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On automorphism groups of simple arguesian lattices

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Abstract. Let $\mathfrak{G}$ be a group. In this paper we prove that there exists a *simple arguesian* lattice M whose automorphism group is isomorphic to $\mathfrak{G}$.

A lattice L is called *interval finite*, if every interval of L is finite. In this note we give a new proof of a theorem of Christian Herrmann [3]. This theorem was proved by G. Grätzer and E. T. Schmidt [2] for finite groups and later by Christian Herrmann [3] in the present form.

Theorem. *Every group $\mathfrak{G}$ can be represented as the automorphism group of an interval finite, simple, arguesian lattice M.*

Let $\mathfrak{G}$ be a given group. By R. Frucht [1], there exists an undirected graph $\langle V, E\rangle$ with no loops whose automorphism group is isomorphic to $\mathfrak{G}$ (that is, V is a set and the set E of edges is a subset of two-elements subsets of V). We begin our construction with this graph.

We consider first a vector space $\mathfrak{V}$ over the two element field Z_2 with a basis V'. We assume that V and V' have the same cardinality, i.e. $|V| = |V'|$. Then we can identify the vertices of the graph with the basis elements of this vector space, that means, we can consider the elements $v_0, v_1, v_2, \dots$ of V as the basis elements of the vector space $\mathfrak{V}$. Let A be the lattice of all finitely generated subspaces of the vector space $\mathfrak{V}$. This lattice A is obviously a simple, atomistic, arguesian lattice. The vector

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space $\mathfrak{V}$ is over the two element field Z_2, consequently every line contains three points. The subspace generated by v_i will be denoted by the same letter v_i. The lattice A has the following three types of atoms:

1. The atoms v_i, $i \in I$ (i.e. the elements of the basis), these form the set V and I an arbitrary index set;
2. Consider the third point $v_i + v_j$ $(i, j \in I)$ of the line $\overline{v_i, v_j}$ spanned by v_i and v_j. Some of these $v_i + v_j$-s correspond to edges of the graph (i.e. $\{v_i, v_j\}$ is an edge), in this case $v_i + v_j$ will be denoted by v_{ij}. All these atoms form a subset W;
3. All other atoms.

We consider the given $\mathfrak{G}$ as a subgroup of the automorphism group of A. To the vertices of the Frucht graph correspond the atoms $v_i \in V$, $i \in I$ and to the edges $\{v_i, v_j\}$ correspond the atoms v_{ij}, these determine the edges in V. Obviously, every permutation of the v_i-s can be extended to an automorphism of A and every automorphism of A is determined by its restriction to the basis V. Indeed, if α and β are two automorphisms of A such that their restrictions to V are the same, then the restriction of $\gamma = \alpha\beta^{-1}$ is the identity map ϵ of V. By any extension of ϵ (i.e. automorphism with the property that its restriction to V is ϵ) the atoms v_i and v_j are fixed elements, conseqently $v_i + v_j$ must be fixed. Similarly, $(v_i + v_j) + v_k$ must be a fixd element. In this way we get that by an extension of ϵ all atoms are fixed elements which means that this extension is the identity mapping of A. It follows that all automorphisms with the property that V and W are invariant form a group isomorphic to $\mathfrak{G}$. To ensure that we have no more automorphisms than the graph we must *label* the vertices and the edges, i.e. the atoms $v_i \in V$ and $v_{ij} \in W$. This will be done by lattices which are glued to A. The idea of the gluing is the following. The ideal $(v_i]$ of A has two elements. We will define a special lattice F_1 with a two element dual ideal D_1 which is therefore isomorphic to $(v_i]$. Similarly, for every $v_{ij} \in W$ we use a lattice F_2 with the dual ideal D_2. For every $i \in I$ we consider an isomorphic copy ${F_1}^i$ of F_1 with the dual ideal ${D_1}^i$ and similarly the lattices ${F_2}^{ij} \cong F_2$ with the dual ideal ${D_2}^{ij}$. We can apply the gluing construction for the lattices A, ${F_1}^i$ and ${F_2}^{ij}$ simultaneously, identifying the ideal $(v_i]$ with ${D_1}^i$ and $(v_{ij}]$ with ${D_2}^{ij}$. On this way we get a join-semilattice and M is the arguesian lattice generated by this configuration. First we define the lattices F_1, F_2. We give the description of M as a sublattice of a vectorspace lattice and prove that this is a simple arguesian lattice with the given automorphism group.

