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On the linear independence of some functions

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Introduction

Let $\mathbf{C}$ be the set of complex numbers and $f, g :]0,1[\to \mathbf{C}$ be measurable functions satisfying the functional equation

$$f(xy) + f(x(1-y)) + f((1-x)y) + f((1-x)(1-y)) = \\ = g(x)g(y) \quad (x, y \in]0,1[\,). \tag{1}$$

From a result of the second author ([3], Theorem 3) it follows that there exist distinct complex numbers $\lambda_1, \dots, \lambda_M$ and natural numbers $m_1, \dots, m_M$ with $\sum_{j=1}^{M} m_j \le 20$ such that

$$f(x) = \sum_{j=1}^{M} \sum_{k=0}^{m_j - 1} c_{jk}\, x^{\lambda_j} \log^k x \qquad (x \in]0,1[\,), \tag{2}$$

where the c_{jk} $(j = 1, \dots M;\ k = 0, \dots, m_j - 1)$ are complex constants. If $g \ne O$, i.e. $g(y_0) \ne 0$ for some $y_0 \in]0,1[$, then (1), (2) yield

$$g(x) = \sum_{j=1}^{M} \sum_{\ell=0}^{m_j - 1} d_{j\ell}\, \alpha_{j\ell}(x) \qquad (x \in]0,1[\,), \tag{3}$$

where

$$\alpha_{j\ell}(x) = x^{\lambda_j} \log^\ell x + (1-x)^{\lambda_j} \log^\ell (1-x) \quad (x \in]0,1[\,), \tag{4}$$

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and $d_{j\ell}$ are constants defined by

$$d_{j\ell} = \sum_{k=\ell}^{m_j-1} c_{jk} \binom{k}{\ell} \alpha_{j,k-\ell}(y_0)/g(y_0)$$

$(j = 1, \dots, M;\ \ell = 0, \dots, m_j - 1)$. To find the constants $c_{jk}, d_{j\ell}$ we substitute (2), (3) into (1). We get the equation

$$\sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} \alpha_{j\ell}(x) D_{j\ell}(y) = 0 \qquad (x, y \in\,]0,1[\,) \tag{5}$$

where

$$D_{j\ell}(y) = \sum_{k=\ell}^{m_j-1} c_{jk} \binom{k}{\ell} \alpha_{j,k-\ell}(y) - d_{j\ell} \sum_{p=1}^{M} \sum_{q=0}^{m_p-1} d_{pq}\, \alpha_{pq}(y).$$

To get further information from (5), one has to know if the functions $\alpha_{j\ell}$ $(j = 1, \dots, M;\ \ell = 0, \dots, m_j - 1)$ are linearly independent or not.

The question of *linear independence* of the functions $\alpha_{j\ell}$ was raised by the second author at the 28th International Symposium on Functional Equations (August 23–Sept 1, 1990, Graz–Mariatrost, Austria). He proved that the functions $\gamma_0, \gamma_2, \dots, \gamma_{2n}$ are linearly independent while $\gamma_0, \gamma_2, \dots, \gamma_{2n}, \gamma_{2n+1}$ are linearly dependent, where

$$\gamma_\ell(x) = x^\ell + (1-x)^\ell \qquad (x \in\,]0,1[\,,\ \ell \in \mathbf{C})$$

(see [4]). It follows that the functions $\gamma_{j_1}, \gamma_{2+j_2}, \gamma_{4+j_3}, \dots, \gamma_{2(n-1)+j_n}, \gamma_{2n}, \gamma_{2n+1}$ are linearly dependent, where $j_k \in \{0,1\}$ for $k = 1, 2, \dots, n$.

