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On the direct decomposition of Pappian projective Veldkamp planes

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Abstract. The theory of projective planes over rings of stable rank 2 was developed by F.D. Veldkamp. In this paper a necessary and sufficient condition will be given for a Veldkamp plane to be a direct product of a collection of n Veldkamp planes.

1. Introduction

The factorisation-problem of Pappian Veldkamp planes is closely related to the decomposition of commutative rings. A commutative ring R is decomposable if and only if it contains a full system of orthogonal idempotent elements. We are looking for a geometric interpretation of this system of elements in case the commutative ring in question is the coordinate-ring of some projective Pappian Veldkamp plane. The goal is that the existence of a certain configuration serves as a neccesary and sufficient condition for the decomposability of projective Pappian Veldkamp planes. The configuration which guarantees the decomposability is the direct product of n copies of the Δ triangle-configuration (Δ^n) joined to the well-known Thomsen-configuration.

2. Definitions and preliminary results

For the sets $\boldsymbol{P}$ and $\boldsymbol{B}$ let us consider the binary relations $\boldsymbol{I} \subseteq \boldsymbol{P} \times \boldsymbol{B}$ and $\approx\, \subseteq \boldsymbol{P} \times \boldsymbol{B}$. The quadruple $\boldsymbol{D} = (\boldsymbol{P}, \boldsymbol{B}, I, \approx)$ will be called an

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incidence-neighbouring-structure or shortly IN-structure. The elements of set $\boldsymbol{P}$ are the points, the elements of the set $\boldsymbol{B}$ are called lines of the structure. A point $p \in \boldsymbol{P}$ and a line $L \in \boldsymbol{B}$ are incident (neighbouring) if $(p, L) \in \boldsymbol{I}$ (resp. $(p, L) \in \approx$). Instead of $(p, L) \in \boldsymbol{I}$ ($(p, L) \in \approx$) we will use the notation $p\boldsymbol{I}L$ (resp. $p \approx L$). The neighbour-relation may be extended to pairs of points (to pairs of lines) by the following definition: Point $p \in \boldsymbol{P}$ is neighbouring to point $q \in \boldsymbol{P}$ (line $L \in \boldsymbol{B}$ to line $M \in \boldsymbol{B}$) if for all lines $L \in \boldsymbol{B}$ (points $p \in \boldsymbol{P}$) for which $q\boldsymbol{I}l$ ($p\boldsymbol{I}m$) holds, $p \approx L$ is valid as well. The fact that the point p is neighbouring to the point q (line L to line M), will be be denoted by $p \approx q$ ($L \approx M$ resp.). Pairs of elements are often called distant if they are not neighbours, i.e., distant means: $\not\approx$.

A trivial example for an IN-structure is the classical projective plane. (Here incidence and neighbouring coincide.) Other examples: The projective Hjelmslev plane (Klingenberg [1954]), the projective Klingenberg plane (Klingenberg [1955, 1956]), and the projective Veldkamp plane (Veldkamp [1981, 1988, 1995]).

Let $\boldsymbol{D} = (\boldsymbol{P}, \boldsymbol{B}, \boldsymbol{I}, \approx)$ be a finite IN-structure with $\boldsymbol{P} = \{p_1, \dots, p_r\}$ and $\boldsymbol{B} = \{L_1, \dots, L_s\}$. The incidence matrix $M(\boldsymbol{D}) = (m_{ij})_{r\times s}$ of $\boldsymbol{D}$ is defined by

$$m_{ij} := 1 \Leftrightarrow p_i\boldsymbol{I}L_j$$
$$m_{ij} := 0 \text{ otherwise.}$$

Similarly, the neighbour-matrix $N(\boldsymbol{D}) = (n_{ij})_{r\times s}$ of $\boldsymbol{D}$ is defined by

$$n_{ij} := 1 \Leftrightarrow p_i \approx L_j$$
$$n_{ij} := 0 \text{ otherwise.}$$

The neighbour-relation in the point-point and the line-line case may be described by neighbour-matrices $N_P(\boldsymbol{D}) = (n^P_{ij})_{r\times r}$ and $N_B(\boldsymbol{D}) = (n^B_{ij})_{s\times s}$ defined by

$$n^P_{ij} := 1 \Leftrightarrow p_i \approx p_j \qquad n^B_{ij} := 1 \Leftrightarrow L_i \approx L_j$$
$$n^P_{ij} := 0 \text{ otherwise} \qquad n^B_{ij} := 0 \text{ otherwise.}$$

