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Bäcklund transformations of n-dimensional constant torsion curves

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Abstract. The Bäcklund transformation of two surfaces of $\mathbf{R}^3$ with the same constant negative Gaussian curvature transforms an asymptotic line of one surface into an asymptotic line of the other. Since by Enneper the asymptotic lines of such a surface have constant torsion, it is natural to restrict the Bäcklund transformations to such curves. This idea was developed by Annalisa Calini and Thomas Ivey in [2]. We shall prove the converse of their theorem and generalize the transformation for the n-dimensional case.

0. Introduction

By the work of Bianchi and Lie it is possible to compute the Gaussian curvature of the focal surfaces of a line congruence in terms of the coefficients of the first fundamental form for the spherical representation and the distance between the corresponding limit points of these surfaces (see [3]).

Bäcklund proved that for pseudospherical congruences satisfying the two additional conditions that the distance r between corresponding limit points is constant and that the normals of the focal surfaces at these points form a constant angle θ, the curvatures must be equal to the same negative constant $-\sin^2\theta/r^2$ (see [3]). By Ennepers relation between the first, second and third fundamental forms of a surface of negative Gaussian curvature the relation $\tau^2 = -K$ between the torsion τ of an asymptotic line

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and the curvature K holds. When the curvature is constant the torsion cannot change. Bearing in mind that under pseudospherical congruences asymptotic lines correspond, these provide a method for restricting the Bäcklund transformation to constant torsion curves. This was done by Annalisa Calini and Thomas Ivey in [2]. They constructed a constant torsion curve from a given one. We prove the converse of their theorem, namely: If there is a correspondence ν between the points of two unit speed curves $\mathbf{c}, \tilde{\mathbf{c}}$ having the property that the line joining the corresponding points $\mathbf{c}(s)$ and $\tilde{\mathbf{c}}(s) = \nu(\mathbf{c}(s))$ is the intersection of the osculating planes of these curves, and this intersection has the same angle with the curves, the line segment $\mathbf{c}(s)\tilde{\mathbf{c}}(s)$ has constant length r and the binormals in corresponding points form a constant angle θ, then the curves have the same constant torsion $\sin\theta/r$.

We could not find a connection between the sectional curvature of an n-dimensional manifold of constant negative curvature in the $2n-1$-dimensional euclidean space and the curvatures of its asymptotic lines in order to restrict the generalized Bäcklund transformation (see [4], and [5]) to curves in $2n-1$-dimensional euclidean spaces.

However if we consider just the transformation of Annalisa Calini and Thomas Ivey [2] for 3-dimensional constant torsion curves we can generalize it for higher dimensions.

1. Bäcklund transformations of 3-dimensional constant torsion curves

In [3] A. Calini and T. Ivey constructed a curve of constant torsion from a given one. We shall prove that under some assumptions made for a transformation between two curves they must have the same constant torsion.

Theorem 1.1. *Suppose that ν is a transformation between two curves $\mathbf{c}$ and $\tilde{\mathbf{c}}$ of $\mathbf{R}^3$ with $\tilde{\mathbf{c}}(s) = \nu(\mathbf{c}(s))$, where s is the arc length of $\mathbf{c}$, such that in corresponding points we have:*

(1) *The line joining these points is the intersection of the osculating planes of the curves, such that the line segment $\mathbf{c}(s)\tilde{\mathbf{c}}(s)$ has constant length r.*

(2) *The vector $\tilde{\mathbf{c}}(s) - \mathbf{c}(s)$ forms the same angle $\beta \neq \pi/2$ with the tangent vectors of the curves.*

(3) *The binormals of the curves form the same constant angle* $\theta \neq 0$.
Then the torsions of the curves are equal to the same constant $\sin\theta/r$.