The first result towards the general problem was found by the first author, proving that the existence of complex nonzero constants $A_1, \dots, A_M$ with the property

$$\sum_{j=1}^{M} A_j \left[x^{\lambda_j} + (1-x)^{\lambda_j}\right] = 0 \qquad (x \in\,]0,1[\,)$$

implies that $\lambda_1, \dots, \lambda_M$ are nonnegative integers. Extending the ideas of the proof the second author showed that if

$$\sum_{j=1}^{M} \sum_{k=0}^{m_j-1} A_{jk}\, \alpha_{jk}(x) = 0 \qquad (x \in\,]0,1[\,)$$

holds with constants $A_{jk} \in \mathbf{C}$ then $A_{jk} = 0$ for $k = 1, \dots, m_j - 1$ (provided that $m_j > 1$) and $A_{j0} = 0$ if $\lambda_j \notin \mathbf{N} \cup \{0\}$.

The aim of this paper is to present this latter result.

In some equations related to (1) the question of linear independence of the functions

$$\delta_{j\ell}(x) = x^{\lambda j} \log^{\ell} x - (1-x)^{\lambda j} \log^{\ell}(1-x) \qquad (x \in \,]0,1[\,)$$

arises. Our result will include this case as well.

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In the sequel $\mathbf{R}$ and $\mathbf{N}$ will denote the sets of real and natural numbers, respectively.

2. The linear independence of the functions $\alpha_{j\ell}, \delta_{j\ell}$

Our main result is the following

Theorem. *Let $M \in \mathbf{N}$ be a natural number, $\lambda_1, \dots, \lambda_M$ distinct complex numbers and $m_1, \dots, m_M$ natural numbers. Suppose that*

$$\sum_{j=1}^{M} \sum_{k=0}^{m_j-1} A_{jk} \left[x^{\lambda_j} \log^{k} x \pm (1-x)^{\lambda_j} \log^{k}(1-x) \right] = 0 \quad (x \in \,]0,1[\,) \tag{6}$$

holds for some constants A_{jk} ($j = 1, \dots, M$; $k = 0, \dots, m_j - 1$) either with the $+$ or with the $-$ sign. For any $j = 1, \dots, M$ we have

(7) if $\lambda_j \notin \mathbf{N} \cup \{0\}$ then $A_{jk} = 0 \quad (k = 0, \dots, m_j - 1)$,

(8) if $\lambda_j \in \mathbf{N} \cup \{0\}$ and $m_j > 1$ then $A_{jk} = 0$ $(k = 1, \dots, m_j - 1)$.

For the proof we need some lemmas.

Lemma 1. *Let $\beta_1, \dots, \beta_n$ $(n \geq 1)$ be distinct real numbers and $x \in \,]0,1[$ a real variable. Then*

$$\lim_{x \to 0+} \sum_{j=1}^{n} a_j x^{i\beta_j} = 0 \tag{9}$$

holds for complex $a_1, \dots, a_n$ if and only if

$$a_1 = \ldots = a_n = 0. \tag{10}$$

Proof of Lemma 1. The "if" part is obvious. We prove the "only if" part by induction on n.

Suppose first that (9) holds for $n = 1$. Then

$$\lim_{x \to 0+} |a_1 x^{i\beta_1}| = 0.$$

Since $|x^{i\beta_1}| = 1$ for $x > 0$, it follows that $a_1 = 0$.

Now suppose that (9) implies (10) for some $n \in \mathbf{N}$ and that

$$\lim_{x\to 0+} \sum_{j=1}^{n+1} a_j x^{i\beta_j} = 0. \tag{11}$$

Then we have

$$\lim_{x\to 0+} \left| \sum_{j=1}^{n+1} a_j x^{i\beta_j} \right| = 0, \qquad |x^{i\beta_{n+1}}| = 1,$$

hence, dividing by $x^{i\beta_{n+1}}$,

$$\lim_{x\to 0+} \left| \sum_{j=1}^{n} a_j x^{i(\beta_j - \beta_{n+1})} + a_{n+1} \right| = 0,$$

and thus

$$\lim_{x\to 0+} \left(\sum_{j=1}^{n} a_j x^{i(\beta_j - \beta_{n+1})} + a_{n+1} \right) = 0. \tag{12}$$

