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Classes of uniformly starlike functions

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Abstract. We give coefficient characterizations for analytic functions with negative coefficients to be in subclasses of uniformly starlike and uniformly convex families. This leads to extremal properties and neighborhood criteria.

1. Introduction

Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. For $0 \le \alpha < 1$, let $S^*(\alpha)$ and $K(\alpha)$ denote the subfamilies of S consisting of functions starlike of order α and convex of order α, respectively. For convenience, we write $S^*(0) = S^*$ and $K(0) = K$. Motivated by geometric considerations, Goodman [1], [2] introduced the classes UCV and UST of uniformly convex and uniformly starlike functions. Ma and Minda [3] and Ronning [5] gave a one-variable analytic characterization for UCV, namely, a function f of the form (1) is in UCV if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \qquad z \in \Delta.$$

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Goodman [1] showed that the classical Alexander result $f \in K \iff zf' \in S^*$ does not hold between the classes UCV and UST. Ronning [5] introduced the class S_p consisting of functions zf', $f \in UCV$, and in [4] the class $S_p(\alpha)$ of functions of the form (1) for which

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \qquad -1 \leq \alpha < 1.$$

He also defined the class $UCV(\alpha)$, uniformly convex functions of order α, of functions f for which $zf' \in S_p(\alpha)$.

The subfamily of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0, \tag{2}$$

is denoted by T. In [7], Silverman investigated functions in the classes $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$. In particular, he proved the following theorem.

Theorem A. *A sufficient condition for f of the form* (1) *to be in $S^*(\alpha)$ is that $\sum_{n=2}^{\infty}((n-\alpha)/(1-\alpha))a_n \leq 1$ and for f to be in $K(\alpha)$ is that $\sum_{n=2}^{\infty}(n(n-\alpha)/(1-\alpha))a_n \leq 1$. If f is of the form* (2), *this sufficient condition is also necessary.*

In this note, we investigate the classes

$$TS_p(\alpha) = T \cap S_p(\alpha), \quad TV(\alpha) = T \cap UCV(\alpha). \tag{3}$$

We will give sufficient conditions for f of the form (1) to be in $S_p(\alpha)$ and $UCV(\alpha)$ and will show these conditions to also be necessary when f is of the form (2). This leads to extremal properties for these classes.

Ruscheweyh in [6] defined a δ-neighborhood for f of the form (1) by

$$N_\delta(F) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{4}$$

We will determine neighborhoods of $TS_p(\alpha)$ and $TV(\alpha)$ that consist of subclasses of starlike functions.

2. The classes $TS_p(\alpha)$ and $TV(\alpha)$

A function f of the form (2) is said to be in $TS_p(\alpha)$, $-1 \le \alpha < 1$, if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge \left|\frac{zf'(z)}{f(z)} - 1\right|$$

and is in $TV(\alpha)$ if $zf' \in TS_p(\alpha)$. Note that these definitions are equivalent to those given in (3).

We begin with some sufficient coefficient conditions.

Theorem 1. (a) *If* $\sum_{n=2}^{\infty}[2n-(\alpha+1)]\,|a_n| \le 1-\alpha$, $-1 \le \alpha < 1$, *then* f *of the form* (1) *is in* $S_p(\alpha)$.

(b) *If* $\sum_{n=2}^{\infty} n[2n-(\alpha+1)]\,|a_n| \le 1-\alpha$, *then* f *of the form* (1) *is in* $UCV(\alpha)$.

Proof of (a). It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| - \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - 1\right\} \le 1-\alpha, \qquad z \in \Delta.$$

We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| - \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - 1\right\} \le 2\left|\frac{zf'(z)}{f(z)} - 1\right|$$

$$\le \frac{2\sum\limits_{n=2}^{\infty}(n-1)|a_n|\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}|a_n|\,|z|^{n-1}} \le \frac{2\sum\limits_{n=2}^{\infty}(n-1)|a_n|}{1-\sum\limits_{n=2}^{\infty}|a_n|}.$$

This last expression is bounded above by $1-\alpha$ if $\sum_{n=2}^{\infty}[2n-(\alpha+1)]|a_n| \le 1-\alpha$, and the proof is complete.

Since by definition $f \in UCV(\alpha)$ if and only if $zf' \in S_p(\alpha)$, (b) follows from (a).

We now look at the subclasses for which these coefficient conditions are also necessary.

Theorem 2. (a) *A necessary and sufficient condition for f of the form* (2) *to be in $TS_p(\alpha)$, $-1 \leq \alpha < 1$, is that $\sum_{n=2}^{\infty}[2n-(\alpha+1)]a_n \leq 1-\alpha$.*

(b) *A necessary and sufficient condition for f of the form* (2) *to be in $TV(\alpha)$ is that $\sum_{n=2}^{\infty} n[2n-(\alpha+1)]a_n \leq 1-\alpha$.*

Proof. In view of Theorem 1, we need only prove the necessity. We do this for (a) only, since (b) then follows from the linear transformation.

If $f \in TS_p(\alpha)$ and z is real, then

$$\frac{1-\sum\limits_{n=2}^{\infty} na_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} a_n z^{n-1}} - \alpha \geq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} a_n z^{n-1}}.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality
$\sum_{n=2}^{\infty}[2n-(\alpha+1)a_n] \leq 1-\alpha$.

Ronning in [4] showed that $S_p(\alpha)$ is properly contained in $S^*\left(\frac{1+\alpha}{2}\right)$. Thus the special case $\alpha = -1$ in Theorem 1 yields the classical sufficient coefficient bounds for starlike and convex functions, respectively. For functions of the form (2) we can do better than inclusion in one direction.

