Publ. Math. Debrecen
53 / 3-4 (1998), 375–381

On the power values of polynomials

By A. BÉRCZES (Debrecen), B. BRINDZA (Debrecen)
and L. HAJDU (Debrecen)

Abstract. In this paper we give a new, generalized version of a result of Brindza, Evertse and Győry, concerning superelliptic equations.

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n and b be a nonzero integer. For effective upper bounds obtained by Baker's method for the exponent z in the equation

$$f(x) = by^z, \quad x, y, z \in \mathbb{Z} \text{ with } |y| > 1, \ z > 1 \tag{1}$$

we refer to [T], [ST], [Tu1], [Tu2], [ShT], [B1], [BEGy], [Bu].

For a polynomial P let $M(P)$ denote the Mahler height of it (cf. [M]). The purpose of this paper, which is related to a recent observation of Brindza on the number of the solutions of a generalization of the Ramanujan–Nagell equation [B3], is to derive a bound for z which is polynomial in $M(f)$. For brevity write $M = M(f)$.

Theorem. *If f has at least two distinct zeros, then*

$$z < cM^{3n} \log^3 |2b|,$$

where c is an effectively computable constant depending only on n.

Mathematics Subject Classification: 11D41.
Key words and phrases: diophantine equations, superelliptic equations.
This research was supported by the Hungarian Grant OTKA No. 023800 (first author), No. D 23992 (second author), No. 023800 and T 016 975 (third author), moreover by the Pro Regione Foundation of the Hajdúsági Agráripari RT and by the Universitas Foundation of the Kereskedelmi Bank RT (first and third author).

Remarks. If f is an irreducible monic and $b = 1$ then this inequality was proved by BRINDZA, GYŐRY and EVERTSE with different constants (see [BEGy], Th. 4). Moreover, if $n > 2$ and f is irreducible then a profound result of GYŐRY (cf. [Gy1] or [Gy2]) makes it possible to substitute cM^{3n} by an effective constant depending only on the discriminant of f.

1. Auxiliary results

To prove our Theorem, we need two lemmas. In what follows, for any non-zero algebraic number α, $h(\alpha)$ and $H(\alpha)$ denotes the logarithmic height and the classical (ordinary) height of α, respectively.

Lemma 1. *Let $\mathbb{K}$ be an algebraic number field of degree n and denote by R and r the regulator and the unit rank of $\mathbb{K}$, respectively. There exists a fundamental sytem of units $\varepsilon_1, \dots, \varepsilon_r$ for $\mathbb{K}$ so that*

$$h(\varepsilon_i) \leq c^* R, \quad i = 1, \dots, r$$

where c^* is an effectively computable constant depending only on n.

PROOF. This statement is a consequence of Lemma 1 in [BGy]. For other versions of this result cf. [B2] or [H]. □

Lemma 2. *Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers and let $A_1, \dots, A_n$ be positive real numbers with $A_i \geq \max\{H(\alpha_i), e\}$ for $i = 1, \dots, n$. Furthermore, let $b_1, \dots, b_n$ be rational integers with $\alpha_1^{b_1} \dots \alpha_n^{b_n} \neq 1$ and suppose that B is a positive real number satisfying $B \geq \max_{i=1,\dots,n} |b_i|$ and $B \geq e$. Now we have*

$$|\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1| \geq B^{-c' \log A_1 \dots \log A_n},$$

where c' is an effectively computable constant depending only on n and on the degree of $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ over $\mathbb{Q}$.

PROOF. This is Theorem 1.2 in [PW]. □

2. Proof of the Theorem

We have two cases to distinguish.

First we assume that f has an irreducible factor $P \in \mathbb{Z}[x]$ of degree $t \geq 2$. Let α be a zero of P, moreover, let R, h, D and r be the regulator, class number, discriminant and unit rank of the field $\mathbb{K} = \mathbb{Q}(\alpha)$, respectively. In the sequel, $c_1, c_2, \dots$ will denote effectively computable positive constants depending only on n. The well-known inequalities

$$hR \leq \sqrt{|D|}\,(\log |D|)^{n-1}, \quad \text{(cf. e.g. [L])}$$

and

$$|D| \leq n^n M(P)^{2n-2} \leq n^n M^{2n-2} \quad \text{(cf. [M])}$$

imply

$$hR < c_1 M^n. \tag{2}$$

Let a denote the leading coefficient of f and $\beta_1, \dots, \beta_n$ be the zeros of $g(x) = a^{n-1} f(\frac{x}{a})$. Set

