM). \tag{3.4}$$

From $Y \in \Gamma(D)$ and $X \in \Gamma(TM)$ we deduce

$$(\nabla_X t)Y = -g(FX,PY)\xi.$$

Thus by using (1.16), from (3.5) we obtain

$$Bh(X,Y) = 0, \quad \forall X \in \Gamma(TM),\ Y \in \Gamma(D).$$

By using Lemma 2.4, our assertion follows from (3.6). Similary we state

Theorem 3.2. *Let M be a cosymplectic CR-submanifold of a quasi-Saskian manifold $\widetilde{M}$, so that $D^{\perp}$ be integrable. Then M is $(D \oplus \{\xi\}, D)$, CR-product.*

Definition 3.1. We say that the f-structure t is η-parallel if we have

$$(\nabla_X t)Y = -g(FX, PY)\xi + \eta(Y)P\alpha X, \quad \forall X, Y \in \Gamma(TM). \tag{3.7}$$

Then from (3.4) we deduce

Corollary 3.1. *Let M be a cosymplectic CR-submanifold of a quasi-Sasakian manifold $\widetilde{M}$. Then the f-structure t is η-parallel.*

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