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Curvature homogeneous unit tangent sphere bundles

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Abstract. We treat the classification of curvature homogeneous unit tangent sphere bundles. For two- and three-dimensional Riemannian manifolds we show that one only obtains the unit tangent sphere bundles of spaces of constant curvature. Moreover, we prove a similar result for conformally flat spaces and Sasakian space forms. Furthermore, we give a complete answer when the Riemannian manifold is a Damek–Ricci harmonic space or a four-dimensional Einstein manifold.

1. Introduction

This paper is a continuation of [6], [7], and is devoted to the study of the unit tangent sphere bundles and is related to our search for non-trivial curvature homogeneous Riemannian manifolds. A Riemannian manifold (M,g) is said to be *curvature homogeneous* ([13]) if and only if, for each pair of points p and q in M, there exists a linear isometry $F: T_pM \to T_qM$ such that $F^*R_q = R_p$, where R is the Riemann curvature tensor of(M,g). Equivalently, there exists a metric connection $\bar{\nabla}$ such that R is $\bar{\nabla}$-parallel. Another useful criterion can be derived from [11]: a Riemannian manifold is curvature homogeneous if and only if all scalar curvature invariants of order zero are global constants. Locally homogeneous spaces are trivially curvature homogeneous, but there exist a lot of examples which are not locally homogeneous. We refer to [5, Chapter 12] for a survey and further references.

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In [7], we started the discussion about the following problem: *which Riemannian manifolds (M, g) have a curvature homogeneous unit tangent sphere bundle (T_1M, g_S)?* Here, g_S denotes the induced metric from the Sasaki metric on the tangent bundle TM. Clearly, this is so when (M, g) is locally isometric to a two-point homogeneous space, since we have

Theorem 1 ([18], [10]). *If (M, g) is a two-point homogeneous space, then (T_1M, g_S) is a homogeneous Riemannian manifold.*

Up to now, we do not know of any other examples.

Since curvature homogeneous spaces have constant scalar curvature, we concentrated in [7] on the problem of determining all manifolds (M, g) such that (T_1M, g_S) has constant scalar curvature. We found a lot of examples and gave a complete classification for the two- and three-dimensional case and for conformally flat spaces. Our method was based on the explicit formulas for the curvature of (T_1M, g_S). (See [6], also for further references.) These formulas are rather complicated, but they are easier to handle when (M, g) has a simple curvature tensor, which is the case for the three cases mentioned.

Continuing our study, we now impose a further necessary condition on the unit tangent sphere bundle (T_1M, g_S). Any curvature homogeneous space is also Ricci-curvature homogeneous, that is, the eigenvalues of the Ricci tensor and their multiplicities are constant on the manifold, or equivalently, there exists a metric connection $\bar{\nabla}$ such that the Ricci tensor is $\bar{\nabla}$-parallel. In dimension two or three, this notion is clearly equivalent to curvature homogeneity, but the equivalence does not hold for higher dimensions.

First, in Section 2, we recall the needed formulas and earlier results from [7]. In Section 3, we classify all (M^2, g) and (M^3, g) with Ricci-curvature homogeneous unit tangent sphere bundle, by showing that this happens if and only if the base manifold (M, g) is of constant curvature. This yields a complete solution of the stated problem. In Section 4, we prove a similar result for conformally flat spaces. Furthermore, and based on the results of [7], we determine in Sections 5, 6 and 7 which of the Damek-Ricci harmonic spaces, the four-dimensional Einstein spaces and the Sasakian space forms are curvature homogeneous or even Ricci-curvature homogeneous. It turns out that in all these cases (M, g) has to be locally isometric to a two-point homogeneous space. As a consequence of these results, the authors feel that there is some support for a positive answer to the following problem:

Problem. *Is any Riemannian manifold (M,g) with (Ricci-)curvature homogeneous (T_1M, g_S) locally isometric to a two-point homogeneous space?*

We are still unable to give a conclusive answer, neither do we know what happens if (T_1M, g_S) is supposed to be (locally) homogeneous. Concerning this last case, we could not find an answer in the literature.

2. Preliminary formulas and results

We first recall the conventions and notations of [6], [7] and collect the formulas and results we need in this paper. We refer to those articles for a more elaborate exposition.

Let (M,g) be a smooth, n-dimensional ($n \geq 2$), connected Riemannian manifold and ∇ its Levi Civita connection. The Riemann curvature tensor R is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for all vector fields X, Y and Z on M. The tangent bundle of (M,g), denoted by TM, consists of pairs (x,u) where x is a point in M and u a tangent vector to M at x. The mapping $\pi : TM \to M : (x,u) \mapsto x$ is the natural projection from TM onto M.

It is well-known that the tangent space to TM at (x,u) splits into the direct sum of the vertical subspace $VTM_{(x,u)} = \ker \pi_{*|(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ with respect to the connection ∇:

$$T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}.$$

For $X \in T_xM$, there exists a unique vector X^h at the point $(x,u) \in TM$ such that $X^h \in HTM_{(x,u)}$ and $\pi_*(X^h) = X$. X^h is called the *horizontal lift* of X to (x,u). There is also a unique vector X^v at the point (x,u) such that $X^v \in VTM_{(x,u)}$ and $X^v(df) = Xf$ for all functions f on M. X^v is called the *vertical lift* of X to (x,u). The map $X \mapsto X^h$, respectively $X \mapsto X^v$, is an isomorphism between T_xM and $HTM_{(x,u)}$, respectively T_xM and $VTM_{(x,u)}$. Similarly, one lifts vector fields on M to horizontal or vertical vector fields on TM. The expressions in local coordinates for these lifts are given in [6].

The tangent bundle TM of a Riemannian manifold (M,g) can be endowed in a natural way with a Riemannian metric Tg, the so-called

Sasaki metric, depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$Tg(X^h, Y^h) = Tg(X^v, Y^v) = g(X, Y) \circ \pi, \qquad Tg(X^h, Y^v) = 0$$

for all vector fields X and Y on M.

In this paper, we consider the hypersurface T_1M, the *unit tangent sphere bundle*, consisting of the unit tangent vectors to (M, g). T_1M is given implicitly by the equation $g_x(u, u) = 1$. A unit normal vector N to T_1M at $(x, u) \in T_1M$ is given by the vertical lift of u to $(x, u) : N_{|(x,u)}{=}u^v$.

As the vertical lift of a vector (field) is not tangent to T_1M in general, we define the *tangential lift* of $X \in T_xM$ to $(x, u) \in T_1M$ by

$$X^t_{(x,u)} = (X - g(X, u)\, u)^v_{(x,u)}.$$

The tangent space to T_1M at (x, u) is spanned by vectors of the form X^h and X^t where $X \in T_xM$.

We endow T_1M with the Riemannian metric g_S induced from the Sasaki metric Tg on TM. It is given explicitly by

$$\begin{aligned}
g_{S|(x,u)}(X^t, Y^t) &= g_x(X, Y) - g_x(X, u) g_x(Y, u),\\
g_{S|(x,u)}(X^t, Y^h) &= 0,\\
g_{S|(x,u)}(X^h, Y^h) &= g_x(X, Y).
\end{aligned}$$

The Riemann curvature tensor $\bar{R}$ associated to this metric has been calculated, e.g., in [6] and [17]. From this, one easily obtains the following expression for the Ricci curvature tensor $\bar{\rho}$:

$$\begin{aligned}
\bar{\rho}_{|(x,u)}(X^t, Y^t) &= (n-2)\big(g_x(X, Y) - g_x(X, u) g_x(Y, u)\big)\\
&\qquad + \frac{1}{4}\sum_{i=1}^{n} g_x(R(u, X)E_i, R(u, Y)E_i),\\
\bar{\rho}_{|(x,u)}(X^t, Y^h) &= \frac{1}{2}\Big((\nabla_u \rho)_x(X, Y) - (\nabla_X \rho)_x(u, Y)\Big),\\
\bar{\rho}_{|(x,u)}(X^h, Y^h) &= \rho_x(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g_x(R(u, E_i)X, R(u, E_i)Y)
\end{aligned} \tag{1}$$

where $\{E_1, \ldots, E_n = u\}$ is an orthonormal basis of T_xM. Then $\{{E_1}^t, \ldots, {E_{n-1}}^t, {E_1}^h, \ldots, {E_n}^h\}$ is an orthonormal basis of $T_{(x,u)}T_1M$. We obtain the scalar curvature $\bar{\tau}$ by making a metric contraction of $\bar{\rho}$:

$$
\begin{aligned}
\bar{\tau}_{|(x,u)} &= \sum_{i=1}^{n-1} \bar{\rho}_{|(x,u)}({E_i}^t, {E_i}^t) + \sum_{i=1}^{n} \bar{\rho}_{|(x,u)}({E_i}^h, {E_i}^h) \\
&= \tau_x + (n-1)(n-2) - \xi_x(u,u)/4
\end{aligned} \tag{2}
$$

where, as in [2], $\xi(u,v) = \sum_{i,j=1}^{n} g(R(u,E_i)E_j, R(v,E_i)E_j)$.