PROOF. Denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the Frenet frame of $\mathbf{c}$ and by $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ that of $\tilde{\mathbf{c}}$ in the corresponding points $\mathbf{c}(s)$ and $\tilde{\mathbf{c}}(s)$. If we denote by $\mathbf{f}_1$ the unit vector of $\tilde{\mathbf{c}}(s) - \mathbf{c}(s)$, then we can complete $\mathbf{f}_1, \mathbf{e}_3$ and $\mathbf{f}_1, \tilde{\mathbf{e}}_3$ to the positively oriented orthonormal frames $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{e}_3)$ and $(\mathbf{f}_1, \tilde{\mathbf{f}}_2, \tilde{\mathbf{e}}_3)$ respectively. Let $\mathbf{f}_3 = \mathbf{e}_3$, $\tilde{\mathbf{f}}_3 = \tilde{\mathbf{e}}_3$ and $-\beta$ be the angle between $\mathbf{f}_1$ and $\mathbf{e}_1$. Then the angle between $\mathbf{f}_1$ and $\tilde{\mathbf{e}}_1$ is also $-\beta$. Thus we can obtain the frames $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ and $(\mathbf{f}_1, \tilde{\mathbf{f}}_2, \tilde{\mathbf{f}}_3)$ by rotating the frames $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ around $\mathbf{e}_3$ and $\tilde{\mathbf{e}}_3$ respectively with angle $-\beta$. Analytically this can be written as:

$$\begin{cases} \mathbf{f}_1 = \cos\beta\mathbf{e}_1 + \sin\beta\mathbf{e}_2, \\ \mathbf{f}_2 = -\sin\beta\mathbf{e}_1 + \cos\beta\mathbf{e}_2, \\ \mathbf{f}_3 = \mathbf{e}_3, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \mathbf{f}_1 = \cos\beta\tilde{\mathbf{e}}_1 + \sin\beta\tilde{\mathbf{e}}_2, \\ \tilde{\mathbf{f}}_2 = -\sin\beta\tilde{\mathbf{e}}_1 + \cos\beta\tilde{\mathbf{e}}_2, \\ \tilde{\mathbf{f}}_3 = \tilde{\mathbf{e}}_3. \end{cases} \tag{1.2}$$

Since $\mathbf{f}_3 = \mathbf{e}_3$, $\tilde{\mathbf{f}}_3 = \tilde{\mathbf{e}}_3$ and the angle between $\mathbf{e}_3$, $\tilde{\mathbf{e}}_3$ is the constant θ, we can obtain the frame $(\mathbf{f}_1, \tilde{\mathbf{f}}_2, \tilde{\mathbf{f}}_3)$ by rotating the frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ around $\mathbf{f}_1$ with angle θ. Thus we have:

$$\begin{cases} \tilde{\mathbf{f}}_2 = \cos\theta\mathbf{f}_2 - \sin\theta\mathbf{f}_3, \\ \tilde{\mathbf{f}}_3 = \sin\theta\mathbf{f}_2 + \cos\theta\mathbf{f}_3. \end{cases} \tag{1.3}$$

Using (1.1), (1.2) and (1.3) we can express $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ in terms of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as follows:

$$\begin{cases} \tilde{\mathbf{e}}_1 = \mathbf{e}_1 + (1-\cos\theta)\sin\beta(\cos\beta\mathbf{e}_2 - \sin\beta\mathbf{e}_1) + \sin\theta\sin\beta\mathbf{e}_3, \\ \tilde{\mathbf{e}}_2 = \mathbf{e}_2 - (1-\cos\theta)\cos\beta(\cos\beta\mathbf{e}_2 - \sin\beta\mathbf{e}_1) - \sin\theta\cos\beta\mathbf{e}_3, \\ \tilde{\mathbf{e}}_3 = \cos\theta\mathbf{e}_3 + \sin\theta(\cos\beta\mathbf{e}_2 - \sin\beta\mathbf{e}_1). \end{cases} \tag{1.4}$$

Bearing in mind that the distance between $\mathbf{c}(s)$ and $\tilde{c}(s)$ is the constant r and $\mathbf{f}_1 = \cos\beta\mathbf{e}_1 + \sin\beta\mathbf{e}_2$, we have:

$$\tilde{\mathbf{c}}(s) = \mathbf{c}(s) + r(\cos\beta\mathbf{e}_1 + \sin\beta\mathbf{e}_2). \tag{1.5}$$

