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Projective modules over twisted group algebras of p-solvable groups

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Abstract. In this paper we generalize some results of P. Fong and I. M. Isaacs on modular representation theory of p-solvable groups to the case of twisted group algebras and the further to the case of strongly group graded algebras. For this, we use a categorical approach to Clifford theory.

1. Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$, and let G be a finite group. If V is an $\mathcal{O}$-module, then either V is $\mathcal{O}$-free of finite rank or $J(\mathcal{O})V = 0$ and V has finite dimension over k.

Let further $R = \bigoplus_{g\in G} R_g$ be a strongly G-graded $\mathcal{O}$-algebra (that is,R_g is an $\mathcal{O}$-summand of R, and $R_gR_h = R_{gh}$ for all $g, h \in G$). For a subset X of G we denote $R_X = \bigoplus_{x\in G} R_x$. If V is a (left) R-module, then $(R|V)$-mod is the full subcategory of R-mod consisting of direct summands of finite direct sums of copies of V. We say that V is *isotypic* if all the indecomposable direct summands of V are isomorphic.

The main result of this paper states as follows. Assume that G is p-solvable and let $1 = N_0 < N_1 < \cdots < N_r = G$ be a chain of normal subgroups of G such N_i/N_{i-1} is either a p-group or a p'-group, for $i = 1, \ldots, r$. Let also H be a Hall p'-subgroup of G. (This means that the index of H in G is a power of p; it is well-known that such an H exists and that any p'-subgroup of G is contained in a G-conjugate of H.) Let finally M be an indecomposable R_1-module.

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1.1. Theorem. (a) *Let* $V \in (R|R\otimes_{R_1} M)$-mod *be an indecomposable* R*-module. Then there is an indecomposable* R_H*-module* $W \in (R_H|R_H\otimes_{R_1} M)$-mod *such that* $V \simeq R\otimes_{R_H} W$.

(b) *Let* $W, W' \in (R_H|R_H\otimes_{R_1} M)$-mod *be indecomposable* R_H*-modules such that the restrictions* $\mathrm{Res}^{R_H}_{R_{H\cap N_i}} W, \mathrm{Res}^{R_H}_{R_{H\cap N_i}} W'$ *are isotypic for* $i = 0, \dots, r-1$*, and* $R\otimes_{R_H} W \simeq R\otimes_{R_H} W'$*. Then* $R\otimes_{R_H} W$ *is an indecomposable* R*-module, and there is* $g \in N_G(H)$ *such that* $W' \simeq R_{gH} \otimes_{R_H} W$ *(as* R_H*-modules).*

These statements are graded versions of the main results of Hubert Fottner and Burkhardt Külshammer [2]. Their proof rely on the theory of G-algebras and on some technical results on lifting idempotents with group actions involved. We shall give here a short proof using the categorical approach to the Clifford theory of indecomposable modules of [5] and [6], and induction, of course. The main results of the above papers will be used in Section 2 to show that it is enough to deal with projective modules over twisted group algebras, and the same results are needed in Section 3 to prove the theorem in this case by induction. It should be noted that in the case of ordinary group algebras 1.1.(a) is due to P. Fong [1], while (b) to I. M. Isaacs [3].

We refer the reader to [7] for general facts on graded rings and to [4] for results on twisted group algebras and projective representations. Our notations tend to follow those of [5] and [6], and we shall recall the needed facts in the next section.

2. Clifford theory for strongly graded algebras

Let $\mathcal{O}$, k, G, R and M be as in the introduction. The purpose of this section is to show that in Theorem 1.1, R can be replaced by a twisted group algebra of G and k.

2.1. If $\alpha \in Z^2(G, k^*)$ then denote, as in [4], by $k^\alpha G$ the twisted group algebra having k-basis $\{\bar{g} \mid g \in G\}$ and multiplication $\bar{g}\bar{h} = \alpha(g,h)\overline{gh}$. Then $k^\alpha G$ is a G-graded algebra in an obvious way. If $\beta \in Z^2(G, k^*)$, then $k^\alpha G \simeq k^\beta G$ as G-graded k-algebras if and only if α and β are cohomologous.