Next choose a real number $c \in \,]0,1[\,$ such that

$$c^{i(\beta_j - \beta_{n+1})} \neq 1 \qquad (j = 1, \dots, n). \tag{13}$$

This is possible because $\beta_1, \dots, \beta_n$ are distinct from β_{n+1}. Replacing x by cx in (12) and subtracting (12) from the equation so obtained we get

$$\lim_{x\to 0+} \sum_{j=1}^{n} a_j [c^{i(\beta_j - \beta_{n+1})} - 1]\, x^{i(\beta_j - \beta_{n+1})} = 0.$$

Since the numbers $\beta_1 - \beta_{n+1}, \dots, \beta_n - \beta_{n+1}$ are distinct we get by the induction hypothesis and by (13) that

$$a_1 = \ldots = a_n = 0.$$

Using (11) or (12) again we obtain $a_{n+1} = 0$, and this completes the proof of Lemma 1.

Remark 1.. Let $\beta_1, \dots, \beta_n$ be distinct real numbers and suppose that

$$\lim_{x\to 0+} \left(\sum_{j=1}^{n} a_j x^{i\beta_j} + a_{n+1} \right) = 0 \tag{14}$$

holds for complex $a_1, \dots , a_n,\ a_{n+1}$.

If $\beta_1\beta_2 \dots \beta_n \neq 0$ then (14) holds if and only if

$$a_1 = \dots = a_n = a_{n+1} = 0. \tag{15}$$

If one of the β_j's is zero, say $\beta_r = 0,\ 1 \le r \le n$, then (14) holds if and only if

$$a_r + a_{n+1} = 0 \text{ and } a_j = 0 \text{ for } j = 1, \dots , n;\ j \neq r. \tag{16}$$

Indeed, in the case $\beta_1\beta_2 \dots \beta_n \neq 0$ we can write (14) in the form

$$\lim_{x\to 0+} \sum_{j=1}^{n+1} a_j x^{i\beta_j} = 0$$

with $\beta_{n+1} = 0$, thus by Lemma 1 (14) and (15) are equivalent.

If $\beta_r = 0$ we write (14) in the form

$$\lim_{x\to 0+} \left[(a_r + a_{n+1})x^{i\beta_r} + \sum_{\substack{j=1 \\ j\neq r}}^{n} a_j\, x^{i\beta_j} \right] = 0$$

hence by Lemma 1 we conclude that (14) holds if and only if (16) is valid.

Lemma 2. *Let $\lambda \in \mathbf{C},\ k \in \mathbf{R}$ be fixed values and let*

$$h(x) = x^\lambda \log^k x \qquad (x \in \,]0, 1[\,). \tag{17}$$

Then h is differentiable arbitrarily many times in $]0, 1[$ and $h^{(n)}$ is given by

(18)

$$h^{(n)}(x) = x^{\lambda-n} \sum_{\ell=0}^{n} (\lambda)_n^{n-\ell}\, (k)_\ell^\ell\, \log^{k-\ell} x \qquad (x \in \,]0, 1[\,;\ n = 0, 1, \dots),$$

where, for any $\lambda \in \mathbf{C}$,

$$(\lambda)_n^0 := 1 \quad \text{for} \quad n = 0, 1, \dots , \tag{19}$$

while for $n \in \mathbf{N},\ 1 \le s \le n$,

$$(\lambda)_n^s := \sigma_{sn}(\lambda, \lambda - 1, \dots , \lambda - n + 1). \tag{20}$$

Here σ_{sn} denotes the sth elementary symmetric polynomial of n variables. Thus, e.g.

$$\begin{aligned}
(\lambda)_n^1 &:= \lambda + (\lambda-1) + \ldots + (\lambda-n+1),\\
(\lambda)_n^2 &:= \lambda(\lambda-1) + \lambda(\lambda-2) + \ldots + (\lambda-n+2)(\lambda-n+1),\\
&\vdots\\
(\lambda)_n^n &:= \lambda(\lambda-1)\ldots(\lambda-n+1).
\end{aligned}$$