N is the chain of all nonnegative integers and N^* denotes the chain of the nonpositive integers. Take the direct product $\mathfrak{C}_2 \times N^*$, (where $\mathfrak{C}_2$ denotes the two element lattice). In this direct product for every $i \in N$ the elements $(0, -i-1)$, $(1, -i-1)$, $(0, -i)$, $(1, -i)$ form a "covering square" (isomorphc to $\mathfrak{C}_2 \times \mathfrak{C}_2$). Into these "covering squares", for $i = 0, 1, \ldots$ we insert one more elemwnt z_i so that a copy of $\mathfrak{M}_3$, the five element non distributive modular lattice, is obtained. The resulting lattice is F_1, see Figure 1a. The lattice F_2 is similar but we don't insert z_0, into the first "covering square", see Figure 1b. The dual ideal consisting of $(0,0)$ and $(1,0)$ etc. of F_1 is D_1. We use isomorphic copies of F_1 and F_2 to label the v_i-s and the v_{ij}-s.

Figure 1a *Figure 1b*

F_1 is a simple arguesian lattice and it has exactly one nontrivial automorphism α, where $\alpha(z_0) = (0,0)$ and $\alpha(0,0) = (z_0)$. F_2 is a rigid (has no nontrivial automorphism) arguesian lattice, its congruence lattice is the four element Boolean lattice.

We define our lattice M as a sublattice of a vectorspace lattice $K = L(\mathfrak{W})$ of a vectorspace $\mathfrak{W}$ over Z_2. Take the set $\{{u_j}^k, v_j; j \in I, k \in N\}$ as a basis of $\mathfrak{W}$. Let ${z_j}^k$ be the third point of the line spanned by ${u_j}^k$ and v_j. Define the following subspaces, (where $[X]$ denotes the subspace spanned by the set X): $\boldsymbol{o} = [u_j{}^k; j \in I, k \in N]$, $\boldsymbol{v_i} = [v_i, u_j{}^k; j \in I, k \in N] = [v_i, \boldsymbol{o}]$. The convex sublattice of K, generated by (as lattice) $\boldsymbol{v_i}$-s form a sublattice isomorphic to A, we identify A with this sublattice.

Set $\boldsymbol{u_i}^0 = \boldsymbol{o}$, $\boldsymbol{u_i}^1 = [u_j{}^k; j \in I, k \in N, u_j{}^k \neq u_i{}^0]$, $\boldsymbol{u_i}^2 = [u_j{}^k; i \in I, k \in N, u_j{}^k \neq u_i{}^0, u_i{}^1] \dots$. Then $\boldsymbol{u_i}^0 > \boldsymbol{u_i}^1 > \boldsymbol{u_i}^2 > \dots$ is a chain of type ω^*. The convex sublattice generated by these chains will be denoted by C. Take the sublattice $A \cup C$, then A is a dual ideal and C is an ideal of this lattice. We adjoin further elements $\boldsymbol{w_i}^0, \boldsymbol{w_i}^1, \boldsymbol{w}_i{}^2, \dots$ and $\boldsymbol{z_i}^1, \boldsymbol{z_i}^2, \boldsymbol{z_i}^3 \dots$, which are defined as follows:.

$$\boldsymbol{w_i}^1 = [\boldsymbol{u_i}^1, v_i], \boldsymbol{w_i}^2 = [\boldsymbol{u_k}^2, v_i], \boldsymbol{w_i}^3 = [\boldsymbol{u_k}^3, v_i] \dots$$

and

$$\boldsymbol{z_i}^1 = [\boldsymbol{u_i}^1, z_i{}^1], \boldsymbol{z_i}^2 = [\boldsymbol{u_i}^2, z_i{}^2], \boldsymbol{z_i}^3 = [\boldsymbol{u_i}^3, z_i{}^3] \dots .$$