We note that the natural projection $\pi_1 : (T_1M, g_S) \to (M, g) : (x,u) \mapsto x$ is a Riemannian submersion with totally geodesic fibres. Hence, one can also use O'Neill's formalism (see, e.g., [3, p. 244]) to obtain the above expressions for the Ricci curvature and the scalar curvature.

From (2), it follows readily

Theorem 2 ([7]). *The unit tangent sphere bundle (T_1M, g_S) has constant scalar curvature $\bar{\tau}$ if and only if on (M, g) it holds*

$$\xi = \frac{|R|^2}{n}\, g, \tag{3}$$

$$4n\tau - |R|^2 = \text{constant}. \tag{4}$$

As examples of Riemannian manifolds whose unit tangent sphere bundle has constant scalar curvature, we mention the irreducible symmetric spaces and harmonic spaces. In particular, this is the case for the Damek-Ricci harmonic spaces. For the definition of this class of manifolds and for some of their geometric properties, see Section 5.

For (locally) reducible manifolds, we have:

Corollary 3 ([7]). *The unit tangent sphere bundle (T_1M, g_S) of a (local) product manifold $(M, g) = (M_1^{n_1}, g_1) \times (M_2^{n_2}, g_2)$ has constant scalar curvature if and only if the unit tangent sphere bundles of both (M_1, g_1) and (M_2, g_2) have constant scalar curvature and, additionally,*

$$\frac{|R_1|^2}{n_1} = \frac{|R_2|^2}{n_2}. \tag{5}$$

In [7], the present authors determined all two- and three-dimensional Riemannian manifolds (M, g) whose unit tangent sphere bundles have constant scalar curvature $\bar{\tau}$. They also considered conformally flat Riemannian manifolds. It holds:

Proposition 4. *(T_1M^2, g_S) has constant scalar curvature $\bar{\tau}$ if and only if (M^2, g) has constant curvature.*

Proposition 5. *(T_1M^3, g_S) has constant scalar curvature $\bar{\tau}$ if and only if (M^3, g) has constant curvature or (M^3, g) is a curvature homogeneous space with constant Ricci roots $\rho_1 = \rho_2 = 0 \neq \rho_3$.*

Proposition 6. *Let (M^n, g) be conformally flat and $n \geq 4$. Then (T_1M, g_S) has constant scalar curvature $\bar{\tau}$ if and only if (M, g) has constant curvature or n is even, say $n = 2k$, and (M, g) is locally isometric to the product manifold $M^k(\kappa) \times M^k(-\kappa)$, $\kappa \neq 0$, or $n = 4$, $|\rho|^2$ is constant and τ is zero.*

Here, $M^k(\kappa)$ denotes a k-dimensional space of constant curvature κ.

The proof of the above three propositions uses the fact that the curvature tensor R can be expressed explicitly using only the scalar curvature τ and the Ricci curvature ρ. Namely, for dimension two, we have

$$R = \frac{\tau}{4}\, g \owedge g, \tag{6}$$

and in dimension three, it holds

$$R = \rho \owedge g - \frac{\tau}{4}\, g \owedge g. \tag{7}$$

For conformally flat manifolds, the curvature is given by

$$R = \frac{1}{n-2}\, \rho \owedge g - \frac{\tau}{2(n-1)(n-2)}\, g \owedge g. \tag{8}$$

Here, $\owedge$ is the Kulkarni-Nomizu product of symmetric $(0,2)$-tensors defined by

$$\begin{aligned}(h \owedge k)(X, Y, Z, V) = {} & h(X, Z)k(Y, V) + h(Y, V)k(X, Z) \\ & - h(X, V)k(Y, Z) - h(Y, Z)k(X, V).\end{aligned}$$

A final result which is useful in the context of this paper concerns four-dimensional Einstein spaces.

Proposition 7 ([7])**.** *Let (M^4, g) be a four-dimensional Einstein space. Then (T_1M, g_S) has constant scalar curvature if and only if $|R|^2$ is constant.*

In the rest of this paper, we consider several classes of Riemannian manifolds whose unit tangent sphere bundle has constant scalar curvature and investigate which of these are curvature or Ricci-curvature homogeneous.

3. Classification in dimension two and three

We start by considering the two- and three-dimensional case. First, combining Theorem 1 with Proposition 4, we obtain easily

Proposition 8. *(T_1M^2, g_S) is Ricci-curvature homogeneous or locally homogeneous if and only if (M^2, g) has constant curvature.*

We have a similar result for three-dimensional spaces:

Proposition 9. *(T_1M^3, g_S) is Ricci-curvature homogeneous or locally homogeneous if and only if (M^3, g) has constant curvature.*

Proof. In view of Theorem 1 and Proposition 5, it suffices to show that a three-dimensional curvature homogeneous space (M^3, g) with constant Ricci roots $\rho_1 = \rho_2 = 0 \neq \rho_3$ cannot have a Ricci-curvature homogeneous unit tangent sphere bundle.

Let (E_1, E_2, E_3) be an orthonormal basis of corresponding eigenvectors for the Ricci tensor ρ and $(\omega^1, \omega^2, \omega^3)$ the dual orthonormal coframe. Then $\tau = \rho_3$ and ρ is given by

$$\rho = \rho_3\, \omega^3 \otimes \omega^3. \tag{9}$$

For its covariant derivative $\nabla\rho$, we have

$$\begin{aligned}\nabla\rho = \rho_3 \Big(&(a\omega^1 + b\omega^2) \otimes (\omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1) \\ &+ (c\omega^1 + e\omega^2) \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2)\Big)\end{aligned} \tag{10}$$

where

$$\begin{aligned} a &= g(\nabla_{E_1}E_3, E_1), \quad b = g(\nabla_{E_2}E_3, E_1), \\ c &= g(\nabla_{E_1}E_3, E_2), \quad e = g(\nabla_{E_2}E_3, E_2). \end{aligned}$$

(See also [9].) As τ is constant, it follows

$$0 = \frac{1}{2}\nabla_{E_3}\tau = \sum_{i=1}^{3} (\nabla_{E_i}\rho)(E_3, E_i) = a + e. \tag{11}$$

We first calculate the Ricci tensor $\bar\rho$ on the unit tangent sphere bundle at the point (x, u) where $u = \cos\theta\, E_1 + \sin\theta\, E_2$. An orthonormal basis of

$T_{(x,u)}T_1M$ is given by

$$F_1 = (\sin\theta\, E_1 - \cos\theta\, E_2)^t,$$

$$F_2 = {E_3}^t, \quad F_3 = {E_1}^h, \quad F_4 = {E_2}^h, \quad F_5 = {E_3}^h.$$