Proof of Lemma 2. It is easy to see that (18) is valid for $n = 0, 1$. Suppose that (18) holds for $n \in \mathbf{N}$. Differentiating (18) with respect to x we get

$$\begin{aligned}
h^{(n+1)}(x) = x^{\lambda-(n+1)} \Bigg[&\sum_{\ell=0}^{n} (\lambda)_n^{n-\ell} (\lambda-n) (k)_\ell^\ell \log^{k-\ell} x + \\
&+ \sum_{\ell=0}^{n} (\lambda)_n^{n-\ell} (k)_\ell^\ell (k-\ell) \log^{k-\ell-1} x \Bigg].
\end{aligned} \tag{21}$$

We can decompose the first sum as

$$\sum_{\ell=1}^{n} (\lambda)_n^{n-\ell} (\lambda-n) (k)_\ell^\ell \log^{k-\ell} x + (\lambda)_n^n (\lambda-n)(k)_0^0 \log^k x. \tag{22}$$

Replacing $\ell+1$ by ℓ in the second sum and separating its last member, we can rewrite the second sum as

$$\sum_{\ell=1}^{n} (\lambda)_n^{n+1-\ell} (k)_{\ell-1}^{\ell-1} (k-\ell+1) \log^{k-\ell} x + (\lambda)_n^0 (k)_n^n (k-n) \log^{k-n-1} x. \tag{23}$$

Since, for any $\lambda \in \mathbf{C}$,

$$\begin{aligned}
&(\lambda)_n^n (\lambda-n) = (\lambda)_{n+1}^{n+1}, \qquad (\lambda)_n^0 = 1 = (\lambda)_{n+1}^0 \quad (n = 0, 1, \ldots),\\
&(\lambda)_n^{n-\ell} (\lambda-n) + (\lambda)_n^{n+1-\ell} = (\lambda)_{n+1}^{n+1-\ell} \quad (n = 1, 2, \ldots;\ \ell = 1, 2, \ldots n)
\end{aligned}$$

and

$$(k)_{\ell-1}^{\ell-1} (k-\ell+1) = (k)_\ell^\ell \qquad (\ell = 1, 2, \ldots, n),$$

using (22), (23) we can write (21) as

$$h^{(n+1)}(x) = x^{\lambda-(n+1)} \sum_{\ell=0}^{n+1} (\lambda)_{n+1}^{n+1-\ell} (k)_\ell^\ell \log^{k-\ell} x$$

which completes the proof.

Remark 2. We shall need (18) in the case when k is a nonnegative integer. Since for $\ell > k$ ($\ell \in \mathbf{N}$) we have

$$(k)_\ell^\ell = 0,$$

(18) can be written as

$$h^{(n)}(x) = x^{\lambda-n} \sum_{\ell=0}^{\min\{n,k\}} (\lambda)_n^{n-\ell} (k)_\ell^\ell \log^{k-\ell} x \qquad (x \in\,]0,1[\,;\ k,n = 0,1,\dots).$$

Writing ℓ for $k-\ell$ here, we obtain

$$h^{(n)}(x) = x^{\lambda-n} \sum_{\ell=\max\{0,k-n\}}^{k} (\lambda)_n^{n-k+\ell}\, (k)_{k-\ell}^{k-\ell} \log^\ell x \qquad (x \in\,]0,1[\,;\ k,n = 0,1,\dots). \tag{24}$$

In the *proof of the main result* we shall need a formula for the nth derivative of equation (6). By (24) and the identity

$$\sum_{k=0}^{m_j-1} \sum_{\ell=\max\{0,k-n\}}^{k} u_{k\ell} = \sum_{\ell=0}^{m_j-1} \sum_{k=\ell}^{\min\{\ell+n,m_j-1\}} u_{k\ell},$$

we have for the nth derivative of (6)

$$\sum_{j=1}^{M} \sum_{\ell=0}^{m_j-1} B_{j\ell}^{(n)} \Big[x^{\lambda_j-n} \log^\ell x \pm (-1)^n (1-x)^{\lambda_j-n} \log^\ell (1-x) \Big] = 0 \tag{25}$$