In our investigations a basic role will be played by the direct product of IN-structures. For every $i = 1, \dots, n$ let $\boldsymbol{D}_i = (\boldsymbol{P}_i, \boldsymbol{B}_i, \boldsymbol{I}_i, \approx_i)$ be an

IN-structure, and put $\boldsymbol{P} = P_1 \times \cdots \times P_n$ and $\boldsymbol{B} = \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_n$. We have to define the incidence- and the neighbour-relations over $\boldsymbol{P} \times \boldsymbol{B}$. The incidence-relation $\boldsymbol{I} \subseteq \boldsymbol{P} \times \boldsymbol{B}$ is given by

$$(p_1, \ldots, p_n)\boldsymbol{I}(L_1, \ldots, L_n) :\Leftrightarrow (p_1\boldsymbol{I}_1 L_1 \wedge \cdots \wedge p_n \boldsymbol{I}_n L_n),$$

and the neighbour-relation $\approx \subseteq \boldsymbol{P} \times \boldsymbol{B}$ by

$$(p_1, \ldots, p_n) \approx (L_1, \ldots, L_n) :\Leftrightarrow (p_1 \approx_1 L_1 \vee \cdots \vee p_n \approx_n L_n).$$

The resulting IN-structure $(\boldsymbol{P}, \boldsymbol{B}, \boldsymbol{I}, \approx)$ will be called the direct product of the IN-structures $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ and denoted by $\boldsymbol{D}_1 \times \cdots \times \boldsymbol{D}_n$. Especially, if $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ are projective Veldkamp planes, then $\boldsymbol{D}_1 \times \cdots \times \boldsymbol{D}_n$ is a projective Veldkamp plane as well. In this structure

$$(p_1, \ldots, p_n) \approx (q_1, \ldots, q_n) \Leftrightarrow (p_1 \approx_1 q_1 \vee \cdots \vee p_n \approx_n q_n)$$

and

$$(L_1, \ldots, L_n) \approx (M_1, \ldots, M_n) \Leftrightarrow (L_1 \approx_1 M_1 \vee \cdots \vee L_n \approx_n M_n).$$

are valid.

3. The Δ- and the Δ^n-configuration

Let $\Delta = (\boldsymbol{P}, \boldsymbol{B}, \boldsymbol{I}, \approx)$ be an IN-structure, where $\boldsymbol{P} = \{p_0, p_1, p_2\}$ is the set of points, $\boldsymbol{B} = \{L_0, L_1, L_2\}$ is the set of lines, and the incidence-relation is given by

$$p_i \boldsymbol{I} L_j \Leftrightarrow i \neq j \quad (i, j = 0, 1, 2).$$

The neighbour-relation coincides with the incidence-relation:

$$p_i \approx L_j :\Leftrightarrow p_i \boldsymbol{I} L_j.$$

By definition, the neighbouring of points resp. lines is given by

$$p_i \approx p_j \Leftrightarrow i = j, \quad (i, j = 0, 1, 2)$$

and by

$$L_i \approx L_j \Leftrightarrow i = j, \quad (i, j = 0, 1, 2).$$

The incidence-matrix $M(\Delta)$ and the neighbour-matrices of Δ are the following:

$$M(\Delta) = N(\Delta) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad N_P(\Delta) = N_B(\Delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\Delta^2 = \Delta \times \Delta$. It is easy to show that the incidence- and neighbour-relations in Δ^2 may be described by

$$\begin{aligned}
(p_{i_1}, p_{i_2})\boldsymbol{I}(L_{j_1}, L_{j_2}) &\Leftrightarrow (i_1 \neq j_1 \wedge i_2 \neq j_2);\\
(p_{i_1}, p_{i_2}) \approx (L_{j_1}, L_{j_2}) &\Leftrightarrow (i_1 \neq j_1 \vee i_2 \neq j_2);\\
(p_{i_1}, p_{i_2}) \approx (q_{j_1}, q_{j_2}) &\Leftrightarrow (i_1 = j_1 \vee i_2 = j_2);\\
(L_{i_1}, L_{i_2}) \approx (M_{j_1}, M_{j_2}) &\Leftrightarrow (i_1 = j_1 \vee i_2 = j_2),
\end{aligned}$$

where $i_1, i_2, j_1, j_2 = 0, 1, 2$.