We compute the components of the Ricci curvature $\bar{\rho}$ in this basis from (1), where we use the curvature expression (7) and the formulas (9) and (10). This yields at (x,u):

$$\bar{\rho}(F_1,F_1) = \bar{\rho}(F_2,F_2) = 1 + ({\rho_3}^2/8),$$

$$\bar{\rho}(F_1,F_i) = \bar{\rho}(F_5,F_i) = 0, \quad i = 2,3,4,$$

$$\bar{\rho}(F_1,F_5) = \rho_3(b-c)/2,$$

$$\bar{\rho}(F_2,F_3) = \rho_3(a\cos\theta + b\sin\theta)/2,$$

$$\bar{\rho}(F_2,F_4) = \rho_3(c\cos\theta + e\sin\theta)/2,$$

$$\bar{\rho}(F_3,F_3) = -{\rho_3}^2(1+\cos^2\theta)/8,$$

$$\bar{\rho}(F_3,F_4) = -{\rho_3}^2\cos\theta\,\sin\theta/8,$$

$$\bar{\rho}(F_4,F_4) = -{\rho_3}^2(1+\sin^2\theta)/8,$$

$$\bar{\rho}(F_5,F_5) = \rho_3 - ({\rho_3}^2/8).$$

The corresponding matrix for $\bar{\rho}_{|(x,u)}$ falls apart in two blocks: the 2×2-matrix associated to (F_1,F_5) and the 3×3-matrix associated to (F_2,F_3,F_4). The first one does not depend on the variable θ, but the second does. Still, the eigenvalues (including multiplicities) must be constant if the unit tangent sphere bundle is Ricci-curvature homogeneous. So, its characteristic polynomial must be independent of θ. This polynomial is given explicitly as

$$-\lambda^3 + (1 - \frac{{\rho_3}^2}{4})\,\lambda^2$$
$$+\frac{{\rho_3}^2}{4}\Big(\frac{{\rho_3}^2}{16} + \frac{3}{2} + (a\cos\theta + b\sin\theta)^2 + (c\cos\theta + e\sin\theta)^2\Big)\,\lambda$$

$$+\frac{{\rho_3}^4}{16}\Big(\frac{{\rho_3}^2}{16}+\frac{1}{2}+\frac{1}{2}\left((a\cos\theta+b\sin\theta)^2+(c\cos\theta+e\sin\theta)^2\right)$$

$$-\cos\theta\sin\theta\,(a\cos\theta+b\sin\theta)(c\cos\theta+e\sin\theta)$$

$$+\frac{1}{2}\left(\sin^2\theta\,(a\cos\theta+b\sin\theta)^2+\cos^2\theta\,(c\cos\theta+e\sin\theta)^2\right)\Big).$$

The necessary and sufficient conditions for the coefficients of this polynomial to be independent of θ are

$$a-e=0,\quad b+c=0. \tag{12}$$

From (11) and (12), it follows $a=e=0$. The (full) characteristic polynomial for $\bar{\rho}_{|(x,u)}$ then reduces to

$$\begin{aligned}&\left(\lambda^2-(1+\rho_3)\lambda+\rho_3\left(1-\left(\frac{1}{8}+b^2\right)\rho_3+\frac{{\rho_3}^2}{8}-\frac{{\rho_3}^3}{64}\right)\right)\\ &\times\left(\lambda^2-\lambda-\frac{{\rho_3}^2}{4}\left(\left(\frac{1}{2}+b^2\right)+\frac{{\rho_3}^2}{16}\right)\right)\times\left(-\frac{{\rho_3}^2}{4}-\lambda\right).\end{aligned} \tag{13}$$

Next, we calculate the Ricci tensor $\bar{\rho}$ at the point (x,u) where now $u=E_3$. With respect to the orthonormal basis $({E_1}^t,{E_2}^t,{E_1}^h,{E_2}^h,{E_3}^h)$ of $T_{(x,u)}T_1M$, the matrix for $\bar{\rho}$ is given by

$$\begin{pmatrix}1+({\rho_3}^2/8) & 0 & 0 & \rho_3 b/2 & 0\\ 0 & 1+({\rho_3}^2/8) & -\rho_3 b/2 & 0 & 0\\ 0 & -\rho_3 b/2 & -{\rho_3}^2/8 & 0 & 0\\ \rho_3 b/2 & 0 & 0 & -{\rho_3}^2/8 & 0\\ 0 & 0 & 0 & 0 & \rho_3-({\rho_3}^2/4)\end{pmatrix}.$$

Its characteristic polynomial is

$$\left(\lambda^2-\lambda-\frac{{\rho_3}^2}{4}\left(\left(\frac{1}{2}+b^2\right)+\frac{{\rho_3}^2}{16}\right)\right)^2\times\left(\rho_3-\frac{{\rho_3}^2}{4}-\lambda\right). \tag{14}$$

The polynomials (13) and (14) must be identical if the unit tangent sphere bundle is Ricci-curvature homogeneous. Hence, $-{\rho_3}^2/4$ must be a root of the quadratic part in (14), and $\rho_3 - ({\rho_3}^2/4)$ must be a root of the first quadratic factor of (13). In this way, we get the conditions

$$3{\rho_3}^2 + 8 = 16b^2 \quad \text{and} \quad \frac{3}{4}\,{\rho_3}^2 - 2\rho_3 + 2 = 16b^2.$$

Eliminating b^2, we obtain a quadratic polynomial in ρ_3 with no real roots. Hence, whatever the value of ρ_3, (T_1M, g_S) is not Ricci-curvature homogeneous. □

4. Conformally flat manifolds

The complete classification results obtained above follow rather easily from (1) because of the simple expressions for R. A similar situation occurs also for conformally flat manifolds. Here, we have

Proposition 10. *Let (M^n, g) be conformally flat and $n \geq 4$. Then (T_1M, g_S) is Ricci-curvature homogeneous or locally homogenenous if and only if (M, g) has constant curvature.*

Proof. From Theorem 1 and Proposition 6, we see that it suffices to consider two cases only: the one where the manifold (M, g) is four-dimensional with $\tau = 0$ and $|\rho|^2$ constant, and the case where (M, g) is a local product $M^k(\kappa) \times M^k(-\kappa)$ of spaces of constant curvature $\kappa \neq 0$. Using the same technique as for the three-dimensional case, we show that neither of those two has Ricci-curvature homogeneous unit tangent sphere bundle. Note that the condition to be conformally flat implies that the curvature operator $\rho - \frac{\tau}{4}\, g$ on (M, g) is a Codazzi tensor. In the cases we are interested in, the scalar curvature τ is constant (actually, $\tau = 0$), hence the Ricci tensor ρ is a Codazzi tensor.

First, take a four-dimensional conformally flat manifold (M^4, g) with zero scalar curvature τ and constant Ricci norm $|\rho|$. Denote the Ricci roots by ρ_1, ρ_2, ρ_3 and ρ_4 and let (E_1, E_2, E_3, E_4) be an orthonormal basis of corresponding eigenvectors for the Ricci tensor ρ. Then $\rho_1 + \rho_2 + \rho_3 + \rho_4 = \tau = 0$ and $R = \frac{1}{2}\, \rho \owedge g$.

We calculate the matrix for the Ricci tensor $\bar{\rho}$ of T_1M^4 at (x, u) where $u = \cos\theta\, E_1 + \sin\theta\, E_2$, with respect to the orthonormal basis $((\sin\theta\, E_1 - \cos\theta\, E_2)^t, {E_3}^t, {E_4}^t, {E_1}^h, {E_2}^h, {E_3}^h, {E_4}^h)$ for $T_{(x,u)}T_1M$. Again this matrix falls apart in two blocks: the 2×2-matrix associated to

$({E_1}^h, {E_2}^h)$ and the remaining diagonal 5×5-matrix with diagonal elements $(2+(\rho_1+\rho_2)^2/2,\ 2+\cos^2\theta(\rho_1+\rho_3)^2/2+\sin^2\theta(\rho_2+\rho_3)^2/2,\ 2+\cos^2\theta(\rho_1+\rho_4)^2/2+\sin^2\theta(\rho_2+\rho_4)^2/2,\ \rho_3-\cos^2\theta(\rho_1+\rho_3)^2/2-\sin^2\theta(\rho_2+\rho_3)^2/2,\ \rho_4-\cos^2\theta(\rho_1+\rho_4)^2/2-\sin^2\theta(\rho_2+\rho_4)^2/2)$.