$$\Delta(g) = \prod_{\beta_i \neq \beta_j} (\beta_i - \beta_j)^2,$$

and write g in the form $g(x) = P_1^{k_1}(x) P_2(x)$ where P_1 and P_2 are relatively prime polynomials in $\mathbb{Z}[x]$ and P_1 is an irreducible monic of degree t; (actually $P_1(x) = a^t P(\frac{x}{a})$). Let $\beta_1, \dots, \beta_t$ be the zeros of P_1 and (x, y) be an arbitrary, however, fixed solution to (1). The g.c.d. of the principal ideals $\langle ax - \beta_1 \rangle$ and $\langle g(ax)(ax - \beta_1)^{-k_1} \rangle$ divides $\Delta^n(g)$, therefore, there are integral ideals A, B, C in $\mathbb{K}$ so that

$$A\langle ax - \beta_1 \rangle = BC^w \quad \text{where } w = \frac{z}{(z, k_1)}, \tag{3}$$

furthermore,

$$\max\{N_{\mathbb{K}/\mathbb{Q}}(A), N_{\mathbb{K}/\mathbb{Q}}(B)\} \leq |a \cdot b \cdot \Delta(g)|^{n^2}.$$

Hence, by a well-known inequality (cf. for example [Gy3], Lemma 3) and by (2), the ideals A^h and B^h have generators α and β, respectively, with

$$\max\left\{\overline{|\alpha|}, \overline{|\beta|}\right\} \leq \exp(c_2 M^{n-1} (\log M)^n \log |2b|).$$

The relation (3) can be written as

$$\alpha(ax-\beta_1)^h = \varepsilon\beta\gamma^w$$

where γ is a generator of C^h and ε is a unit. Let $\varepsilon_1,\dots,\varepsilon_r$ be a fundamental system of units for $\mathbb{K}$ satisfying Lemma 1. Then we can express ε as $\varepsilon = \rho\varepsilon_1^{l_1}\dots\varepsilon_r^{l_r}$ where ρ is a root of unity and we may assume that $\max\limits_{1\le i\le r}|l_i| < w$ (the remaining factors, if any, are incorporated in γ).

If $|ax| \le M(g)+1$ then

$$2^z \le |y|^z \le (2M(g)+1)^n$$

and the Theorem is proved. Otherwise, $|ax| > M(g)+1$ and $|ax-\beta_i| > 1$, $i=1,\dots,n$ implies

$$|ax-\beta_i| \le |a^{n-1}by^z|, \quad i=1,\dots,n,$$

$$|a^{n-1}by^z|^h \ge \max_{1\le i\le t}|ax-\beta_i|^h \ge \overline{|\varepsilon_1|}^{-nw}\dots\overline{|\varepsilon_r|}^{-nw}\overline{|\alpha|}^{-n}\overline{|\beta|}^{-n}\overline{|\gamma|}^{w}$$

and

$$\overline{|\gamma|} \le |a^{n-1}b|^{\frac{h}{w}}|y|^{nh}\overline{|\alpha|}^{\frac{n}{w}}\overline{|\beta|}^{\frac{n}{w}}\prod_{i=1}^{r}\overline{|\varepsilon_i|}^{n}.$$

If $w < nh$ then by $0.056 < R$ (cf. [Z]) we obtain $w < 20nhR$ and

$$z < c_3M^{n-1}(\log(2M))^{n-1}.$$

In case of $w \ge nh$

$$\overline{|\gamma|} \le M|b|^{\frac{1}{n}}|y|^{nh}\overline{|\alpha|}\,\overline{|\beta|}\prod_{i=1}^{r}\overline{|\varepsilon_i|}^{n},$$

and we get

$$\log H\left(\frac{\gamma}{\gamma^{(2)}}\right) \le c_4\log|2b|M^{n-1}(\log(2M))^n\log|y|.$$

We may assume that $|ax| \ge \frac{1}{2}|y|^{\frac{z}{n}}$. Indeed, otherwise $\max\limits_{1\le i\le n}|ax-\beta_i| \ge |y|^{\frac{z}{n}}$ yields

$$|ax| \ge |y|^{\frac{z}{n}} - M(g)$$

and the Theorem is proved. Supposing

$$\frac{|\beta_i-\beta_j|}{|ax-\beta_i|}\geq\frac{|\beta_2-\beta_1|}{|ax-\beta_2|},\quad 1\leq i,j\leq t,\ i\neq j$$

we have

$$\prod_{\substack{1\leq i,j\leq t\\ \beta_i\neq\beta_j}}\frac{|\beta_i-\beta_j|}{|ax-\beta_i|}\leq\frac{|\Delta(g)|\cdot 2^n}{|y|^z}.$$

Then

$$\frac{|\beta_2-\beta_1|}{|ax-\beta_2|}\leq|y|^{-\frac{z}{4}},$$

or else we can derive a bound for z better than stated in the Theorem. Avoiding the trivial case $\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h=1$, whenever $\frac{1}{2}|y|^{\frac{z}{n}}\leq|\Delta(g)|^{n^2}$ we obtain

$$\log\left|\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h-1\right|\leq\log\left(h\left|\frac{ax-\beta_1}{ax-\beta_2}-1\right|\right)\leq-\frac{z}{8}\log|y|.$$