Let us introduce the following notation:

$$q_{3i+j} := (p_i, p_j) \qquad \text{and} \quad G_{3i+j} := (L_i, L_j) \quad (i, j = 0, 1, 2).$$

Using this notation the point-set and the line-set of Δ^2 are $\{q_0, \dots, q_8\}$ and $\{G_0, \dots, G_8\}$ respectively. The incidence- and neighbour-matrices of Δ^2 are:

$$M(\Delta^2) = \begin{pmatrix}
0 & 0 & 0 & \cdot & 0 & 1 & 1 & \cdot & 0 & 1 & 1 \\
0 & 0 & 0 & \cdot & 1 & 0 & 1 & \cdot & 1 & 0 & 1 \\
0 & 0 & 0 & \cdot & 1 & 1 & 0 & \cdot & 1 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 1 & \cdot & 0 & 0 & 0 & \cdot & 0 & 1 & 1 \\
1 & 0 & 1 & \cdot & 0 & 0 & 0 & \cdot & 1 & 0 & 1 \\
1 & 1 & 0 & \cdot & 0 & 0 & 0 & \cdot & 1 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 1 & \cdot & 0 & 1 & 1 & \cdot & 0 & 0 & 0 \\
1 & 0 & 1 & \cdot & 1 & 0 & 1 & \cdot & 0 & 0 & 0 \\
1 & 1 & 0 & \cdot & 1 & 1 & 0 & \cdot & 0 & 0 & 0
\end{pmatrix}$$

$$N(\Delta^2) = \begin{pmatrix} 0 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 1 & 0 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & 0 & 1 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 0 & 1 & \cdot & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 & 0 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 0 \end{pmatrix}$$

$$N_P(\Delta^2) = N_B(\Delta^2) = \begin{pmatrix} 1 & 1 & 1 & \cdot & 1 & 0 & 0 & \cdot & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdot & 0 & 1 & 0 & \cdot & 0 & 1 & 0 \\ 1 & 1 & 1 & \cdot & 0 & 0 & 1 & \cdot & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & 1 & 1 & 1 & \cdot & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdot & 1 & 1 & 1 & \cdot & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdot & 1 & 1 & 1 & \cdot & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & 1 & 0 & 0 & \cdot & 1 & 1 & 1 \\ 0 & 1 & 0 & \cdot & 0 & 1 & 0 & \cdot & 1 & 1 & 1 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 1 & \cdot & 1 & 1 & 1 \end{pmatrix}$$

(cf. Figure 1).

Similarly, we define the direct product Δ^n as the direct product of n copies of Δ. The incidence- and neighbour-relations may be described by indices:

$$\begin{aligned} (p_{i_1}, \dots, p_{i_n}) \boldsymbol{I} (L_{j_1}, \dots, L_{j_n}) &\Leftrightarrow (i_1 \neq j_1 \wedge \dots \wedge i_n \neq j_n); \\ (p_{i_1}, \dots, p_{i_n}) \approx (L_{j_1}, \dots, L_{j_n}) &\Leftrightarrow (i_1 \neq j_1 \vee \dots \vee i_n \neq j_n); \\ (p_{i_1}, \dots, p_{i_n}) \approx (q_{j_1}, \dots, q_{j_n}) &\Leftrightarrow (i_1 = j_1 \vee \dots \vee i_n = j_n); \\ (L_{i_1}, \dots, L_{i_n}) \approx (M_{j_1}, \dots, M_{j_n}) &\Leftrightarrow (i_1 = j_1 \vee \dots \vee i_n = j_n); \end{aligned}$$

$$i_1, \dots, i_n, j_1, \dots, j_n = 0, 1, 2; \quad (p_{i_1}, \dots, p_{i_n}), (q_{j_1}, \dots, q_{j_n}) \in \boldsymbol{P};$$
$$(L_{i_1}, \dots, L_{i_n}), (M_{j_1}, \dots, M_{j_n}) \in \boldsymbol{B}.$$