For (T_1M, g_S) to be Ricci-curvature homogeneous, this diagonal matrix must have constant eigenvalues. Hence, we have $(\rho_1+\rho_3)^2=(\rho_2+\rho_3)^2$ and $(\rho_1+\rho_4)^2=(\rho_2+\rho_4)^2$. Actually, using $\rho_1+\rho_2+\rho_3+\rho_4=0$, both are equivalent to

$$(\rho_1-\rho_2)(\rho_3-\rho_4)=0.$$

If we start with the choices $u=\cos\theta\, E_1+\sin\theta\, E_3$ and $u=\cos\theta\, E_1+\sin\theta\, E_4$, we obtain similarly the conditions

$$\begin{aligned}(\rho_1-\rho_3)(\rho_2-\rho_4)&=0,\\ (\rho_1-\rho_4)(\rho_2-\rho_3)&=0.\end{aligned}$$

Hence, at least three of the Ricci roots are equal. We can suppose that $\rho_1=\rho_2=\rho_3$. Then $\rho_4=-3\rho_1$ and $|\rho|^2=12{\rho_1}^2$. As a consequence, all Ricci roots are constant and the manifold (M^4, g) is curvature homogeneous.

In [15], H. Takagi gives an explicit classification of conformally flat locally homogeneous spaces of arbitrary dimension. He shows that such a space is locally isometric to one of the following locally symmetric spaces:

1. a space of constant curvature;
2. the Riemannian product of a space of non-zero constant curvature κ and a space of constant curvature $-\kappa$;
3. the Riemannian product of a space of non-zero constant curvature κ and a one-dimensional space.

As his proof uses only curvature homogeneity, this classification is also valid for conformally flat curvature homogeneous spaces. (See also [8].) In the present situation, we see that (M^4, g) is locally isometric to

1. a space of constant curvature: this corresponds to the value $\rho_1=0$ and $(M, g)$is flat;
2. the product of a space of non-zero constant curvature κ and a space of non-zero constant curvature $-\kappa$: this is incompatible with the Ricci roots $(\rho_1, \rho_1, \rho_1, -3\rho_1)$;

3. the product of a space of non-zero constant curvature κ and a one-dimensional space: again this is not compatible with the specific Ricci roots we have here.

So, in dimension four, the only conformally flat manifolds having curvature homogeneous unit tangent sphere bundles are the spaces of constant curvature.

The remaining case when (M,g) is locally isometric to $M^k(\kappa) \times M^k(-\kappa)$, is a special case of the more general Proposition 11 on product manifolds, which immediately follows this proof. $\square$

Proposition 11. *Let (M,g) be locally isometric to the Riemannian product $(M_1,g_1)\times(M_2,g_2)$ and suppose that the Ricci tensor ρ_1 of (M_1,g_1) is a Codazzi tensor. If (T_1M,g_S) is Ricci-curvature homogeneous, then (M,g) is flat.*

Proof. Let $\pi_i : M \to M_i$ denote the natural projection from M to M_i, $i=1,2$. If (T_1M,g_S) is Ricci-curvature homogeneous, then it has constant scalar curvature $\bar{\tau}$. By Corollary 3, we have $\xi_i = (|R_i|^2/n_i)\, g_i$, $i=1,2$, and $|R_1|^2/n_1 = |R_2|^2/n_2$. In particular, if one of the factors is flat, so is the other. For this reason, we suppose that $n_1, n_2 \geq 2$.

Let (x,y) be a point in M and take an arbitrary unit vector $u_1 \in T_xM_1$ and an arbitrary unit vector $v_1 \in T_yM_2$. Extend these to an orthonormal basis $(u_1,u_2,\ldots,u_{n_1})$ of T_xM_1, respectively $(v_1,v_2,\ldots,v_{n_2})$ of T_yM_2. We calculate $\bar{\rho}$ at $((x,y),u)$ where $u = \cos\theta\, u_1 + \sin\theta\, v_1$. By the expressions (1) for the Ricci curvature and the assumption that ρ_1 is a Codazzi tensor, we have at $((x,y),u)$

$$\begin{aligned}
\bar{\rho}(X^t,Y^t) &= (n-2)\big(g(X,Y) - g_x(X,u)g_x(Y,u)\big)\\
&\qquad + \frac{1}{4}\cos^2\theta \sum_{k=1}^{n_1} g_1(R_1(u_1,\pi_{1*}X)u_k, R_1(u_1,\pi_{1*}Y)u_k)\\
&\qquad + \frac{1}{4}\sin^2\theta \sum_{k=1}^{n_2} g_2(R_2(v_1,\pi_{2*}X)v_k, R_2(v_1,\pi_{2*}Y)v_k),\\
\bar{\rho}(X^t,Y^h) &= \frac{1}{2}\sin\theta\big((\nabla_{v_1}\rho_2)(\pi_{2*}X,\pi_{2*}Y) - (\nabla_{\pi_{2*}X}\rho_2)(v_1,\pi_{2*}Y)\big).
\end{aligned}$$

It follows that the matrix of $\bar{\rho}$ at $((x,y),u)$ with respect to the orthonormal basis $({u_2}^t,\ldots,{u_{n_1}}^t,{v_2}^t,\ldots,{v_{n_2}}^t,(\sin\theta\, u_1 - \cos\theta\, v_1)^t,{u_1}^h,\ldots,{u_{n_1}}^h,{v_1}^h,$

$\dots, {v_{n_2}}^h)$ has the form

$$\begin{pmatrix} \bar\rho({u_i}^t, {u_j}^t)_{|i,j=2,\dots,n_1} & 0 \\ 0 & * \end{pmatrix}.$$

For (T_1M, g_S) to be Ricci-curvature homogeneous, the eigenvalues of the matrix

$$\begin{aligned} &(\bar\rho({u_i}^t, {u_j}^t)_{|i,j=2,\dots,n_1}) \\ &= \Big((n-2)\delta_{ij} + \frac{\cos^2\theta}{4} \sum_{k=1}^{n_1} g_1(R_1(u_1, u_i)u_k, R_1(u_1, u_j)u_k) \Big) \end{aligned}$$

must be independent of θ. Equivalently, the eigenvalues of the symmetric operator

$$(X, Y) \mapsto \cos^2\theta \sum_{k=1}^{n_1} g_1(R_1(u_1, X)u_k, R_1(u_1, Y)u_k)$$

must be independent of θ. This is possible only if u_1 belongs to the nullity vector space of R_1 at x. But x and u_1 are arbitrary, hence $R_1 = 0$ and (M_1, g_1) is flat. By the introductory comments to this proof, also $R_2 = 0$ and (M, g) is flat. □

5. Damek-Ricci spaces

In the next two sections, we consider some cases already mentioned in Section 2 for which we know that the unit tangent sphere bundle has constant scalar curvature. We start with the Damek-Ricci harmonic spaces and first briefly recall their definition and some results about these remarkable manifolds that we need in this section. We refer the reader to [1] for an extensive treatment and further references.

Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces of dimensions n and m respectively, and $\beta : \mathfrak{v}\times\mathfrak{v} \to \mathfrak{z}$ a skew-symmetric bilinear map. We endow the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with an inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{n}}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular to each other. For each $Z \in \mathfrak{z}$, we define an operator $J_Z : \mathfrak{v} \to \mathfrak{v}$ by $\langle J_Z U, V \rangle_{\mathfrak{n}} = \langle \beta(U, V), Z \rangle_{\mathfrak{n}}$ for all $U, V \in \mathfrak{v}$. Next, we make $\mathfrak{n}$ into a Lie algebra with the bracket

$$[U + X, V + Y]_{\mathfrak{n}} := \beta(U, V)$$

for $X, Y \in \mathfrak{z}$ and $U, V \in \mathfrak{v}$. This Lie algebra is said to be a *generalized Heisenberg algebra* if $J_Z{}^2 = -\langle Z, Z\rangle_{\mathfrak{n}} \,\mathrm{Id}_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$. This condition puts strong restrictions on the dimension n of $\mathfrak{v}$ once we fix the dimension m of $\mathfrak{z}$. (See the table in [1, p. 23].)