If G is a p-group, then it is well-known that $Z^2(G, k^*) = 1$ and $k^\alpha G \simeq kG$ is a local ring with $k^\alpha G/J(k^\alpha G) \simeq k$.

If H is a subgroup of G, we shall denote $k^\alpha H = k^{\operatorname{res}^G_H \alpha}H$, where $\operatorname{res}^G_H \alpha \in H^2(H,k^*)$ is the restriction of α to H. If W is a $k^\alpha H$-module and V is a $k^\alpha G$-module, then $\operatorname{Ind}^{k^\alpha G}_{k^\alpha H} W = k^\alpha G \otimes_{k^\alpha H} W$ is the induced module, and $\operatorname{Res}^{k^\alpha G}_{k^\alpha H} V$ is the $k^\alpha G$-module obtained by scalar restriction via the inclusion $k^\alpha H \hookrightarrow k^\alpha G$.

If N is a normal subgroup of G, it will be useful to regard $k^\alpha G$ as a G/N-graded algebra, where for $x = gN \in G/N$, $(k^\alpha G)_x = \bar{g}k^\alpha N$.

2.2. The *graded Jacobson radical* of R is, by definition, the intersection of maximal graded left ideals of R, and it coincides with the two-sided graded ideal $J_{\mathrm{gr}}(R) = J(R_1)R = RJ(R_1)$. Then $R/J_{\mathrm{gr}}(R)$ is a strongly graded k-algebra with $(R/J_{\mathrm{gr}}(R)_1 \simeq R_1/J(R_1)$. It is well-known that $J_{\mathrm{gr}}(R) \subseteq J(R)$, and if G is a p'-group, then $J_{\mathrm{gr}}(R) = J(R)$.

2.3. *Remark.* The following observation will be crucial in the next section. Let N be a normal p-subgroup of G and H a p'-subgroup of G such that $G = HN$. Let $\alpha \in Z^2(G,k^*)$ and regard $k^\alpha G$ as a G/N-graded k-algebra. Let f denote the composition $k^\alpha H \hookrightarrow k^\alpha G \to k^\alpha G/J_{\mathrm{gr}}(k^\alpha G)$ (where, using the G/N-grading, $J_{\mathrm{gr}}(k^\alpha G) = J(k^\alpha N)J(k^\alpha G)$, and by the last statement of (2.2) we have $J_{\mathrm{gr}}(k^\alpha G) = J(k^\alpha G)$ since G/N is a p'-group). Clearly, f is a homomorphism of G/N-graded k-algebras, and since $k^\alpha N$ is a local ring, we have that $k^\alpha G/J_{\mathrm{gr}}(k^\alpha G)$ is a twisted group algebra, so there is $\beta \in Z^2(H,k^*)$ such that $k^\alpha G/J_{\mathrm{gr}}(k^\alpha G) \simeq k^\beta H$. It follows immediately that f is an isomorphism of H-graded k-algebras. Consequently, we have the injective map $k^\alpha H \to k^\alpha G$ and the surjective map $k^\alpha G \to k^\alpha H$ given by the composition $k^\alpha G \to k^\beta H \xrightarrow{f} k^\alpha H$.

2.4. Consider now the indecomposable R_1-module M and let

$$G_M = I_G(M) = \{g \in G \mid R_g \otimes_{R_1} M \simeq M \text{ in } R_1\text{-mod}\}$$

be the stabilizer (inertia group) of M. Let further $E = \operatorname{End}_{R_1}(R \otimes_{R_1} M)^{op}$. Then it is well-known that E is a G-graded $\mathcal{O}$-algebra (not necessarily strongly graded) with

$$E_g \simeq \{f \in E \mid f(R_x \otimes_{R_1} M) \subseteq R_{xg} \otimes_{R_1} M \text{ for all } x \in G\}.$$

In particular, $E_1 \simeq \operatorname{End}_{R_1}(M)^{op}$, and moreover, for any subgroup H of G, $E_H \simeq \operatorname{End}_{R_H}(R_H \otimes_{R_1} M)^{op}$.