For the sake of a simple description of IN-matrices on Δ^n we introduce the notations

$$\begin{aligned} q_{3^{n-1}i_1+3^{n-2}i_2+\cdots+i_n} &:= (p_{i_1},\ldots,p_{i_n}) \\ G_{3^{n-1}i_1+3^{n-2}i_2+\cdots+i_n} &:= (L_{i_1},\ldots,L_{i_n}), \end{aligned}$$

where $i_1,\ldots,i_n = 0,1,2$. We shall give the IN-matrices $M(\Delta^n)$, $N(\Delta^n)$, $N_P(\Delta^n)$ and $N_B(\Delta^n)$ by recursion. It is easy to show that if $M(\Delta^{n-1})$ is already given then

$$M(\Delta^n) = \begin{pmatrix} \mathbf{0} & M(\Delta^{n-1}) & M(\Delta^{n-1}) \\ M(\Delta^{n-1}) & \mathbf{0} & M(\Delta^{n-1}) \\ M(\Delta^{n-1}) & M(\Delta^{n-1}) & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ denotes the zero-matrix of order 3^{n-1}. Naturally, the order of $M(\Delta^n)$ is 3^n.

Similarly, denote the neighbour-matrix of Δ^{n-1} by $N(\Delta^{n-1})$, the point-point neighbour-matrix and the line-line neighbour-matrix of Δ^{n-1} by $N_P(\Delta^{n-1})$ and by $N_B(\Delta^{n-1})$ respectively, then

$$N(\Delta^n) = \begin{pmatrix} N(\Delta^{n-1}) & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & N(\Delta^{n-1}) & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & N(\Delta^{n-1}) \end{pmatrix}$$

$$N_P(\Delta^n) = \begin{pmatrix} \mathbf{1} & N_P(\Delta^{n-1}) & N_P(\Delta^{n-1}) \\ N_P(\Delta^{n-1}) & \mathbf{1} & N_P(\Delta^{n-1}) \\ N_P(\Delta^{n-1}) & N_P(\Delta^{n-1}) & \mathbf{1} \end{pmatrix}$$

$$N_B(\Delta^n) = \begin{pmatrix} \mathbf{1} & N_B(\Delta^{n-1}) & N_B(\Delta^{n-1}) \\ N_B(\Delta^{n-1}) & \mathbf{1} & N_B(\Delta^{n-1}) \\ N_B(\Delta^{n-1}) & N_B(\Delta^{n-1}) & \mathbf{1} \end{pmatrix}$$

where $\mathbf{1}$ denotes the $3^{n-1}\times 3^{n-1}$-matrix every component of which is 1. Naturally, the order of $N(\Delta^n)$, $N_P(\Delta^n)$ and $N_B(\Delta^n)$ is 3^n.

4. Pappian projective Veldkamp planes

An important type of IN-structures is the projective Veldkamp plane, introduced by F. D. Veldkamp [1981]. The IN-relations of such a plane are given by seven axioms. Accepting two further axioms concerning central transvections and affine dilatations and their duals we get a Desarguesian projective Veldkamp plane (Veldkamp [1981]). Every Desarguesian projective Veldkamp plane may be coordinatized by a ring of stable rank two, which is not necessarily commutative and is unique up to isomorphism. The Desarguesian projective Veldkamp planes, which are coordinatizable by a commutative ring of stable rank 2 will be called Pappian projective Veldkamp planes. It is easy to show that the direct product of Desarguesian (Pappian) Veldkamp planes as IN-structure is again a Desarguesian (Pappian) Veldkamp plane. The coordinate ring of this direct product is the direct product of the coordinate rings of the single components. (Veldkamp [1988])

Closing this section we note that as in case of classical projective planes, the notion of affine Veldkamp planes may be defined on projective Veldkamp planes as well. A more detailed discussion of Veldkamp planes resp. spaces may be found in the work of F. D. Veldkamp [1995].

5. The configurations T_L, $\{\Delta^2, T_L\}$ and $\{\Delta^n, (2^{n-1}-1)T_L\}$

Let $\boldsymbol{V}$ be a projective Veldkamp plane, L an arbitrary but fixed line of $\boldsymbol{V}$, and $\boldsymbol{V}_L$ an affine plane corresponding to the ideal line L. Consider the points a_0, a_1, and a_2, incident to the ideal line L, lying pairwise distant from each other, i.e. let $a_0, a_1, a_2 \in L$ $a_i \not\approx a_j$ if $i \neq j$ $(i, j = 0, 1, 2)$. The set of lines incident to the point a_i $(i = 0, 1, 2)$ and lying distant from the ideal line L will be denoted by $[a_i]$ and called the affine pencil with support a_i.