Finally, Lemma 2 yields

$$0\neq\left|\left(\frac{ax-\beta_1}{ax-\beta_2}\right)^h-1\right|=\left|\left(\frac{\varepsilon_1}{\varepsilon_1^{(2)}}\right)^{l_1h}\cdots\left(\frac{\varepsilon_r}{\varepsilon_r^{(2)}}\right)^{l_rh}\frac{\beta/\alpha}{\beta^{(2)}/\alpha^{(2)}}\left(\frac{\gamma}{\gamma^{(2)}}\right)^{wh}-1\right|$$
$$\geq\exp\left(-c_5\log|2b|M^{3n-3}(\log|2M|)^{3n-1}\log|y|\log w\right)\}$$

and the comparision of the upper and lower bounds completes the proof (in the first case).

In the easier second case all the zeros of g are integral. Let k_i denote the multiplicities of β_i, $i=1,2$.

Repeating the argument one can have

$$u_i(ax-\beta_i)=v_iy_i^w$$

where $w=\frac{z}{(a,k_1k_2)}$ and $u_i,v_i,y_i\in\mathbb{Z}$, $|y_i|>1$, $i=1,2$.

To derive a bound for w from the equation

$$Ay_1^w - By_2^w = C$$

$(A = u_2v_1,\ B = u_1v_2,\ C = u_1u_2(\beta_2 - \beta_1))$ one can apply Lemma 2 again, and we have

$$\frac{z}{\log z} \le c_6 \log M \log |2b|,$$

and the Theorem is proved. □

References

[B1] B. Brindza, On S-integral solutions of the equation $y^m = f(x)$, *Acta Math. Hung.* **44** (1984), 133–139.

[B2] B. Brindza, On the generators of S-unit groups in algebraic number fields, *Bull. Austral Math. Soc.* **43** (1991), 325–329.

[B3] B. Brindza, On the generalized Ramanujan-Nagell equation, *Publ. Math. Debrecen* (*to appear*).

[BEGy] B. Brindza, J.-H. Evertse and K. Győry, Bounds for the solutions of some diophantine equations in terms of the discriminants, *J. Austral Math. Soc.* **51** (1991), 8–26.

[Bu] Y. Bugeaud, Sur la distance entre deux puissances pures, *C. R. Acad. Sci. Paris* **322**, Série I (1996), 1119–1121.

[BGy] Y. Bugeaud and K. Győry, Bounds for the solutions of unit equations, *Acta Arith.* **74** (1996), 67–80.

[Gy1] K. Győry, Sur les polynômes à coefficients entiérs et de discriminant donné, *Acta Arith.* **23** (1973), 419–426.

[Gy2] K. Győry, Sur les polynômes à coefficients entiérs et de discriminant donné II., *Publ. Math. Debrecen* **21** (1974), 125–144.

[Gy3] K. Győry, On the solutions of linear diophantine equations in algebraic integers of bounded norm, *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* **22/23** (1980), 225–233.

[H] L. Hajdu, A quantitative version of Dirichlet's S-unit theorem in algebraic number fields, *Publ. Math. Debrecen* **42** (1993), 239–246.

[L] H. W. Lenstra Jr., Algorithms in algebraic number theory, *Bull. Amer. Math. Soc.* **26** (1992), 211–244.

[M] K. Mahler, An inequality for the discriminant of a polynomial, *Michigan Math. J.* **11** (1964), 257–262.

[PW] P. Philippon and M. Waldschmidt, Lower bounds for linear forms in logarithms, New Advances in Transcendence Theory (A. Baker, ed.), *Cambridge Univ. Press*, *Cambridge*, 1988, 280–312.

[ST] A. Schinzel and R. Tijdeman, On the equation $y^m = P(x)$, *Acta Arith.* **31** (1976), 199–204.

[ShT] T.N. Shorey and R. Tijdeman, Exponential diophantine equations, *Cambridge Univ. Press*, *Cambridge*, 1986.

[T] R. Tijdeman, Applications of the Gelfond-Baker method to rational number theory, Topics in Number Theory, Proceedings of the Conference held in Debrecen 1974, Colloq. Math. Soc. János Bólyai 13, *North-Holland, Amsterdam*, 399–416.

[Tu1] J. Turk, Polynomial values and almost powers, *Michigan Math. J.* **29** (1982), 213–220.

[Tu2] J. Turk, On the difference between perfect powers, *Acta Arith.* **45** (1986), 289–307.

[Z] R. Zimmert, Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung, *Invent. Math.* **62** (1981), 367–380.

A. BÉRCZES, B. BRINDZA, L. HAJDU
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

(Received September 2, 1997)