Next, let $\mathfrak{n}$ be a generalized Heisenberg algebra, $\mathfrak{a}$ a one-dimensional real vector space and A a non-zero vector in $\mathfrak{a}$. Define a new vector space $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$. In what follows, we always use U, V, W for vectors in $\mathfrak{v}$, X, Y, Z for vectors in $\mathfrak{z}$ and r, s, t for real numbers. On $\mathfrak{s}$, we define an inner product $\langle \cdot\,, \cdot\rangle$ by

$$\langle U + X + rA, V + Y + sA\rangle := \langle U + X, V + Y\rangle_{\mathfrak{n}} + rs$$

and a Lie bracket $[\cdot\,, \cdot]$ by

$$[U + X + rA, V + Y + sA] := [U, V]_{\mathfrak{n}} + \frac{1}{2}\, rV - \frac{1}{2}\, sU + rY - sX.$$

The simply connected Lie group S, attached to the Lie algebra $\mathfrak{s}$ and with the left-invariant metric induced from the inner product on $\mathfrak{s}$, is called a *Damek-Ricci space*.

The Damek-Ricci spaces were the first counterexamples to the fundamental Lichnerowicz conjecture, as they are all harmonic but only symmetric in special cases (see, e.g., [1]):

Theorem 12. *A Damek-Ricci space S is a Riemannian symmetric space if and only if the attached generalized Heisenberg algebra $\mathfrak{n}$ satisfies the J^2-condition: for all $X, Y \in \mathfrak{z}$ with $\langle X, Y\rangle = 0$ and all non-zero $U \in \mathfrak{v}$, there exists a vector $Z \in \mathfrak{z}$ such that $J_X J_Y U = J_Z U$. In this case, S is two-point homogeneous.*

We need two more ingredients. First, let $V \in \mathfrak{v}$ be a non-zero vector. Denote by $\ker \mathrm{ad}(V)$ the kernel of the linear map

$$\mathrm{ad}(V) : \mathfrak{v} \to \mathfrak{z} : U \mapsto [U, V],$$

and by $\ker \mathrm{ad}(V)^{\perp}$ the orthogonal complement to $\ker \mathrm{ad}(V)$ in $\mathfrak{v}$. Since $U \in \ker \mathrm{ad}(V)$ if and only if $0 = \langle [V, U], Z\rangle = \langle J_Z V, U\rangle$ for all $Z \in \mathfrak{z}$, we see that

$$\ker \mathrm{ad}(V)^{\perp} = J_{\mathfrak{z}} V. \tag{15}$$

In particular, $\dim\ker\mathrm{ad}(V) = n - \dim\ker\mathrm{ad}(V)^{\perp} = n - \dim J_{\mathfrak{z}}V = n - m$.

Secondly, one can express the curvature tensor R of a Damek-Ricci space S completely in terms of the bracket on $\mathfrak{s}$ and the operators J_Z. If we denote the left-invariant vector field on S associated to the vector $U + X + rA$ in $\mathfrak{s}$ by the same expression, we have

$$
\begin{aligned}
(16)\qquad & R(U+X+rA, V+Y+sA)(W+Z+tA) \\
&= \frac{1}{2}J_X J_Y W + \frac{1}{4}J_Z J_Y U - \frac{1}{4}J_Z J_X V + \frac{1}{2}J_{[U,V]}W \\
&\quad - \frac{1}{4}J_{[V,W]}U + \frac{1}{4}\,J_{[U,W]}V + \frac{1}{2}rJ_Y W - \frac{1}{2}sJ_X W \\
&\quad - \frac{1}{4}sJ_Z U + \frac{1}{4}tJ_Y U + \frac{1}{4}rJ_Z V - \frac{1}{4}tJ_X V \\
&\quad + \frac{1}{2}\langle X,Y\rangle W - \frac{1}{4}(\langle V,W\rangle + st)U + \frac{1}{4}(\langle U,W\rangle + rt)V \\
&\quad - \frac{1}{2}[U,J_Z V] - \frac{1}{4}[U,J_Y W] + \frac{1}{4}[V,J_X W] \\
&\quad + \frac{1}{2}t[U,V] + \frac{1}{4}s[U,W] - \frac{1}{4}r[V,W] \\
&\quad - \langle V+Y+sA, W+Z+tA\rangle X \\
&\quad + \langle U+X+rA, W+Z+tA\rangle Y \\
&\quad + \frac{1}{2}\langle V,W\rangle X - \frac{1}{2}\langle U,W\rangle Y + \frac{1}{2}\langle U,V\rangle Z \\
&\quad + \Big\{ -\frac{1}{2}\langle J_Z U,V\rangle - \frac{1}{4}\langle J_Y U,W\rangle + \frac{1}{4}\langle J_X V,W\rangle \\
&\quad - r\left(\frac{1}{4}\langle V,W\rangle + \langle Y,Z\rangle\right) + s\left(\frac{1}{4}\langle U,W\rangle + \langle X,Z\rangle\right)\Big\}\, A.
\end{aligned}
$$

With these preliminaries, we are ready to prove

Proposition 13. *The unit tangent sphere bundle (T_1S, g_S) of a Damek-Ricci space S is Ricci-curvature homogeneous or locally homogeneous if and only if S is a symmetric space.*

Proof. We show that, if (T_1S, g_S) is Ricci-curvature homogeneous, then the attached generalized Heisenberg algebra $\mathfrak{n}$ must satisfy the J^2-condition. The proposition then follows easily from Theorem 12 and Theorem 1.

For a start, we note that S is an Einstein manifold, hence $\bar{\rho}_{|(x,u)}(F^t, G^h) = 0$ for all tangent vectors $F, G \in T_xS$ and for every unit vector u at x. So, for whatever choice of $(x,u) \in T_1S$, the matrix of $\bar{\rho}_{|(x,u)}$ is of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 is the matrix of $\bar{\rho}_{|(x,u)}$ restricted to $\mathfrak{s}^t \times \mathfrak{s}^t$ and A_2 is the matrix of $\bar{\rho}_{|(x,u)}$ restricted to $\mathfrak{s}^h \times \mathfrak{s}^h$. If we suppose that (T_1S, g_S) is Ricci-curvature homogeneous, then the eigenvalues of A_1 and of A_2 must be independent of the choice of $(x,u) \in T_1S$. In what follows, we concentrate only on $\bar{\rho}_{|(x,u)}$ restricted to $\mathfrak{s}^t \times \mathfrak{s}^t$.

First, we take the unit vector A at x. At (x,A), the tangential part of $T_{(x,A)}T_1S$ is given by $\mathfrak{v}^t \oplus \mathfrak{z}^t$. From the formula (16) for the curvature and the expression for the Ricci tensor $\bar{\rho}$ (see again (2) in [7]), it follows by a straightforward calculation:

$$\bar{\rho}_{|(x,A)}(U^t, V^t) = \left(m + n - 1 + \frac{m+1}{32} \right) g_S(U^t, V^t),$$

$$\bar{\rho}_{|(x,A)}(U^t, X^t) = 0,$$

$$\bar{\rho}_{|(x,A)}(X^t, Y^t) = \left(m + n - 1 + \frac{n+8}{16} \right) g_S(X^t, Y^t).$$

Hence, the eigenvalues of $\bar{\rho}_{|(x,A)}$ are $m+n-1+\frac{m+1}{32}$ with multiplicity n and $m+n-1+\frac{n+8}{16}$ with multiplicity m. We note that these values are always distinct, as $m < 2n+15$ (see again [1, p. 23]).