The affine pencils $[a_i]$ $(i = 0, 1, 2)$ of an affine Veldkamp plane $\boldsymbol{V}_L$ have the following properties:

(1) If a is a point of a Veldkamp plane lying distant from the ideal line L then for $i = 0, 1, 2$ the pencil $[a_i]$ has exactly one line incident to a.

(2) Every line of the pencil $[a_i]$ lies distant from every line of the pencil $[a_j]$ if $i \neq j$ $(i, j = 0, 1, 2)$.

(3) On any pair of lines of the pencil $[a_i]$ $(i = 0, 1, 2)$ there exists no other common point lying distant from L except a_i.

By (1)–(3) we can state that the lines of pencils $[a_i]$ $(i = 0, 1, 2)$ form a three-web on the plane $\boldsymbol{V}_L$. On the basis of this three-web we will define the T_L- and T-configurations. The incidence-structure $\{\boldsymbol{P}, \boldsymbol{B}, \boldsymbol{I}\}$ is a T_L-configuration, if

$$\boldsymbol{P} = \{p_{ijk} \mid i, j, k = 0, 1, 2;\ i \neq j \neq k \neq i \wedge p_{ijk} \not\approx L\}$$

is the set of points,

$$\boldsymbol{B} = \{L_{mn} \mid m, n = 0, 1, 2;\ L_{mn} \in [a_m]\}$$

is the set of lines, and the incidence $\boldsymbol{I}$ is defined by

$$p_{ijk}\boldsymbol{I}L_{0i},\ L_{1j},\ L_{2k} \quad (i, j, k = 0, 1, 2 \wedge i \neq j \neq k \neq i).$$

Completing the T_L-configuration by points a_0, a_1, a_2 and by line L we get a configuration denoted by T (cf. Figure 2).

We note that the additive structure of the cooordinate-ring of the plane $\boldsymbol{V}$ is an Abelian group, therefore every T_L-configuration of the plane is closing. The configurations T_L and T on the planes $\boldsymbol{V}_L$ and $\boldsymbol{V}$ respectively correspond to the Thomsen-configuration well-known in web-geometry.

In what follows let $\boldsymbol{V}$ be such a projective Veldkamp plane, some affine plane $\boldsymbol{V}_L$ of which contains a Δ^2-configuration. Let us select a Δ-configuration in Δ^2. It is easy to show that there exist exactly six such configurations in Δ^2, one of them is the configuration determined by the point-set $\{q_0, q_4, q_8\}$. (Here we use the notations of Figure 2.) If the remaining six points: q_1, q_2, q_3, q_5, q_6 and q_7 are simultaneously the points of a T_L-configuration, then we say that the configurations Δ^2 and T_L are joined. This situation will be denoted by $\{\Delta^2, T_L\}$ (cf. Figure 3).

As a generalization of the construction given above let now $\boldsymbol{V}$ denote such a projective Veldkamp plane, some affine plane $\boldsymbol{V}_L$ of which contains a Δ^n-configuration $n \geq 2$. Let us select a Δ-configuration in Δ^n. There exist several such configurations in Δ^n, one of them is determined by the point-set $\{q_0, q_{(3^n-1)/2}, q_{3^n-1}\}$. If the remaining $3 \cdot 2^n - 6 = 6(2^{n-1} - 1)$ points lying on the sides of Δ are simultaneously the points of $2^{n-1} - 1$ T_L-configurations, then we say that the confugurations Δ^n and the "concentric" T_L-configurations mentioned above are joined. This situation will be denoted by $\{\Delta^n, (2^{n-1} - 1)T_L\}$ (cf. Figure 4).

6. Decomposition of Pappian projective Veldkamp planes

In this section we will prove our main result the Decomposability Theorem. We will investigate the question, what is the necessary and sufficient condition for the decomposability of a Pappian projective Veldkamp plane into n direct components, which are copies of a plane of the same type as the original one. We shall demonstrate the methods used in the proof in the case $n = 2$.