2.5. Let $D = E/J_{\mathrm{gr}}(E)$. Then, since E_1 is a local ring and k is algebraically closed, we have that D is a twisted group algebra of G_M and k, so there is $\alpha \in H^2(G_M, k^*)$ such that $D = k^\alpha G_M$ (see [5], Section 3 for details).

2.6. By [5, Theorem 4.1], we have that the additive functor

$$D \otimes_E \mathrm{Hom}_R(R \otimes_{R_1} M, -) \colon (R|R \otimes_{R_1} M)\text{-mod} \to (D|D)\text{-mod}$$

induces an isomorphism between the Grothendieck groups associated to these categories, where $(D|D)$-mod is the category of (finitely generated) projective D-modules. Moreover, this functor commutes with induction from subgroups, restriction, truncation and conjugation (see [5, Theorem 4.1 and Remark 4.5.b]), and the functors

$$(-)_{G_M} \colon (R|R \otimes_{R_1} M)\text{-mod} \to (R_{G_M}|R_{G_M} \otimes_{R_1} M)\text{-mod}$$

and

$$R \otimes_{R_H} - \colon (R_{G_M}|R_{G_M} \otimes_{R_1} M)\text{-mod} \to (R|R \otimes_{R_1} M)\text{-mod}$$

induce isomorphisms between the Grothendieck groups of these categories ([5, Theorem 4.1.c]).

This implies (see [5], Corollary 4.4) that if $V \in (R|R \otimes_{R_1} M)$-mod is indecomposable, then $\mathrm{Res}^R_{R_1} V$ is an isotypic R_1-module if and only if $G_M = G$.

The above discussion implies immediately that Theorem 1.1 is equivalent to the following theorem.

2.7. Theorem. *Assume that G is a p-solvable group, H is a Hall p'-subgroup of G, and $\alpha \in Z^2(G, k^*)$. Let $1 = N_0 < N_1 < \cdots < N_r = G$ be a chain of normal subgroups of G such that N_i/N_{i-1} is either a p-group or a p'-group.*

(a) *Let V be a projective indecomposable $k^\alpha G$-module. Then there is a simple $k^\alpha H$-module W such that $V \simeq \mathrm{Ind}^{k^\alpha G}_{k^\alpha H} W$.*

(b) *Let W, W' be simple $k^\alpha H$-modules such that $\mathrm{Res}^{k^\alpha H}_{k^\alpha (H\cap N_i)} W$ and $\mathrm{Res}^{k^\alpha H}_{k^\alpha (H\cap N_i)} W'$ are isotypic for $i = 0, \dots, r-1$, and $\mathrm{Ind}^{k^\alpha G}_{k^\alpha H} W \simeq \mathrm{Ind}^{k^\alpha G}_{k^\alpha H} W'$ as $k^\alpha G$-modules. Then $\mathrm{Ind}^{k^\alpha G}_{k^\alpha H} W$ is an indecomposable (and* *projective) $k^\alpha G$-module, and there is $g \in N_G(H)$ such that $k^\alpha H\bar{g} \otimes_{k^\alpha H} W$ as $k^\alpha H$-modules.*

The proof of this theorem will be given in the next section. It will require Clifford theory of projective modules, so we recall the basic result [6, Theorem 2.3] in the general context of this section.