Next, we take a unit vector $U \in \mathfrak{v}$ at x. At (x,U), the tangential part of $T_{(x,U)}T_1S$ is given by $(\mathfrak{v} \cap U^{\perp})^t \oplus \mathfrak{z}^t \oplus \mathfrak{a}^t$. Using the formula (16) again, we find

$$\bar{\rho}_{|(x,U)}(X^t, Y^t) = \left(m + n - 1 + \frac{m+1}{32} \right) g_S(X^t, Y^t),$$

$$\bar{\rho}_{|(x,U)}(X^t, V^t) = 0,$$

$$\bar{\rho}_{|(x,U)}(X^t, A^t) = 0,$$

$$\bar{\rho}_{|(x,U)}(A^t, A^t) = \left(m + n - 1 + \frac{m+1}{32} \right) g_S(A^t, A^t),$$

$$\bar{\rho}_{|(x,U)}(A^t, V^t) = 0.$$

So, we have already the eigenvalue $m+n-1+\frac{m+1}{32}$ with multiplicity $m+1$. As a consequence, $\bar{\rho}_{|(x,U)}$ restricted to $(\mathfrak{v}\cap U^{\perp})^t \times (\mathfrak{v}\cap U^{\perp})^t$ must have the eigenvalue $m+n-1+\frac{n+8}{16}$ with multiplicity m and the eigenvalue $m+n-1+\frac{m+1}{32}$ with multiplicity $n-m-1$.

Now, $\bar{\rho}_{|(x,U)}(V^t,W^t) = (m+n-1)g_S(V^t,W^t) + \frac{1}{4}F(V,W)$ where $F : U^{\perp}\times U^{\perp} \to \mathbb{R} : (V,W) \mapsto \langle R(U,V)e_i, R(U,W)e_i\rangle$ with $\{e_i\}$ an orthonormal basis of $\mathfrak{s}$. Explicitly, from (16), F is given by

$$\begin{aligned} F(V,W) = \frac{m+1}{8}\,\langle V,W\rangle + \frac{n-2m+9}{4}\,\langle [U,V],[U,W]\rangle \\ + \frac{3}{8}\sum_{j=1}^{m}\langle [V,J_{Z_j}U],[W,J_{Z_j}U]\rangle \end{aligned}$$

where $\{Z_1,\ldots,Z_m\}$ is an orthonormal basis of $\mathfrak{z}$ with respect to the inner product induced by $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$. F is a symmetric, bilinear form on $U^{\perp}$, hence diagonalizable. Moreover, we know that its eigenvalues are $\frac{n+8}{4}$ and $\frac{m+1}{8}$ with multiplicities m and $n-m-1$ respectively.

Denote the $(n-m-1)$-dimensional eigenspace to the eigenvalue $\frac{m+1}{8}$ by $\mathrm{Eig}(\frac{m+1}{8})$. If $W\in \mathrm{Eig}\left(\frac{m+1}{8}\right)$, then $F(W,W) = \frac{m+1}{8}\,\langle W,W\rangle$. Hence,

$$\frac{n-2m+9}{4}\,|[U,W]|^2 + \frac{3}{8}\sum_{j=1}^{m}|[W,J_{Z_j}U]|^2 = 0.$$

Note that $n-2m+9>0$: this follows again from [1, p. 23]. So, $W\in\ker\mathrm{ad}(U)$ and $W\in\ker\mathrm{ad}(J_ZU)$ for every $Z\in\mathfrak{z}$. But $\dim\ker\mathrm{ad}(U) = n-m$ and hence, $\mathrm{Eig}\left(\frac{m+1}{8}\right) = \ker\mathrm{ad}(U)\cap U^{\perp}$.

So, if $W\in\ker\mathrm{ad}(U)\cap U^{\perp}$, then $W\in\ker\mathrm{ad}(J_ZU)$ for every $Z\in\mathfrak{z}$, or equivalently, $\ker\mathrm{ad}(U)\cap U^{\perp}\subset\ker\mathrm{ad}(J_ZU)$ for every $Z\in\mathfrak{z}$. Taking orthogonal complements on both sides and using (15), we get that $J_{\mathfrak{z}}J_ZU\subset \mathbb{R}U\oplus J_{\mathfrak{z}}U$. Hence, for $X\in\mathfrak{z}$, there exists $Z'\in\mathfrak{z}$ and $\alpha\in\mathbb{R}$, such that $J_XJ_ZU = J_{Z'}U+\alpha U$, where $\alpha = \langle J_XJ_ZU,U\rangle = -\langle J_XU,J_ZU\rangle = -\langle X,Z\rangle$. As U is an arbitrary unit vector in $\mathfrak{v}$, it follows that the J^2-condition holds. $\square$

6. Four-dimensional Einstein spaces

From Proposition 7, we know that a four-dimensional Einstein space (M, g) has a unit tangent sphere bundle with constant scalar curvature if and only if the norm of its Riemann curvature tensor $|R|$ is constant. Now, we determine when (T_1M, g_S) is Ricci-curvature homogeneous.

Proposition 14. *Let (M^4, g) be a four-dimensional Einstein space. Its unit tangent sphere bundle (T_1M, g_S) is locally homogeneous or Ricci-curvature homogeneous if and only if (M, g) is locally isometric to one of the two-point homogeneous spaces $\mathbb{R}^4$, S^4, H^4, $\mathbb{C}P^2$ or $\mathbb{C}H^2$.*

Proof. Let $\{E_1, E_2, E_3, E_4\}$ be a Singer-Thorpe orthonormal basis ([2], [14]), that is, the components of the Riemann curvature tensor with respect to this basis are given by

$$\begin{aligned}
&R_{1212} = R_{3434} = \lambda_1, \qquad R_{1234} = \mu_1,\\
&R_{1313} = R_{2424} = \lambda_2, \qquad R_{1342} = \mu_2,\\
&R_{2323} = R_{1414} = \lambda_3, \qquad R_{1423} = \mu_3,\\
&R_{ijkl} \ = 0 \quad \text{otherwise},
\end{aligned}$$

with $\mu_1 + \mu_2 + \mu_3 = 0$ by the first Bianchi identity. Further, $\rho = -(\lambda_1 + \lambda_2 + \lambda_3)\, g$ and $\tau = -4(\lambda_1 + \lambda_2 + \lambda_3)$.

Now take $u = \cos\theta\, E_1 + \sin\theta\, E_2$. A long but routine calculation computes the components of the Ricci tensor $\bar\rho$ of (T_1M, g_S) at the point (x, u) with respect to the orthonormal basis $\{(\sin\theta\, E_1 - \cos\theta\, E_2)^t, {E_3}^t, {E_4}^t, {E_1}^h, {E_2}^h, {E_3}^h, {E_4}^h\}$. The matrix for $\bar\rho_{|(x,u)}$ has the form

$$\begin{pmatrix}
2 + \frac{{\lambda_1}^2 + {\mu_1}^2}{2} & 0 & 0 & 0\\
0 & 2\,\mathrm{Id}_2 + \frac{1}{2}\, A_1 & 0 & 0\\
0 & 0 & \frac{\tau - 2{\lambda_1}^2}{4}\,\mathrm{Id}_2 - \frac{1}{2}\, A_2 & 0\\
0 & 0 & 0 & \frac{\tau - 2{\mu_1}^2}{4}\,\mathrm{Id}_2 - \frac{1}{2}\, A_3
\end{pmatrix}$$

where the (2×2)-matrices A_1, A_2 and A_3 are given explicitly by

$$\begin{pmatrix}
({\lambda_2}^2 + {\mu_2}^2)\cos^2\theta + ({\lambda_3}^2 + {\mu_3}^2)\sin^2\theta & 2(\lambda_2\mu_2 + \lambda_3\mu_3)\cos\theta\sin\theta\\
2(\lambda_2\mu_2 + \lambda_3\mu_3)\cos\theta\,\sin\theta & ({\lambda_3}^2 + {\mu_3}^2)\cos^2\theta + ({\lambda_2}^2 + {\mu_2}^2)\sin^2\theta
\end{pmatrix},$$

$$\begin{pmatrix} (\lambda_2{}^2+\lambda_3{}^2)\cos^2\theta+(\mu_2{}^2+\mu_3{}^2)\sin^2\theta & 2(\lambda_2\lambda_3+\mu_2\mu_3)\cos\theta\sin\theta \\ 2(\lambda_2\lambda_3+\mu_2\mu_3)\cos\theta\sin\theta & (\mu_2{}^2+\mu_3{}^2)\cos^2\theta+(\lambda_2{}^2+\lambda_3{}^2)\sin^2\theta \end{pmatrix},$$

$$\begin{pmatrix} (\lambda_2{}^2+\mu_3{}^2)\cos^2\theta+(\lambda_3{}^2+\mu_2{}^2)\sin^2\theta & 2(\lambda_2\mu_3+\lambda_3\mu_2)\cos\theta\sin\theta \\ 2(\lambda_2\mu_3+\lambda_3\mu_2)\cos\theta\sin\theta & (\lambda_3{}^2+\mu_2{}^2)\cos^2\theta+(\lambda_2{}^2+\mu_3{}^2)\sin^2\theta \end{pmatrix}.$$