for $x \in\,]0,1[$, where

$$B_{j\ell}^{(n)} := \sum_{k=\ell}^{\min\{\ell+n,m_j-1\}} A_{jk} (\lambda_j)_n^{n-k+\ell}\, (k)_{k-\ell}^{k-\ell} \tag{26}$$

for $j = 1,\dots,M;\ \ell = 0,\dots,m_j-1;\ n = 0,1,\dots.$

Lemma 3. *Assume the hypotheses of the Theorem. Suppose that for some $j \in \{1,\dots,M\}$ and for some $n \in \mathbf{N} \cup \{0\}$ we have $(\lambda_j)_n^n = \lambda_j(\lambda_j-1)\dots(\lambda_j-n+1) \neq 0$ and*

$$B_{j\ell}^{(n)} = 0 \quad \text{for} \quad \ell = m_j-1, m_j-2, \dots, 1, 0, \tag{27}$$

where $B_{j\ell}^{(n)}$ is defined by (26). Then

$$A_{j\ell} = 0 \quad \text{for} \quad \ell = m_j-1, m_j-2, \dots, 1, 0. \tag{28}$$

Proof of Lemma 3. By (26), (27) we see that

$$B^{(n)}_{j,m_j-1} = A_{j,m_j-1}\,(\lambda_j)^n_n\,(m_j-1)^0_0 = 0.$$

Since $(m_j-1)^0_0 = 1$ and $(\lambda_j)^n_n \neq 0$, this implies (28) for $\ell = m_j - 1$. Assume that we have proved (28) for $\ell = m_j-1, m_j-2, \dots, m_j-q$ where $1 \le q \le m_j - 1$. Then by (26), (27) and by our hypothesis we have

$$B^{(n)}_{j,m_j-q-1} = A_{j,m_j-q-1}\,(\lambda_j)^n_n\,(m_j-q-1)^0_0 = 0$$

hence $A_{j,m_j-q-1} = 0$. Letting q run over $\{1,\dots,m_j-1\}$, the proof of Lemma 3 is completed.

Remark 3. If (under the assumptions of Lemma 3) (27) holds only for $\ell = m_j-1, m_j-2, \dots, 1$ then (28) also holds for these subscripts only. This is clear from the proof of Lemma 3.

Lemma 4. *Let* $x \in]0,1[$ *be a real variable,* $\lambda \in \mathbf{C}$, $k \in \mathbf{R}$. *If* $\operatorname{Re}\lambda > 0$, $k \in \mathbf{R}$ *or* $Re\lambda = 0$, $k < 0$ *we have*

$$\lim_{x\to 0+} x^{\lambda}\log^k x = 0; \tag{29}$$

and for any $\lambda \in \mathbf{C}$,

$$\lim_{x\to 0+}(1-x)^{\lambda}\log^k(1-x) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0. \end{cases} \tag{30}$$

Proof of Lemma 4. Let $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}$, then

$$x^{\lambda}\log^k x = x^{i\beta}(x^{\alpha}\log^k x),$$

where by hypothesis either $\alpha > 0$ and k is arbitrary, or $\alpha = 0$ and $k < 0$. The first factor $x^{i\beta}$ has absolute value 1 while the limit of the second factor (in parantheses) is zero if $x \to 0+$. This is obvious if $\alpha = 0$, $k < 0$ or $\alpha > 0$, $k \le 0$, while for $\alpha > 0$, $k > 0$ it can be proved by applying L'Hospital's rule.

Concerning (30), we have $(1-x)^{\lambda} \to 1$ if $\lambda \in \mathbf{C}$, $x \to 0+$, and the limit of $\log^k(1-x)$ as $x \to 0+$ is 1 if $k = 0$ and 0 if $k > 0$.