Theorem 1. *The Pappian projective Veldkamp plane $\boldsymbol{V}$ is isomorphic to the direct product of the Pappian projective Veldkamp planes $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ if and only if some affine plane $\boldsymbol{V}_L$ of $\boldsymbol{V}$ contains a joined $\{\Delta^2, T_L\}$-configuration.*

Proof. As it is well-known from the general theory of commutative rings with unit element, for such rings the following assertions are equivalent (Lambek [1966]):

(1) The ring R contains a pair $e_1, e_2 \in R$ of orthogonal idempotent elements i.e. such elements, for which $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$, $e_1 + e_2 = 1$, and $e_1, e_2 \neq 0, 1$ hold.

(2) The ring R is the direct sum of the principal ideals Re_1 and Re_2, i.e., $R = Re_1 \oplus Re_2$;

(3) The ring R is isomorphic to the direct product of the rings Re_1 and Re_2, i.e.: $R \cong Re_1 \times Re_2$.

The following assertion was proved by Veldkamp [1988]: Let R, R_1 and R_2 be the coordinate rings of the Desarguesian Veldkamp planes $\boldsymbol{V}$, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ respectively. Then $R \cong R_1 \times R_2$ iff $\boldsymbol{V} \cong \boldsymbol{V}_1 \times \boldsymbol{V}_2$.

By this fact and by (1)–(3) it is enough to prove that the coordinate ring of the Pappian projective Veldkamp plane $\boldsymbol{V}$ contains a pair of orthogonal idempotent elements if and only if there exists an affine plane $\boldsymbol{V}_L$ in $\boldsymbol{V}$ containing a joined $\{\Delta^2, T_L\}$-configuration.

Assume that the affine plane $\boldsymbol{V}_L$ of $\boldsymbol{V}$ contains the joined $\{\Delta^2, T_L\}$-configuration. We select the following points as a coordinate-quadrangle for $\boldsymbol{V}$: $o := q_0$, $e_x := q_4$, $e_y := q_8$, $x := a_2$, $y := a_1$. Let further $L_\infty := L$. Using this coodinate system the points and line of the $\{\Delta^2, T_L\}$ get the homogeneous coordinates given in Figure 5.

Examining the IN-matrices of the Δ^2-configuration we can make the following statements:

– The point $\lceil 1,0,a\rceil$ lies distant from the point $\lceil 1,1,0\rceil$, and their unique connecting line is $\lfloor -a,a,1\rfloor$. The point $\lceil 1,a,0\rceil$ is incident to the line $\lfloor -a,a,1\rfloor$, therefore $1(-a)+a^2+0\cdot 1=0$, hence $a^2=a$ holds. It is easy to show that $a\neq 0,1$.

– The point $\lceil 1,0,1-a\rceil$ lies distant from the point $\lceil 1,1,0\rceil$, and their unique connecting line is $\lfloor a-1,1-a,1\rfloor$. The point $\lceil 1,1-a,0\rceil$ is incident to the line $\lfloor a-1,1-a,1\rfloor$, therefore $1\cdot(a-1)+(1-a)^2+0\cdot 1=0$ hence $(1-a)^2=(1-a)$ holds. By $a\neq 0,1$ follows that $1-a\neq 0,1$.

– Finally, the point $\lceil 1,1-a,a\rceil$ is incident to the line $\lfloor a-1,1,1-a\rfloor$ therefore $1\cdot(a-1)+(1-a)\cdot 1+a(1-a)=0$, hence $a(1-a)=0$.

By arguments discussed above we can state that the pair a, $1-a$ is an orthogonal and idempotent pair of elements in the coordinate-ring of the plane.

Conversely, if $\boldsymbol{V}$ is coordinatized by a commutative ring of stable rank two, and this ring contains an orthogonal and idempotent pair a, b of elements, then $b=1-a$ and the coordinatized points and lines of Figure 5 form a joined $\{\Delta^2,T_L\}$-configuration.

Finally, we note that if $\boldsymbol{V}$ is a Pappian projective Veldkamp plane and $\boldsymbol{V}\cong\boldsymbol{V}_1\times\boldsymbol{V}_2$ then both $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are Pappian projective Veldkamp planes, since if R is the coordinate-ring of $\boldsymbol{V}$ which is commutative and of stable rank two, and for $i=1,2$ R_i is the coordinate-ring of $\boldsymbol{V}_i$, then by $R\cong R_1\times R_2$, R_1 and R_2 are necessarily commutative rings of stable rank two (Velkamp [1988]). □

We can copy the previous proof for the case of n components.