Differentiating $(\tilde{\mathbf{c}}-\mathbf{c})^2 = r^2$ we obtain $2(\tilde{\mathbf{c}}-\mathbf{c})(|\dot{\tilde{\mathbf{c}}}|\tilde{\mathbf{e}}_1 - \mathbf{e}_1) = 0$. Since $\tilde{\mathbf{c}}-\mathbf{c} = r\mathbf{f}_1$, this yields $|\dot{\tilde{\mathbf{c}}}|\cos\beta - \cos\beta = 0$, so that $\tilde{\mathbf{c}}$ is also of unit speed.

Using the Frenet formulae for $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ it is easy to see that $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ given by (1.4) satisfy the Frenet formulae if and only if: $\tilde{K}_1 = K_1 - 2C\sin\beta$, $\tilde{K}_2 = K_2$ and $d\beta/ds = C\sin\beta - K_1$, where K_1, $\tilde{K}_1$ and K_2, $\tilde{K}_2$ are the curvatures and torsions of $\mathbf{c}$ and $\tilde{\mathbf{c}}$ respectively, and $C = K_2\tan\theta/2$.

By (1.5) we have:

$$\tilde{\mathbf{e}}_1 = \left(1 - rK_2\tan\frac{\theta}{2}\sin^2\beta\right)\mathbf{e}_1 + rK_2\tan\frac{\theta}{2}\sin\beta\cos\beta\mathbf{e}_2 + rK_2\sin\beta\mathbf{e}_3.$$

Comparing this with $(1.4)_1$ we obtain $K_2 = \sin\theta/r$. Thus the the curves $\mathbf{c}$ and $\tilde{\mathbf{c}}$ have the same constant torsion

$$\tilde{K}_2 = K_2 = \frac{\sin\theta}{r},$$

and the transformation can be given by

$$\tilde{\mathbf{c}} = \mathbf{c} + \frac{2C}{C^2 + K_2^2}(\cos\beta\mathbf{e}_1 + \sin\beta\mathbf{e}_2),$$

where

$$\frac{d\beta}{ds} = C\sin\beta - K_1.$$

The last two equations are the defining relations of the transformation given by A. Calini and T. Ivey in Theorem 1.1 of [3]. Examples for such transformation are given in [3].

2. Bäcklund transformations of n-dimensional constant torsion curves

From now on we mean by the torsion of a curve its last curvature. Let $\mathbf{c}$ and $\tilde{\mathbf{c}}$ be two curves in $\mathbf{R}^n$, with curvatures $K_1, \dots, K_{n-1}$ and $\tilde{K}_1, \dots, \tilde{K}_{n-1}$ respectively. Then the main theorem of Section 1 can be generalized as follows:

Theorem 2.1. *Suppose that ν is a transformation between $\mathbf{c}$ and $\tilde{\mathbf{c}}$ with $\tilde{\mathbf{c}}(s) = \nu(\mathbf{c}(s))$, where s is the arclength of $\mathbf{c}$ such that for correspond-*

ing points we have:

(1) *The line joining these points is contained in the intersection of the osculating hyperplanes and the line segment* $\mathbf{c}(s)\tilde{\mathbf{c}}(s)$ *has constant length* r.

(2) (a) *The angle between the vectors* $\mathbf{f}_1$ *and* $\mathbf{e}_{n-1}$ *is complementary to the angle between the vectors* $\mathbf{e}_1$ *and* $\mathbf{f}_{n-1}$, *where* $\mathbf{f}_1$ *is the unit vector of* $\tilde{\mathbf{c}}(s)-\mathbf{c}(s)$, $(\mathbf{f}_1,\dots,\mathbf{f}_{n-2})$ *and* $(\mathbf{f}_1,\dots,\mathbf{f}_{n-2},\mathbf{f}_{n-1},\mathbf{e}_n)$ *are positively oriented frames of the intersection of the osculating planes and the whole space respectively and* $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ *is the Frenét frame of* $\mathbf{c}$.