2.8. Assume in addition that M is a projective R_1-module, and let $S = M/J(R_1)M$ and $R' = R/J_{\mathrm{gr}}(R)$. It follows that S is a simple R'_1-module and $\mathrm{End}_{R'}(R' \otimes_{R'_1} S)^{op} \simeq D$ as G_M-graded k-algebras. Moreover, there is a commutative diagram of categories

$$\begin{array}{ccc} (R|R \otimes_{R_1} M)\text{-mod} & \xrightarrow{\mathrm{Hom}_R(R\otimes_{R_1} M,-)} & (E|E)\text{-mod} \\ {\scriptstyle R'\otimes_R -}\big\downarrow & & \big\downarrow{\scriptstyle D\otimes_E -} \\ (R'|R' \otimes_{R'_1} S) & \xrightarrow{\mathrm{Hom}_{R'}(R'\otimes_{R'_1} S,-)} & (D|D)\text{-mod} \end{array}$$

and this diagram is also compatible with induction from subgroups.

3. A proof of Theorem 2.7

Assume that G is p-solvable, H a Hall p'-subgroup of G, and let $1 = N_0 < N_1 < \cdots < N_r = G$ be a chain of normal subgroups as in (2.7).

(a) We prove by induction on G that if $\alpha \in Z^2(G, k^*)$ and V is a projective $k^\alpha G$-module, then there is a simple $k^\alpha H$-module W such that $V \simeq \mathrm{Ind}_{k^\alpha H}^{k^\alpha G} W$.

The statement is trivial if G is a p'-group (since then $H = G$) or if G is a p-group (then $H = 1$ and $V = k^\alpha G$ is the unique projective indecomposable $k^\alpha G$-module).

Denote $N = N_1$ and assume first that $N \neq 1$ is a p'-group. Since N is a normal subgroup of G, we have that $N \subseteq H$. Since V is a projective indecomposable $k^\alpha G$-module, by Clifford theory there is a projective (and simple) $k^\alpha N$-module M such that $V \in (k^\alpha G|\,\mathrm{Ind}_{k^\alpha N}^{k^\alpha G} M)$-mod. Let $I/N = (G/N)_M$ be the stabilizer of M. By (2.6) there is a projective $k^\alpha I$-module V_0 such that $V \simeq \mathrm{Ind}_{k^\alpha I}^{k^\alpha G} V_0$, and moreover $\mathrm{Res}_{k^\alpha N}^{k^\alpha I} V_0$ is an isotypic $k^\alpha N$-module. By (2.6) and (2.5) V_0 corresponds to a projective indecomposable $k^\beta(I/N)$-module $\tilde{V}_0$, where $\beta \in Z^2(I/N, k^*)$. Denote

$H_0 = I \cap H$, so H_0/N is a Hall p'-subgroup of I/N. Since $|H_0/N| < |G|$, by induction there is a projective and simple $k^\beta H_0$-module $\tilde{W}_0$ such that $\tilde{V}_0 \simeq \mathrm{Ind}_{k^\beta(H_0/N)}^{k^\beta(I/N)} \tilde{W}_0$. Again by (2.6), $\tilde{W}_0$ corresponds to a projective $k^\alpha H_0$-module $W_0 \in (k^\alpha H_0 | \mathrm{Ind}_{k^\alpha N}^{k^\alpha H_0} M)$-mod and $V_0 \simeq \mathrm{Ind}_{k^\alpha H_0}^{k^\alpha I} W_0$. Denoting $W = \mathrm{Ind}_{k^\alpha H_0}^{k^\alpha H} W_0$, we finally have

$$\begin{aligned} V &\simeq \mathrm{Ind}_{k^\alpha I}^{k^\alpha G} V_0 \simeq \mathrm{Ind}_{k^\alpha I}^{k^\alpha G} \mathrm{Ind}_{k^\alpha H_0}^{k^\alpha I} W_0 \\ &\simeq \mathrm{Ind}_{k^\alpha H}^{k^\alpha G} \mathrm{Ind}_{k^\alpha H_0}^{k^\alpha H} W_0 \simeq \mathrm{Ind}_{k^\alpha H}^{k^\alpha G} W. \end{aligned}$$