Theorem 2. *A Pappian projective Veldkamp plane $\boldsymbol{V}$ is isomorphic to the direct product of the Pappian projective Veldkamp planes $\boldsymbol{V}_1,\dots,\boldsymbol{V}_n$ if and only if some affine plane $\boldsymbol{V}_L$ of $\boldsymbol{V}$ contains a joined $\{\Delta^n,(2^{n-1}-1)T_L\}$-configuration.*

Proof. As it is well-known from the theory of commutative rings with unit element, for such rings the following assertions are equivalent (Lambek [1966]):

(1') The ring R contains a system $e_1,\dots,e_n\in R$ of orthogonal idempotent elements, i.e. such elements, for which $e_i^2=e_i$, $e_ie_j=0$ $(i\neq j)$, $e_1+\dots+e_n=1$, and $e_i\neq 0,1$ holds if $i,j=1,\dots,n$;

(2') The ring R is the direct sum of the principal ideals $Re_1,\dots,Re_n$, i.e., $R=Re_1\oplus\dots\oplus Re_n$;

(3') The ring R is isomorphic to the direct product of the rings $Re_1, \dots, Re_n$, i.e.: $R \cong Re_1 \times \cdots \times Re_n$.

The following assertion was proved by Veldkamp [1988]: Let R and R_i be the coordinate rings of Desarguesian Veldkamp planes $\boldsymbol{V}$ and $\boldsymbol{V}_i$ respectively $(i = 1, \dots, n)$. Then $R \cong R_1 \times \cdots \times R_n$ iff $\boldsymbol{V} \cong \boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n$.

By this fact and by (1')–(3') it is enough to prove that the coordinate ring of the Pappian projective Veldkamp plane $\boldsymbol{V}$ contains a system of n orthogonal idempotent elements if and only if there exists an affine plane $\boldsymbol{V}_L$ in $\boldsymbol{V}$ containing a joined $\{\Delta^n, (2^{n-1} - 1) \cdot T_L\}$-configuration.

Assume that the affine plane $\boldsymbol{V}_L$ of $\boldsymbol{V}$ contains the joined $\{\Delta^n, (2^{n-1} - 1) \cdot T_L\}$-configuration. We select the following points as a coordinate-quadrangle for $\boldsymbol{V}$: $o := q_0$, $e_x := q_{(3^n-1)/2}$, $e_y := q_{3^n-1}$, $x := a_2$, $y := a_1$. Let further $L_\infty := L$. The graphic description of the resulting coordinates is given in Figure 5 for $n = 2$ and in Figure 6 for $n = 3$.

Examining the IN-matrices of the Δ^n-configuration we can make the following statements:

– The point $\lceil 1, 0, e_i \rceil$ lies distant from the point $\lceil 1, 1, 0 \rceil$, and their unique connecting line is $\lfloor -e_i, e_i, 1 \rfloor$. The point $\lceil 1, e_i, 0 \rceil$ is incident to the line $\lfloor -e_i, e_i, 1 \rfloor$, therefore $1(-e_i) + e_i^2 + 0 \cdot 1 = 0$ hence $e_i^2 = e_i$ holds if $i = 1, \dots, n$. It is easy to show that $e_i \neq 0, 1$ $(i = 1, \dots, n)$.

– The point $\lceil 1, e_i, e_j \rceil$ $(i \neq j,\ i, j = 1, \dots, n)$ is incident to the line $\lfloor -e_i, 1, e_i \rfloor$ $(i = 1, \dots, n)$, therefore $1(-e_i) + e_i \cdot 1 + e_i e_j = 0$ hence $e_i e_j = 0$ holds.

Considering the "concentric" position of the $(2^{n-1} - 1)T_L$ configurations in the joined $\{\Delta^n, (2^{n-1} - 1)T_L\}$-configuration we can state that $e_1 + \cdots + e_n = 1$ holds as well, therefore the set $\{e_1, \dots, e_n\}$ is a full system of orthogonal idempotent elements in the coordinate-ring R of the plane $\boldsymbol{V}$.