(b) $\langle \mathbf{e}_1,\mathbf{f}_1\rangle \neq 0$, *where* $\langle .,.\rangle$ *is the standard scalar product of* $\mathbf{R}^n$.

(3) *The Frenét frame of* $\tilde{\mathbf{c}}$ *can be obtained from that of* $\mathbf{c}$ *by a rotation with constant angle* $\theta \neq 0$ *around a plane which contains* $\mathbf{e}_n$.

Then the curves have the same constant torsion $\sin\theta/r$. *Moreover for* $n \geq 4$ *we have that*

$$K_1 = \tilde{K}_1,\ \dots,\ K_{n-3} = \tilde{K}_{n-3}.$$

Proof. From (3) we have $\tilde{E} = A^T\Theta AE$, where $E^T = (\mathbf{e}_1,\dots,\mathbf{e}_n)$ and $\tilde{E}^T = (\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n)$ are the Frenét frames of $\mathbf{c}$ and $\tilde{\mathbf{c}}$,

$$\Theta = \begin{vmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

and $A \in SO(n)$ such that $a_{in} = a_{ni} = \delta_{in}$, where δ_{ij} is the Kronecker symbol. In terms of the entries this can be written as:

$$\text{(2.1)}\qquad \begin{cases} \tilde{\mathbf{e}}_i = \sum\limits_{j=1}^{n-1}[\delta_{ij} - a_{n-1,i}a_{n-1,j}(1-\cos\theta)]\mathbf{e}_j - a_{n-1,i}\sin\theta\mathbf{e}_n; \\ i = \overline{1,n-1}, \\ \tilde{\mathbf{e}}_n = \sin\theta\sum\limits_{j=1}^{n-1} a_{n-1,j}\mathbf{e}_j + \cos\theta\mathbf{e}_n. \end{cases}$$

Since the first $n-2$ columns and rows of Θ form an identity matrix and $a_{in} = a_{ni} = \delta_{in}$, $A\tilde{E} = \Theta AE$ implies that the first $n-2$ rows of AE are equal to the first $n-2$ columns of $A\tilde{E}$ and form the basis $(\mathbf{f}_1,\dots,\mathbf{f}_{n-2})$ for the intersection of the osculating hyperplanes. Thus $F^T = (\mathbf{f}_1,\dots,\mathbf{f}_{n-1},\mathbf{e}_n)$, where $F = AE$.

By (2)(a) we also have

$$a_{1,n-1} = -a_{n-1,1}, \tag{0}$$

where $A = (a_{ij})_{1\le i,j\le n}$. Differentiating (2.1) and using the Frenét formulae for E, $\tilde{E} = (\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n)$ satisfies the Frenét formulae of $\tilde{\mathbf{c}}$ if and only if the following groups of relations hold:

$$\text{(I)} \qquad a_{n-1,2} = a_{n-1,n-1} = 0, \quad \text{i}f \ n \ge 4,$$

$$\tilde{K}_{n-2} = K_{n-2} + 2K_{n-1} \tan\frac{\theta}{2} a_{n-1,n-2},$$

$$\text{(II)} \qquad \begin{cases} \dot{a_{n-1,1}} = -K_1 a_{n-1,2} + K_{n-1} a_{n-1,n-1} a_{n-1,1} \tan\dfrac{\theta}{2}, \\ \dot{a_{n-1,j}} = K_j a_{n-1,j+1} - K_{j-1} a_{n-1,j-1} \\ \quad + K_{n-1} a_{n-1,n-1} a_{n-1,j} \tan\dfrac{\theta}{2}; \quad j = \overline{2, n-2}, \\ \dot{a_{n-1,n-1}} = -K_{n-2} a_{n-1,n-2} - K_{n-1} \tan\dfrac{\theta}{2}(1 - a_{n-1,n-1}^2), \end{cases}$$

$$\text{(III)} \qquad K_i = \tilde{K}_i; \quad i = \overline{1, n-3}, \quad \text{i}f \ n \ge 4.$$