3. The proof of the main result

Now we are ready to prove our theorem. Let $\lambda_j = \alpha_j + i\beta_j$ ($\alpha_j, \beta_j \in \mathbf{R}$; $j = 1, \dots, M$) and arrange the λ_j's such that

$$\alpha_1 = \ldots = \alpha_{p_1} < \alpha_{p_1+1} = \ldots = \alpha_{p_2} < \ldots < \alpha_{p_{s-1}+1} = \ldots = \alpha_{p_s}. \tag{31}$$

Here $p_1 < p_2 \ldots < p_s$ are natural numbers, $p_s = M$. We may suppose that

$$m_1 \geq m_2 \geq \ldots \geq m_{p_1}; \tag{32}$$

moreover, we may choose the natural numbers $q_1 < q_2 < \ldots < q_u,\ q_u = p_1$ such that

$$m_1 = \ldots = m_{q_1} > m_{q_1+1} = \ldots = m_{q_2} > \ldots > m_{q_{u-1}+1} = \ldots = m_{q_u}. \tag{33}$$

First we show that the statement of the theorem holds for the subscripts $j = 1, \ldots, p_1$.

We shall distinguish two cases:

Case 1: $\alpha_1 = \ldots = \alpha_{p_1} \notin \mathbf{N} \cup \{0\}$.

Case 2: $\alpha_1 = \ldots = \alpha_{p_1} \in \mathbf{N} \cup \{0\}$.

In both cases we shall calculate the limit of the nth derivative of (6) divided by suitable functions as $x \to 0+$. Then by Lemma 1 we conclude that for some $n \in \mathbf{N} \cup \{0\}$, $B_{j\ell}^{(n)} = 0$, $(\lambda_j)_n^n \neq 0$ and by Lemma 3 we arrive at the statements (7), (8).

In case 1, choose $n \in \mathbf{N} \cup \{0\}$ such that $n - \alpha_1 > 0$. Differentiating (6) n times we obtain (25). Dividing it by $x^{\alpha_1 - n} \log^{m_1 - 1} x$ and taking its limit as $x \to 0+$ we have

$$\lim_{x \to 0+} \sum_{j=1}^{M} \sum_{\ell=0}^{m_j - 1} B_{j\ell}^{(n)} \Big[x^{\alpha_j - \alpha_1}\, x^{i\beta_j} \log^{\ell - m_1 + 1} x \pm$$
$$\pm (-1)^n\, x^{n - \alpha_1} \log^{1 - m_1} x\, (1 - x)^{\lambda_j - n}\, \log^{\ell}(1 - x) \Big] = 0.$$

For all subscripts $j = 1, \ldots, M;\ \ell = 0, \ldots, m_j - 1$, the limit of the second term in the bracket is zero by (29), (30) and $n - \alpha_1 > 0$. For $j > p_1;\ \ell = 0, \ldots, m_j - 1$, the limit of the first term in the bracket is zero, since for these subscripts $\alpha_j - \alpha_1 > 0$. If $j = 1, \ldots, p_1;\ \ell < m_1 - 1$ or $j = q_1 + 1, \ldots, q_u (= p_1)$ then again the limit of the first term of the bracket is zero, since the exponent of $\log x$ is negative (by the condition $\ell < m_1 - 1$ or by (33)). Thus we have finally

$$\lim_{x \to 0+} \sum_{j=1}^{q_1} B_{j, m_1 - 1}^{(n)}\, x^{i\beta_j} = 0.$$

Note that $\beta_1, \ldots, \beta_{q_1}$ are distinct, because of $\alpha_1 = \ldots = \alpha_{q_1}$ (cf. (31)) and the distinctness of $\lambda_1, \ldots, \lambda_M$. By Lemma 1 we obtain

$$B_{j, m_1 - 1}^{(n)} = 0 \qquad (j = 1, \ldots, q_1)$$

and thus by (33)

$$B^{(n)}_{j,m_j-1} = B^{(n)}_{j,m_1-1} = 0 \qquad (j = 1, \ldots, q_1).$$

If $m_1 - 1 > 0$ then we omit the terms with zero coefficients from (25), divide it by $x^{\alpha_1 - n} \log^{m_1-2} x$ and take its limit as $x \to 0+$. In the same way as above we get