Assume now that $N \neq 1$ is a p-group. Since V is a projective indecomposable $k^\alpha G$-module, there is a projective indecomposable $k^\alpha N$-module M such that $V \in (k^\alpha G | \mathrm{Ind}_{k^\alpha N}^{k^\alpha G} M)$-mod. But N is a p-group, so $M \simeq k^\alpha N$ is the unique projective indecomposable $k^\alpha N$-module, hence M is G/N-invariant. By (2.6) and (2.7) V corresponds to a projective indecomposable $k^\beta(G/N)$-module $\tilde{V}$, where $\beta \in Z^2(G/N, k^*)$. We have that $|G/N| < |G|$ and HN/N is a Hall p'-subgroup of G/N. By induction it follows that there is a projective simple $k^\beta(HN/N)$-module $\tilde{W}$ such that $\tilde{V} \simeq \mathrm{Ind}_{k^\beta(HN/N)}^{k^\beta(G/N)} \tilde{W}$. By (2.6) again, $\tilde{W}$ corresponds to a projective indecomposable $k^\alpha(HN)$-module $P \in (k^\alpha(HN) | \mathrm{Ind}_{k^\alpha N}^{k^\alpha(HN)} M)$-mod such that $V \simeq \mathrm{Ind}_{k^\alpha(HN)}^{k^\alpha G} P$.

Since $\tilde{W}$ is a simple $k^\beta(HN/N)$-module, by (2.8) it correponds to a simple module W over $k^\alpha(HN)/J_{\mathrm{gr}}(k^\alpha(HN))$-module, where $k^\alpha(HN)$ is regarded as a HN/N-graded algebra. By Remark 2.3, W is a projective simple $k^\alpha H$-module. Clearly, $k^\alpha(HN) \otimes_{k^\alpha H} W$ is a projective $k^\alpha(HN)$-module, and multiplication induces an epimorphism $\mathrm{Ind}_{k^\alpha H}^{k^\alpha(HN)} W \to W$. By the commutativity of the diagram (2.8) we have that P is the projective cover of W (where W is regarded as a simple $k^\alpha(HN)$-module), so P is a direct summand of $\mathrm{Ind}_{k^\alpha H}^{k^\alpha(HN)} W$. Since $W \simeq P/J(k^\alpha H)P$, we have that $\dim_k P = |N| \dim_k W$; comparing dimensions, it follows that $P \simeq \mathrm{Ind}_{k^\alpha H}^{k^\alpha N} W$, hence $V \simeq \mathrm{Ind}_{k^\alpha H}^{k^\alpha G} W$.

(b) We use again induction on G, and we see that the statements are trivial if G is a p'-group or a p-group.

Suppose first that $N = N_1 \neq 1$ is a p'-group. By the assumption on isotypy, there is a G/N-invariant $k^\alpha N$-module M such that W and W' belong to the same category $(k^\alpha H | \mathrm{Ind}_{k^\alpha N}^{k^\alpha G} M)$-mod. Denoting

$V = \operatorname{Ind}_{k^\alpha H}^{k^\alpha G} W \simeq \operatorname{Ind}_{k^\alpha H}^{k^\alpha G} W'$, we have that V belongs to the category $(k^\alpha G|\operatorname{Ind}_{k^\alpha N}^{k^\alpha G} M)$-mod. By (2.6) and (2.4) V corresponds to a projective $k^\beta(G/N)$-module, W and W' correspond to projective simple $k^\beta(H/H)$-modules $\tilde{W}$ and $\tilde{W}'$ respectively, such that

$$\tilde{V} \simeq \operatorname{Ind}_{k^\beta(H/N)}^{k^\beta(G/N)} \tilde{W} \simeq \operatorname{Ind}_{k^\beta(H/N)}^{k^\beta(G/N)} \tilde{W}',$$

and $\operatorname{Res}_{k^\beta(H\cap N_i/N)}^{k^\beta(H/N)} \tilde{W}$, $\operatorname{Res}_{k^\beta(H\cap N_i/N)}^{k^\beta(H/N)} \tilde{W}'$ are isotypic for $i = 1, \dots, r-1$.