Since the distance between corresponding points is the constant r, we have

$$\tilde{\mathbf{c}} = \mathbf{c} + r\mathbf{f}_1. \tag{2.2}$$

Using $F = AE$ we have

$$\mathbf{f}_1 = \sum_{j=1}^{n-1} a_{1j} \mathbf{e}_j,$$

which implies

$$\tilde{\mathbf{c}} = \mathbf{c} + r \sum_{j=1}^{n-1} a_{1j} \mathbf{e}_j.$$

We have already seen that the first $n-2$ rows of $A\tilde{E}$ coincide with the first $n-2$ columns of AE. Thus

$$\langle \mathbf{f}_1, \tilde{\mathbf{e}}_1 \rangle = \langle \mathbf{f}_1, \mathbf{e}_1 \rangle = a_{11}. \tag{2.3}$$

Differentiating $(\tilde{\mathbf{c}} - \mathbf{c})^2 = r^2$ we obtain $2\langle \tilde{\mathbf{c}} - \mathbf{c}, |\dot{\tilde{\mathbf{c}}}|\tilde{\mathbf{e}}_1 - \mathbf{e}_1\rangle = 0$, which by (2.2) and (2.3) becomes $|\dot{\tilde{\mathbf{c}}}| = 1$, hence $\tilde{\mathbf{c}}$ is also of unit speed. Differentiating (2.3) and using the Frenét formulae for $\mathbf{e}_j$ we obtain

$$\tilde{\mathbf{e}}_1 = (1 + r\dot{a_{11}} - rK_1 a_{12})\mathbf{e}_1 + r\sum_{j=2}^{n-2}(\dot{a_{1j}} - K_j a_{1,j+1} + K_{j-1}a_{1,j-1})\mathbf{e}_j$$
$$+ r(\dot{a_{1,n-1}} + a_{1,n-2}K_{n-2})\mathbf{e}_{n-1} + r a_{1,n-1}K_{n-1}\mathbf{e}_n.$$

Comparing this with (2.1) and using $a_{1,n-1} = -a_{n-1,1}$, we obtain

$$\text{(IV)} \qquad \begin{cases} K_1 = \dfrac{r\dot{a_{11}} + a_{n-1,1}^2(1-\cos\theta)}{r a_{12}}, \\ K_j = \dfrac{\dot{a_{1j}} + a_{n-1,1}a_{n-1,j}(1-\cos\theta) + K_{j-1}a_{1,j-1}}{a_{1,j+1}}; \\ \qquad j = \overline{2, n-2}, \\ K_{n-1} = \dfrac{\sin\theta}{r}. \end{cases}$$

In conclusion, (III) and $(\mathrm{IV})_3$ are exactly the assertions of our theorem.

For $n > 4$ if we fix a unit speed curve $\mathbf{c}$ in $\mathbf{R}^n$ with a given constant torsion, the system $(0) + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{IV})$ is underdetermined.

But this system is equivalent to the conditions (1), (2), (3) of Theorem 3.1. In conclusion, for every curve $\mathbf{c} \subset \mathbf{R}^n$; $n > 4$ with $K_{n-1} = a$, where a is a given constant, and every vector $\mathbf{v} \in T_{\mathbf{c}(0)}\mathbf{R}^n$, we can find an infinite number of curves $\tilde{\mathbf{c}} \subset \mathbf{R}^n$ satisfying the conditions (1), (2), (3) of Theorem 3.1 and such that $\sin\theta = ar$ and $\tilde{\mathbf{c}}(0) - \mathbf{c}(0) = \mathbf{v}$, where $T_{\mathbf{c}(0)}\mathbf{R}^n$ is the tangent space of $\mathbf{R}^n$ in $\mathbf{c}(0)$.