Conversely, if $\boldsymbol{V}$ is coordinatized by a commutative ring of stable rank two, and this ring contains a full system of orthogonal idempotent elements then the point-set defined by the following recursion yields the points of a joined $\{\Delta^n, (2^{n-1} - 1)T_L\}$-configuration (cf. Figure 5 and Figure 6):

$$P_0 := \{\lceil 1, 0, 0 \rceil\}$$

$$P_i := P_{i-1} \cup \{\lceil 1, x + e_i, y \rceil, \lceil 1, x, y + e_i \rceil \mid \lceil 1, x, y \rceil \in P_{i-1}\} \ (1 \leq i \leq n).$$

Note further that if $\boldsymbol{V}$ is a Pappian projective Veldkamp plane and $\boldsymbol{V} \cong \boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n$ then the $\boldsymbol{V}_i$ $(i = 1, \dots, n)$ are also Pappian projective Veldkamp planes. Indeed, if R is the commutative coordinate-ring of $\boldsymbol{V}$ having stable rank two, and R_i is the coodinate-ring of $\boldsymbol{V}_i$ of the same type as that of R, then by $R \cong R_1 \times \cdots \times R_n$, the R_i are necessarily commutative rings of stable rank two. □

7. Corollaries

It is well-known from the theory of finite commutative rings that every finite commutative ring with unit element is isomorphic to the finite direct product of commutative local rings (McDonald [1974]). The components of this direct product are the coordinate-rings of finite Pappian projective Klingenberg planes.

By this note and by Theorem 2 the following statement holds.

Corollary 1. *Every finite Pappian projective Veldkamp plane is isomorphic to the direct product of a finite number of Pappian projective Klingenberg planes. This direct product has exactly n components iff the Veldkamp plane contains a joined $\{\Delta^n, (2^{n-1} - 1)T_L\}$-confuguration.*

From Corollary 1 it follows further that among finite Pappian projective Veldkamp planes the Pappian projective Klingenberg planes are those planes, which contain no joined $\{\Delta^n, (2^{n-1} - 1)T_L\}$-configuration for $n \geq 2$.

Indeed, if a finite Pappian Veldkamp plane $P_2(R)$ contains no $\{\Delta^n, (2^{n-1} - 1)T_L\}$-configuration in case of $n \geq 2$ and the coordinate-ring of the plane is not a local ring, then R is isomorphic to the direct product of finitely many commutative local rings $L_1, \dots, L_n$, therefore $P_2(R) \cong P_2(L_1) \times \cdots \times P_2(L_n)$. Then by Theorem 2, $P_2(R)$ contains a $\{\Delta^n, (2^{n-1} - 1)T_L\}$-configuration, a contradiction. Conversely, if for $n \geq 2$ the finite Pappian projective Klingenberg plane $P_2(L)$ contains a $\{\Delta^n, (2^{n-1} - 1)T_L\}$-configuration then by Theorem 2, $P_2(L)$ is isomorphic to direct product of finitely many Pappian projective Klingenberg planes $P_2(L_1), \dots, P_2(L_n)$, hence $L \cong L_1 \times \cdots \times L_n$. But then $K \cong L/\operatorname{rad}(L) \cong L_1 \times \cdots \times L_n/\operatorname{rad}(L_1 \times \cdots \times L_n) \cong L_1 \times \cdots \times L_n/\operatorname{rad}(L_1) \times \cdots \times \operatorname{rad}(L_n) \cong L_1/\operatorname{rad}(L_1) \times \cdots \times L_n/\operatorname{rad}(L_n) \cong K_1 \times \cdots \times K_n$ where $\operatorname{rad}(L)$ and

$\mathrm{rad}(L_i)(1 \leq i \leq n)$ are the Jacobson radicals of L and L_i, and K, K_i are the facor-fields corresponding to the radicals. But then $K \cong K_1 \times \cdots \times K_n$, a contradiction by $n \geq 2$.

Therefore the following assertion holds:

Corollary 2. *A finite Pappian projective Veldkamp plane is a Pappian projective Klingenberg plane if and only if it contains no joined* $\{\Delta^n, (2^{n-1}-1)T_L\}$*-configuration if* $n > 1$.

Figure 1:
The Δ^2-configuration.

Figure 2:
The T_L- and the T-configuration.

Figure 3:
The $\{\Delta^2, T_L\}$-configuration.

Figure 4:
The points of the $\{\Delta^3, T_L\}$-configuration.

Figure 5:
The coordinatization of the $\{\Delta^2, T_L\}$-configuration.

Figure 6:
The partial coordinatization of the $\{\Delta^3, T_L\}$-configuration.

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