$$\lim_{x\to 0+} \sum_{j=1}^{q_1} B^{(n)}_{j,m_1-2}\, x^{i\beta_j} = 0$$

hence by Lemma 1 and (33)

$$B^{(n)}_{j,m_j-2} = 0 \qquad (j = 1, \ldots, q_1).$$

Similarly

$$B^{(n)}_{j\ell} = 0 \qquad (j = 1, \ldots, q_1;\ \ell = 0, \ldots, m_1 - 1).$$

Continuing this process with the next groups $j = q_1+1, \ldots, q_2; \ldots ; q_{u-1}+1, \ldots, q_u = p_1$ of subscripts (i.e. division by $x^{\alpha_1-n} \log^k x$, $k = m_{q_1+1} - 1, m_{q_1+1} - 2, \ldots, 1, 0; \ldots ; m_{q_{u-1}+1} - 1, m_{q_{u-1}+1} - 2, \ldots, 1, 0$, and taking the limit as $x \to 0+$) we obtain after similar arguments that

$$B^{(n)}_{j\ell} = 0 \qquad (j = 1, \ldots, p_1;\ \ell = 0, \ldots, m_j - 1).$$

By Lemma 3 this implies that

$$A_{j\ell} = 0 \qquad (j = 1, \ldots, p_1;\ \ell = 0, \ldots, m_j - 1), \tag{34}$$

since $(\lambda_j)^n_n \neq 0$ for $j = 1, \ldots, p_1$ because of $\alpha_1 \notin \mathbf{N} \cup \{0\}$, $n - \alpha_1 > 0$. Thus we have proved (7) for $j = 1, \ldots, p_1$.

In case 2 suppose first that *there is a natural number* q_t $(1 \le t \le u)$ among the $q_1, \ldots, q_u$ such that

$$m_{q_t} > 1 \text{ and } m_j = 1 \text{ for } q_t < j \le q_u (= p_1). \tag{35}$$

Let $n = \alpha_1$, differentiate (6) n times, divide it by $x^{\alpha_1-n} \log^k x = \log^k x$ with $k = m_1 - 1, m_1 - 2, \ldots, 1; \ldots ; m_{q_{t-1}+1} - 1, m_{q_{t-1}+1} - 2, \ldots, 1$ and take the limit as $x \to 0+$ for each value of k in the indicated order. With the same reasoning as in case 1 we get

$$B^{(n)}_{j\ell} = 0 \qquad (j = 1, \ldots, q_t;\ \ell = 1, \ldots, m_j - 1). \tag{36}$$

If there is not any q_t with property (35) then by (32), (33)

$$m_1 = \ldots = m_{p_1} = 1 \tag{37}$$

i.e. there is no coefficient $A_{j\ell}$ with $j = 1, \dots, p_1$, $\ell \geq 1$.

Let us now take the limit of (25) with $n = \alpha_1$ as $x \to 0+$. By (35) – (36) or (37), together with (29), (30), (31), we obtain

$$\lim_{x \to 0+} \sum_{j=1}^{p_1} B_{j,0}^{(n)} \, x^{i\beta_j} \pm \sum_{j=1}^{M} B_{j,0}^{(n)} \, (-1)^n = 0. \tag{38}$$

If $\beta_1 \beta_2 \dots \beta_{p_1} \neq 0$ then by Remark 1, (38) implies that

$$B_{j,0}^{(n)} = 0 \qquad (j = 1, \dots, p_1). \tag{39}$$

If $\beta_1 \beta_2 \dots \beta_{p_1} = 0$ then only one of the factors here can be zero, otherwise $\lambda_1, \dots, \lambda_{p_1}$ were not distinct. Let $\beta_r = 0$, $1 \leq r \leq p_1$, then again by Remark 1