Remark 2.2. The case $n = 4$ is a special one, since for this dimension the condition (I) implies that the matrix A must be of the form

$$A = \begin{bmatrix} 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

hence the discussion of this case reduces to the 3-dimensional one.

Example 2.3. The following example will be given for $n = 5$. Let $r > 0$, a, b, c, θ, be five constants, such that

$$4(a^2+b^2)\sin^4\frac{\theta}{2} = c^2r^2.$$

Consider the curve $\mathbf{c} : s \mapsto \exp(s\,\Omega)$, where

$$\Omega = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & b & 0 & 0 \\ 0 & -b & 0 & c & 0 \\ 0 & 0 & -c & 0 & \frac{\sin\theta}{r} \\ 0 & 0 & 0 & -\frac{\sin\theta}{r} & 0 \end{bmatrix}.$$

By the Frenét formulae, the curvatures of $\mathbf{c}$ are $K_1 = a$, $K_2 = b$, $K_3 = c$, $K_4 = r^{-1}\sin\theta$. Integrating the system $(0)+(\mathrm{I})+(\mathrm{II})+(\mathrm{IV})$, we obtain

$$\tilde{\mathbf{c}}(s) = c(s) - \left[\frac{4}{r^3}\frac{b^2}{a^2}\sin^6\frac{\theta}{2}\,s - a\alpha(s)\right]e_1$$
$$+\dot{\alpha}(s)e_2 + \left[\left(\frac{2}{r}\frac{b}{a}\sin^2\frac{\theta}{2} - \frac{4}{r^2}\sin^6\frac{\theta}{2}\right)s - b\alpha(s)\right]e_3 + \frac{2}{r}\frac{b}{ac}\sin^2\frac{\theta}{2}e_4,$$

and

$$a_{43} = -\frac{2}{r}\frac{1}{c}\sin^2\frac{\theta}{2},$$

where α satisfies the following differential equation:

$$\dot{\alpha}^2(s) + \left[\frac{4}{r^3}\frac{b^2}{a^2}\sin^6\frac{\theta}{2}\,s - a\alpha(s)\right]^2$$
$$+\left[\left(\frac{1}{r}\frac{2b}{a}\sin^2\frac{\theta}{2} - \frac{4}{r^2}\sin^6\frac{\theta}{2}\right)s - b\alpha(s)\right]^2 + \frac{1}{r^2}\frac{4b^2}{a^2c^2}\sin^4\frac{\theta}{2} = 1$$

In particular the constant solutions of this equation can be found explicitly by solving a quadratic polinomial equation.

Using the formulae $\tilde{K}_1 = K_1, \dots, \tilde{K}_{n-3} = K_{n-3}$, $\tilde{K}_{n-1} = K_{n-2}$ and $\tilde{K}_{n-2} = K_{n-2} + 2K_{n-1}\tan\frac{\theta}{2}a_{n-1,n-2}$, we obtain

$$\tilde{K}_1 = a, \quad \tilde{K}_2 = b, \quad \tilde{K}_4 = \frac{\sin\theta}{r},$$

and

$$\tilde{K}_3 = c - \frac{8}{r^2}\,\frac{1}{c}\sin^4\frac{\theta}{2},$$

respectively.

Hence, if we impose the initial conditions $\tilde{E}(0) = I$, where I is the identical matrix, the Frenét formulae of $\tilde{\mathbf{c}}$ yields

$$\tilde{\mathbf{c}}(s) = \exp s\,\tilde{\Omega},$$

where

$$\tilde{\Omega} = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & b & 0 & 0 \\ 0 & -b & 0 & c - \frac{1}{c}\frac{8}{r^2}\sin^4\frac{\theta}{2} & 0 \\ 0 & 0 & -c + \frac{1}{c}\frac{8}{r^2}\sin^4\frac{\theta}{2} & 0 & \frac{\sin\theta}{r} \\ 0 & 0 & 0 & -\frac{\sin\theta}{r} & 0 \end{bmatrix}.$$

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