Now, we note that the eigenvalues of the matrices A_i depend in general on θ. But if we suppose that (T_1M, g_S) is Ricci-curvature homogeneous, only a discrete set of eigenvalues is possible. Hence, the continuous eigenvalue functions are in fact constant.

For $\theta = 0$, the matrix A_1 has eigenvalues $\lambda_2{}^2+\mu_2{}^2$ and $\lambda_3{}^2+\mu_3{}^2$. So, these must also be the eigenvalues for all other values of θ. If we express that

$$\det(A_1 - (\lambda_2{}^2+\mu_2{}^2)\,\mathrm{Id}_2) = 0,$$
$$\det(A_1 - (\lambda_3{}^2+\mu_3{}^2)\,\mathrm{Id}_2) = 0,$$

we find the condition

$$\Big((\lambda_3-\mu_3)^2-(\lambda_2+\mu_2)^2\Big)\Big((\lambda_3+\mu_3)^2-(\lambda_2-\mu_2)^2\Big) = 0. \tag{17}$$

Similarly, we see that the matrix A_2 has eigenvalues $\lambda_2{}^2+\lambda_3{}^2$ and $\mu_2{}^2+\mu_3{}^2$, and we obtain the condition

$$\Big((\mu_2-\mu_3)^2-(\lambda_2+\lambda_3)^2\Big)\Big((\mu_2+\mu_3)^2-(\lambda_2-\lambda_3)^2\Big) = 0. \tag{18}$$

Finally, the matrix A_3 has eigenvalues $\lambda_2{}^2+\mu_3{}^2$ and $\mu_2{}^2+\lambda_3{}^2$ and we get the condition

$$\Big((\lambda_3-\mu_2)^2-(\lambda_2+\mu_3)^2\Big)\Big((\lambda_3+\mu_2)^2-(\lambda_2-\mu_3)^2\Big) = 0. \tag{19}$$

Clearly, if we start from a different choice of unit vector u of the form $u = \cos\theta\, E_i + \sin\theta\, E_j$, $i, j \in \{1,2,3,4\}$ and $i \neq j$, we obtain the same eigenvalues and symmetric analogues to the conditions (17)–(19). In particular, we see that $\bar\rho$ has the constant eigenvalues

$$\begin{gathered} 2+(\lambda_1{}^2+\mu_1{}^2)/2, 2+(\lambda_2{}^2+\mu_2{}^2)/2, 2+(\lambda_3{}^2+\mu_3{}^2)/2, \\ (\tau-2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2))/4, \\ (\tau-2(\lambda_1{}^2+\mu_2{}^2+\mu_3{}^2))/4, (\tau-2(\mu_1{}^2+\lambda_2{}^2+\mu_3{}^2))/4, \\ (\tau-2(\mu_1{}^2+\mu_2{}^2+\lambda_3{}^2))/4, \end{gathered} \tag{20}$$

and that the following nine conditions must hold

$$
\begin{aligned}
&\Big((\lambda_3-\mu_3)^2-(\lambda_2+\mu_2)^2\Big)\Big((\lambda_3+\mu_3)^2-(\lambda_2-\mu_2)^2\Big)=0,\\
&\Big((\lambda_1-\mu_1)^2-(\lambda_3+\mu_3)^2\Big)\Big((\lambda_1+\mu_1)^2-(\lambda_3-\mu_3)^2\Big)=0,\\
&\Big((\lambda_2-\mu_2)^2-(\lambda_1+\mu_1)^2\Big)\Big((\lambda_2+\mu_2)^2-(\lambda_1-\mu_1)^2\Big)=0,\\
&\Big((\mu_2-\mu_3)^2-(\lambda_2+\lambda_3)^2\Big)\Big((\mu_2+\mu_3)^2-(\lambda_2-\lambda_3)^2\Big)=0,\\
&\Big((\mu_3-\mu_1)^2-(\lambda_3+\lambda_1)^2\Big)\Big((\mu_3+\mu_1)^2-(\lambda_3-\lambda_1)^2\Big)=0,\\
&\Big((\mu_1-\mu_2)^2-(\lambda_1+\lambda_2)^2\Big)\Big((\mu_1+\mu_2)^2-(\lambda_1-\lambda_2)^2\Big)=0,\\
&\Big((\lambda_3-\mu_2)^2-(\lambda_2+\mu_3)^2\Big)\Big((\lambda_3+\mu_2)^2-(\lambda_2-\mu_3)^2\Big)=0,\\
&\Big((\lambda_1-\mu_3)^2-(\lambda_3+\mu_1)^2\Big)\Big((\lambda_1+\mu_3)^2-(\lambda_3-\mu_1)^2\Big)=0,\\
&\Big((\lambda_2-\mu_1)^2-(\lambda_1+\mu_2)^2\Big)\Big((\lambda_2+\mu_1)^2-(\lambda_1-\mu_2)^2\Big)=0.
\end{aligned}
\tag{21}
$$

Taking also the condition $\mu_1+\mu_2+\mu_3=0$ into account, the solutions of the system (21) are given, up to choice of Singer-Thorpe basis, by

(1) $\lambda_1,\ \lambda_2$ and λ_3 are arbitrary,
$\mu_1=\lambda_2-\lambda_3,\quad \mu_2=\lambda_3-\lambda_1,\quad \mu_3=\lambda_1-\lambda_2;$

(2) λ_1 and λ_2 are arbitrary,
$\lambda_3=\lambda_2,$
$\mu_1=0,\quad \mu_2=\lambda_1+\lambda_2,\quad \mu_3=-\lambda_1-\lambda_2;$

(3) λ_1 and λ_2 are arbitrary,
$\lambda_3=-\lambda_1-\lambda_2,$
$\mu_1=-\dfrac{\lambda_1+2\lambda_2}{3},\quad \mu_2=\dfrac{2\lambda_1+\lambda_2}{3},\quad \mu_3=-\dfrac{\lambda_1-\lambda_2}{3};$

(4) λ_2 and λ_3 are arbitrary,
$\lambda_1=\lambda_2-3\lambda_3,$
$\mu_1=\lambda_2-\lambda_3,\quad \mu_2=\lambda_3-\lambda_2,\quad \mu_3=0;$

(5) λ_1 and λ_2 are arbitrary,

$\lambda_3 = 0,$

$$\mu_1 = \frac{2\lambda_1 + \lambda_2}{3}, \quad \mu_2 = -\frac{\lambda_1 + 2\lambda_2}{3} \quad \mu_3 = -\frac{\lambda_1 - \lambda_2}{3};$$

(6) λ_2 and λ_3 are arbitrary,

$\lambda_1 = 0,$

$$\mu_1 = \lambda_2 - \lambda_3, \quad \mu_2 = -\lambda_3, \quad \mu_3 = 2\lambda_3 - \lambda_2.$$