Theorem 3. *For $-1 \leq \alpha < 1$, $TS_p(\alpha) = T^*\left(\frac{1+\alpha}{2}\right)$ and $TV(\alpha) = C\left(\frac{1+\alpha}{2}\right)$.*

Proof. Replacing α with $\frac{1+\alpha}{2}$ in the necessary and sufficient coefficient conditions in Theorem A, we obtain the corresponding coefficient condition of Theorem 2.

Remark. In [9], the family $UCT(\alpha)$, $\alpha \geq 0$, is defined for functions of the form (2) by $\mathrm{Re}\{1+zf''(z)/f'(z)\} \geq \alpha|zf''(z)/f'(z)|$, $z \in \Delta$. It can be shown from the coefficient conditions there that $TV(\alpha) = UCT\left(\frac{1+\alpha}{1-\alpha}\right)$, $-1 \leq \alpha < 1$.

A consequence of Theorem 3 is that any information known about $T^*(\alpha)$ and $C(\alpha)$ can be transformed into corresponding results for $TS_p(\alpha)$ and $TV(\alpha)$. We state two such corollaries based on theorems that may be found in [7].

Corollary 1. *The extreme points of* $TS_p(\alpha)$, $-1 \le \alpha < 1$, *are* $f_1(z)=z$ *and* $f_n(z) = z - \frac{1-\alpha}{2n-(\alpha+1)} z^n$, $n = 2, 3, \ldots$. *The extreme points of* $TV(\alpha)$ *are* $f_1(z) = z$ *and* $f_n(z) = z - \frac{1-\alpha}{n(2n-(\alpha+1))} z^n$, $n = 2, 3, \ldots$.

Corollary 2. (a) *If* $f \in TS_p(\alpha)$, $-1 \le \alpha < 1$, *then*

$$r - \frac{1-\alpha}{3-\alpha} r^2 \le |f(z)| \le r + \frac{1-\alpha}{3-\alpha} r^2,$$

$$1 - \frac{2(1-\alpha)}{3-\alpha} r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{3-\alpha} r, \quad |z| = r.$$

(b) *If* $f \in TV(\alpha)$, *then*

$$r - \frac{1-\alpha}{2(3-\alpha)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{2(3-\alpha)} r^2,$$

$$1 - \frac{1-\alpha}{3-\alpha} r \le |f'(z)| \le 1 + \frac{1-\alpha}{3-\alpha} r.$$

The results are best possible.

3. Neighborhoods

For δ-neighborhoods defiend by (4), the following was proved in [8].

Theorem B. *For* $0 \le \alpha < 1$, $N_\delta(T^*(\alpha)) \subset S^*(\beta)$ *when* $\delta = \frac{\alpha}{2-\alpha} - \beta$ *and* $N_\delta(C(\alpha)) \subset S^*(\beta)$ *when* $\delta = \frac{1}{2-\alpha} - \beta$.

In view of Theorem 3, a consequence of Theorem B is that $N_\delta(TS_p(\alpha)) = N_\delta\left(T^*\left(\frac{1+\alpha}{2}\right)\right) \subset S^*(\beta)$ when $\delta = \frac{1+\alpha}{3-\alpha} - \beta$, $-1 \le \alpha < 1$, and $N_\delta(TS_p(\alpha)) \subset S^*\left(\frac{1+\beta}{2}\right)$ when $\delta = \frac{1+\alpha}{3-\alpha} - \frac{1+\beta}{2} = \frac{3\alpha-1-\beta(3-\alpha)}{2(3-\alpha)}$. Noting that $S_p(\beta) \subset S^*\left(\frac{1+\beta}{2}\right)$, we can improve on this last inclusion. But first we need the following lemma.

Lemma A [8]. *For* f *of the form* (2), *if* $f \in T^*(\alpha)$ *then* $\sum_{n=2}^{\infty} na_n \le 2(1-\alpha)/(2-\alpha)$ *and if* $f \in C(\alpha)$ *then* $\sum_{n=2}^{\infty} na_n \le (1-\alpha)/(2-\alpha)$.

Theorem 4. *For* $-1 \le \alpha < 1$ *and* $-1 \le \beta \le (3\alpha - 1)/(3-\alpha)$, $N_\delta(TS_p(\alpha)) \subset S_p(\beta)$ *when* $\delta = \frac{3\alpha-1-\beta(3-\alpha)}{2(3-\alpha)}$.

Proof. Applying Lemma A to Theorem 3, we see that $\sum_{n=2}^{\infty} na_n \leq 2(1-\alpha)/(3-\alpha)$ when $f \in TS_p(\alpha)$. From (4) we have,

$$\sum_{n=2}^{\infty} n|b_n| \leq \sum_{n=2}^{\infty} na_n + \delta \leq 2(1-\alpha)/(3-\alpha) + \delta = (1-\beta)/2.$$

We may now apply Theorem 1 to this last inequality to conclude that $g \in S_p(\beta)$.

Corollary. *For* $-1 \leq \alpha < 1$ *and* $-1 \leq \beta \leq (1+\alpha)/(3-\alpha)$, $N_\delta(TV(\alpha)) \subset S_p(\beta)$ *when* $\delta = \frac{1+\alpha-\beta(3-\alpha)}{2(3-\alpha)}$.

Proof. From Lemma A we see that

$$\sum_{n=2}^{\infty} n|b_n| \leq \sum_{n=2}^{\infty} na_n + \delta \leq (1-\alpha)/(3-\alpha) + \delta = (1-\beta)/2,$$

and the result follows.

Finally, we remark that there is no δ-neighborhood of $TV(\alpha)$ for any α that contains only convex functions. For $f(z) = z$ and $g_n(z) = z + z^n/n^{3/2}$ we have $g_n \in N_\delta(f)$ when $n > 1/\delta^2$ even though $g_n \notin K$.

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