$$B_{j,0}^{(n)} = 0 \qquad (j = 1, \dots, p_1;\ j \neq r). \tag{40}$$

Since $(\lambda_j)_n^n \neq 0$ for $j = 1, \dots, p_1$, applying Lemma 3 and Remark 3 together with either (35) – (36) or (37), we obtain from (39) that

$$A_{j\ell} = 0 \quad \text{for} \quad j = 1, \dots, p_1;\ \ell = 0, \dots, m_j - 1 \tag{41}$$

if $\alpha_1 \in \mathbf{N} \cup \{0\}$ and $\beta_1 \beta_2 \dots \beta_{p_1} \neq 0$ (i.e. if none of the numbers $\lambda_1, \dots, \lambda_{p_1}$ is a nonnegative integer), and we obtain from (40) that

$$\begin{aligned} A_{j\ell} = 0 \quad \text{for} \quad & j = 1, \dots, p_1;\ j \neq r,\ \ell = 0 \\ & \text{and for } j = 1, \dots, q_t;\ \ell = 1, \dots, m_j - 1 \text{ (if (35) holds)} \end{aligned} \tag{42}$$

provided that $\alpha_1 \in \mathbf{N} \cup \{0\}$ and $\beta_r = 0$.

Thus we have proved the theorem for $j = 1, \dots, p_1$.

Suppose now that we have proved our theorem for subscripts $j = 1, \dots, p_1, \dots, p_m$. We claim that for the subscripts $j = p_m + 1, p_m + 2, \dots, p_{m+1}$ $(m = 1, 2, \dots, s-1)$ the same method can be applied as the one we used for $j = 1, 2, \dots, p_1$.

Differentiate (6) n times, where now $n - \alpha_{p_m+1} > 0$ if $\alpha_{p_m+1} = \alpha_{p_m+2} = \dots = \alpha_{p_{m+1}} \notin \mathbf{N} \cup \{0\}$ and $n = \alpha_{p_m+1}$ if $\alpha_{p_m+1} = \alpha_{p_m+2} = \dots = \alpha_{p_{m+1}} \in \mathbf{N} \cup \{0\}$. Due to the arrangement (31), in both cases n is greater than any of the nonnegative integers among the exponents $\lambda_1, \dots, \lambda_{p_m}$. Hence, after the differentation both x^{λ_j} and $(1-x)^{\lambda_j}$ $(\lambda_j \in \mathbf{N} \cup \{0\}, 1 \leq j \leq p_m,\ m_j = 1)$ will disappear and we get

$$\sum_{j=p_m+1}^{M} \sum_{\ell=0}^{m_j-1} B_{j\ell}^{(n)} \left[x^{\lambda_j - n} \log^\ell x \pm (-1)^n (1-x)^{\lambda_j - n} \log^\ell (1-x) \right] = 0.$$

This is an equation of the form (25) with subscripts j shifted by p_m. Repeating the arguments used for subscripts $j = 1, \dots, p_1$ for the subscript groups $j = p_m + 1, \dots, p_{m+1}$ $(m = 1, 2, \dots, s-1)$ we can complete the proof.

4. Further remarks

As stated in the introduction, the motivation for this work comes from the functional equation (1). This equation arises from characterization problems for information measures having the sum property with measurable generating function which satisfy a generalized (2, 2)–additivity (see [1], [2]). Applications of the results in this paper to such characterization problems will be pursued elsewhere.

References

[1] B. Ebanks, P. K. Sahoo and W. Sander, Determination of measurable sum form information measures satisfying (2, 2)–additivity of degree (α, β), *Radovi Matematički* **6** (1990), 77–96.

[2] L. Losonczi, On a functional equation of sum form, *Publ. Math. Debrecen* **36** (1989), 167–177.

[3] L. Losonczi, Measurable solutions of functional equations of sum form, *To appear in Acta Math. Hung.*

[4] L. Losonczi, 39. Remark. In: Report of Meeting. The Twenty-eight International Symposium on Functional Equations, Graz–Mariatrost, 1990., *Aequationes Math.* **41** (1991), 302–303.

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