From the constancy of the eigenvalues (20), it follows for all six cases above that the λ_i and the μ_i, $i = 1, 2, 3$, are constant. Hence, (M^4, g) is a curvature homogeneous four-dimensional Einstein manifold. It is an immediate consequence of an unpublished theorem by A. Derdziński (see [12]) that (M^4, g) must be locally symmetric and, hence, locally isometric to one of the two-point homogeneous spaces $\mathbb{R}^4$, S^4, H^4, $\mathbb{C}P^2$ or $\mathbb{C}H^2$, or to one of the product manifolds $S^2(c^2) \times S^2(c^2)$ or $H^2(-c^2) \times H^2(-c^2)$. The proposition now follows from Proposition 11 and Theorem 1. □

7. Sasakian space forms

Finally, we consider another case where the curvature tensor is simple enough to handle the formulas (1) and (2). Let $(M^{2k+1}, g, \xi, \eta, \varphi)$ be a Sasakian space form with constant φ-sectional curvature c, that is, $g(R(X, \varphi X)\varphi X, X) = c$ for every unit vector X perpendicular to the characteristic vector field ξ. (Clearly, the vector field ξ has nothing to do with the curvature operator ξ in (3).) The Riemann curvature tensor R is given in explicit form (see [4, p. 97]) as

$$\begin{aligned}(22)\qquad R(X,Y)Z = {} & \frac{c+3}{4}\big(g(Y,Z)X - g(X,Z)Y\big) \\ & + \frac{c-1}{4}\big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\big).\end{aligned}$$

We first determine for which values for the constant φ-sectional curvature c the unit tangent sphere bundle (T_1M, g_S) has constant scalar curvature $\bar{\tau}$. As (M, g) is locally homogeneous, condition (4) holds. We check condition (3) now.

Using (22), we calculate easily, for X and Y perpendicular to ξ:

$$\begin{aligned} g(R(\xi, E_i)E_j, R(\xi, E_i)E_j) &= 4k\, g(\xi, \xi), \\ g(R(\xi, E_i)E_j, R(X, E_i)E_j) &= 0, \\ g(R(X, E_i)E_j, R(Y, E_i)E_j) &= \big(3k - 1 + c^2(k+1)\big)\, g(X, Y). \end{aligned}$$

Hence, (3) holds if and only if $4k = 3k - 1 + c^2(k+1)$, or $c = \pm 1$. This proves the claim in [7]:

Proposition 15. *The unit tangent sphere bundle of a Sasakian space form with constant φ-sectional curvature c has constant scalar curvature if and only if $c = \pm 1$.*

In the complete and simply connected case, both of these Sasakian space forms are principal circle bundles over $\mathbb{C}P^n$ with constant holomorphic sectional curvature 4 (if $c = 1$) or 2 (if $c = -1$), and they are related by a D-homothetic transformation ([16]). Further, the Sasakian space form with constant φ-sectional curvature $c = 1$ is locally isometric to a sphere $S^{2n+1}(1)$. By Theorem 1, its unit tangent sphere bundle is locally homogeneous. This contrasts with

Proposition 16. *The unit tangent sphere bundle of a Sasakian space form $(M^{2k+1}, g, \xi, \eta, \varphi)$ with constant φ-sectional curvature $c = -1$ is not even Ricci-curvature homogeneous.*

Proof. Fix a point $x \in M$. We calculate $\bar{\rho}$ at (x, ξ) and at (x, X) with X perpendicular to ξ and show that the eigenvalues are different. We note beforehand that the Ricci curvature ρ of (M, g) is given by

$$\rho(U, V) = (k-1)g(U, V) + (k+1)g(U, \xi)g(V, \xi)$$

(see [4]). So,

$$\begin{aligned} (\nabla_W \rho)(U, V) &= (k+1)\big(g(U, \nabla_W \xi)g(V, \xi) + g(U, \xi)g(V, \nabla_W \xi)\big) \\ &= -(k+1)\big(g(U, \varphi W)g(V, \xi) + g(U, \xi)g(V, \varphi W)\big) \end{aligned}$$

as $\nabla_W \xi = -\varphi W$. (See again [4].)

With these formulas and with (22), where we put $c=-1$, the expressions (1) for $\bar\rho$ at (x,ξ) are given by

$$\bar\rho_{|(x,\xi)}(U^t,V^t)=\left(2k-\frac{1}{2}\right)\big(g(U,V)-g(U,\xi)g(V,\xi)\big),$$
$$\bar\rho_{|(x,\xi)}(U^t,V^h)=-\frac{k+1}{2}\,g(U,\varphi V),$$
$$\bar\rho_{|(x,\xi)}(U^h,V^h)=\left(k-\frac{3}{2}\right)g(U,V)+\frac{3}{2}\,g(U,\xi)g(V,\xi).$$

In particular, we see that ξ^h is an eigenvector for $\bar\rho_{|(x,\xi)}$ with eigenvalue k:

$$\bar\rho_{|(x,\xi)}(\xi^h,\xi^h)=k\,g_S(\xi^h,\xi^h).$$

Next, we take $u=X$, a unit vector at x perpendicular to ξ. A straightforward calculation yields

$$\begin{aligned}\bar\rho_{|(x,X)}(U^t,V^t)&=\left(2k-\frac{3}{4}\right)\big(g(U,V)-g(U,X)g(V,X)\big)\\&\quad+\frac{1}{4}\,g(U,\xi)g(V,\xi)+\frac{2k-1}{4}\,g(U,\varphi X)g(V,\varphi X),\\ \bar\rho_{|(x,X)}(U^t,V^h)&=-\frac{k+1}{2}\big(g(U,\xi)g(V,\varphi X)+2g(V,\xi)g(U,\varphi X)\big),\\ \bar\rho_{|(x,X)}(U^h,V^h)&=\left(k-\frac{7}{4}\right)g(U,V)-\frac{k}{4}\,g(U,X)g(V,X)\\&\quad+\left(k+\frac{5}{4}\right)g(U,\xi)g(V,\xi)-\frac{k-2}{4}\,g(U,\varphi X)g(V,\varphi X).\end{aligned}$$

At x, we take an orthonormal basis $\{E_1=X,E_2=\varphi E_1,E_3,E_4=\varphi E_3,\ldots,E_{2k}=\varphi E_{2k-1},E_{2k+1}=\xi\}$. Then, at (x,X), an orthonormal basis for $T_{(x,X)}T_1M$ is given by $\{{E_2}^t,\xi^h,{E_2}^h,\xi^t,{E_1}^h,{E_3}^t,\ldots,{E_{2k}}^t,{E_3}^h,\ldots,{E_{2k}}^h\}$. With respect to this basis, $\bar\rho_{|(x,X)}$ is given in matrix form by

$$\begin{pmatrix}\frac14 A_1 & 0 & 0 & 0 & 0\\ 0 & \frac14 A_2 & 0 & 0 & 0\\ 0 & 0 & \frac{3k-7}{4} & 0 & 0\\ 0 & 0 & 0 & \frac{8k-3}{4}\,\mathrm{Id}_{2(k-1)} & 0\\ 0 & 0 & 0 & 0 & \frac{4k-7}{4}\,\mathrm{Id}_{2(k-1)}\end{pmatrix}$$

where the (2×2)-matrices A_1 and A_2 are given explicitly by

$$A_1 = \begin{pmatrix} 2(5k-2) & -4(k+1) \\ -4(k+1) & 2(4k-1) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2(4k-1) & -2(k+1) \\ -2(k+1) & 3k-5 \end{pmatrix}.$$

The eigenvalues of $\bar{\rho}_{|(x,X)}$ are $(3k-7)/4$ with multiplicity 1, $(8k-3)/4$ with multiplicity $2(k-1)$, $(4k-7)/4$ with multiplicity $2(k-1)$, and the solutions of the quadratic equations $\lambda^2 - 3(3k-1)/2\ \lambda + (16k^2 - 21k - 2)/4 = 0$ and $\lambda^2 - (11k-7)/4\ \lambda + (10k^2 - 27k + 3)/8 = 0$. The eigenvalue k of $\bar{\rho}_{|(x,\xi)}$ is not among these and hence, (T_1M, g_S) is not Ricci-curvature homogeneous. □

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