Then the join of the chains $\boldsymbol{u_i}^0 > \boldsymbol{u}_i{}^1 > \boldsymbol{u}_i{}^2 > \dots$ and $\boldsymbol{w_i}^0 > \boldsymbol{w_i}^1 > \boldsymbol{w_i}^2 > \dots$ form a sublattice isomorphic to $\mathfrak{C}_2 \times N^*$. For every j, $\boldsymbol{u_i}^j$, $\boldsymbol{z_i}^{j+1}$ and $\boldsymbol{w_i}^{j+1}$ generete $\mathfrak{M}_3$. For every $i \in I$ all these elements form a sublattice, the flap

$F_1{}^i = \{\boldsymbol{u_i}^0, \boldsymbol{u_i}^1, \boldsymbol{u_i}^2 \dots\} \cup \{\boldsymbol{w_i}^0, \boldsymbol{w_i}^1, \boldsymbol{w_i}^2 \dots\} \cup \{\boldsymbol{z_i}^1, \boldsymbol{z_i}^2, \boldsymbol{z_i}^3 \dots\}$ isomorphic to the lattice F_1.

Similarly, we define for the elements v_{ij} the flaps

$F_2{}^{ij} = \{\boldsymbol{u_{ij}}^0, \boldsymbol{u_{ij}}^1, \boldsymbol{u_{ij}}^2 \dots\} \cup \{\boldsymbol{w_{ij}}^0, \boldsymbol{w_{ij}}^1, \boldsymbol{w_{ij}}^2 \dots\} \cup \{\boldsymbol{z_{ij}}^2, \boldsymbol{z_{ij}}^3, \boldsymbol{z_{ij}}^4 \dots\}$ isomorphic to F_2.

Let M be $A \cup C \cup \bigcup(F_1{}^i, F_2{}^{ij} \mid i, j \in I)$.

M can be vizulised as follows, see Figure 2.

It is easy to see that M is a sublattice of K. The lattice K is an arguesian lattice, consequently M is again arguesian. We prove that M is simple. We know that A and the $F_1{}^i$-s are simple lattices and the intervals $[\boldsymbol{u_i}^k, \boldsymbol{u_i}^{k+1}]$ and $[\boldsymbol{u_j}^k, \boldsymbol{u_j}^{k+1}]$ resp. $[\boldsymbol{u_{ij}}^k, \boldsymbol{u_{ij}}^{k+1}]$ and $[\boldsymbol{u_j}^k, \boldsymbol{u_j}^{k+1}]$ are projective in C. These imply that any two prime intervals are projective, which proves that M is a simple lattice.

M contains the chains $\boldsymbol{w_i}^1 > \boldsymbol{w_i}^2 > \boldsymbol{w_i}^3 > \dots$ and $\boldsymbol{w_{ij}}^0 > \boldsymbol{w_{ij}}^1 > \boldsymbol{w_{ij}}^2 \dots$, where $\boldsymbol{w_i}^1, \boldsymbol{w_i}^2, \dots$ resp. $\boldsymbol{w_{ij}}^1, \boldsymbol{w_{ij}}^2 \dots$, $(i, j \in I)$ are meet irreducible elements.and M has no other chains of this type. Then for any automorphism the image of $\boldsymbol{w_i}^1$ must be $\boldsymbol{w_j}^1$ for some j and similarly the image of $\boldsymbol{u_{ij}}^1$ is some $\boldsymbol{u_{k\ell}}^1$. This yields that the restriction of an automorphism to the atoms of the dual ideal A of M is a permutation, where V and W are invariant. This proves that the automorphim group of M is isomorphic to $\mathfrak{G}$.

Figure 2

References

[1] R. Frucht, Herstellung von graphen mit vorgegebener abstrakter gruppe, *Compos. Math.* **6** (1938), 239–250.

[2] G. Grätzer and E. T. Schmidt, On finite automorphism groups of simple arguesian lattices, *Submitted for publication in Studia Sci. Math.*

[3] Ch. Herrmann, On automorphism groups of Arguesian lattices, *Acta Math. Acad. Sci. Hungar.*

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