Since $|G/N| < |G|$ and H/N is a Hall p'-subgroup of G/N, by the induction hypothesis we have that $\tilde{V}$ is an indecomposable $k^\beta(G/N)$-module, and there is $gN \in N_{G/N}(H/N)$ such that

$$\tilde{W}' \simeq k^\beta(H/N)\overline{gN} \otimes_{k^\beta(H/N)} \tilde{W}.$$

By (2.6) it follows that V is an indecomposable $k^\alpha G$-module and $W' \simeq k^\alpha H\bar{g} \otimes_{k^\alpha H} W$.

Finally, assume that $N \neq 1$ is a p-group. By Remark 2.3 we have an epimorphism $k^\alpha(HN) \to k^\alpha H$, and regard W, W' as simple $k^\alpha(HN)$-modules by scalar restriction. Let $P = P(W)$ and $P' = P(W')$ be the projective covers of W and W' respectively, so P and P' are projective indecomposable $k^\alpha(HN)$-modules. By the last part of the proof of (a) we have that $P \simeq \operatorname{Ind}_{k^\alpha H}^{k^\alpha(HN)} W$ and $P' \simeq \operatorname{Ind}_{k^\alpha H}^{k^\alpha(HN)} W'$. Moreover, by assumption

$$V \simeq \operatorname{Ind}_{k^\alpha(HN)}^{k^\alpha G} P \simeq \operatorname{Ind}_{k^\alpha(HN)}^{k^\alpha G} P'.$$

Clearly, V belongs to $(k^\alpha G|\operatorname{Ind}_{k^\alpha N}^{k^\alpha G} M)$-mod while P and P' belong to the category $(k^\alpha(HN)|\operatorname{Ind}_{k^\alpha N}^{k^\alpha(HN)} M)$-mod, where $M \simeq k^\alpha N$ is the unique projective indecomposable $k^\alpha N$-module. (Hence M is G-invariant and we do not have to use the assumption on isotypy when we restrict to normal p-groups.) By (2.6) and (2.4) V corresponds to a projective $k^\beta(G/N)$-module $\tilde{V}$, while W and W' correspond to projective $k^\beta(HN/N)$-modules W and W' respectively, where $\beta \in Z^2(G/N, k^*)$. Moreover, by assumption it follows that

$$\tilde{V} \simeq \operatorname{Ind}_{k^\beta(HN/N)}^{k^\beta(G/N)} \tilde{W} \simeq \operatorname{Ind}_{k^\beta(HN/N)}^{k^\beta(G/N)} \tilde{W}'.$$

Since $|G/N| < G$ and HN/N is a Hall p'-subgroup of G/N, the induction hypothesis can be applied, hence $\tilde{V}$ is an indecomposable $k^\beta(G/N)$-module, and there is $gN \in N_{G/N}(HN/N)$ (so $g \in N_G(H)$) such that

$\tilde{W}' \simeq k^{\beta}(H/N)\overline{gN} \otimes_{k^{\beta}(H/N)} \tilde{W}$. By (2.6) it follows that V is an indecomposable $k^{\alpha}G$-module and $P' \simeq k^{\alpha}(HN)\bar{g} \otimes_{k^{\alpha}(HN)} P$ as $k^{\alpha}(HN)$-modules. Consequently, by Remark 2.3,

$$W' \simeq P'/J(k^{\alpha}N)P' \simeq k^{\alpha}H\bar{g} \otimes_{k^{\alpha}H} P/J(k^{\alpha}H)P \simeq k^{\alpha}H\bar{g} \otimes_{k^{\alpha}H} W$$

as $k^{\alpha}H$-modules.

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