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On the uniqueness problem for continuous convolution semigroups of probability measures on simply connected nilpotent Lie groups

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Abstract. Let G be a simply connected nilpotent Lie group and assume $\{\mu_t^{(i)}\}_{t\geq 0}$ $(i=1,2)$ are Poisson semigroups of probability measures on G with boundedly supported Lévy measures. We prove that if $\mu_1^{(1)}=\mu_1^{(2)}$, then $\mu_t^{(1)}=\mu_t^{(2)}$ for all $t\geq 0$. As a consequence, e.g. a convergent triangular system of rowwise i.i.d. probability measures on G which are concentrated on a fixed circular annulus automatically converges functionally.

1. Introduction

One of the most important problems in probability theory on groups G (e.g.) is the so-called *embedding problem*, i.e. the question if a given probability measure μ on G may be embedded into a continuous convolution semigroup of probability measures (c.c.s. for short) $\{\mu_t\}_{t\geq 0}$ on G (i.e. $\mu=\mu_1$). In this note we are interested in the class of simply connected nilpotent Lie groups G. It has been proved by Burrell, McCrudden (1974) that any infinitely divisible probability measure on G is embeddable into a c.c.s. on G. But now not only the question of existence of a c.c.s. in which a given probability measure can be embedded is important, but also that of its uniqueness, i.e. the following problem: If $\{\mu_t^{(i)}\}_{t\geq 0}$ $(i=1,2)$ are c.c.s. such that $\mu_1^{(1)}=\mu_1^{(2)}$, do then the c.c.s. have to coincide (i.e. $\mu_t^{(1)}=\mu_t^{(2)}$ for every $t\geq 0$)? If this is true, then

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this has important consequences for limit theorems, namely in this case, a convergent triangular system of rowwise i.i.d. random variables automatically converges functionally, in detail: For an increasing sequence $\{k(n)\}_{n\geq 1}$ of natural numbers, the relation $\nu_n^{*k(n)} \xrightarrow{w} \mu_1$ $(n \to \infty)$ implies $\nu_n^{*\lfloor k(n)t\rfloor} \xrightarrow{w} \mu_t$ $(t \geq 0)$ (cf. NOBEL (1991), Remark 2.(a)). It is well-known that this uniqueness property is true for $(\mathbb{R}^d, +)$. Finite groups satisfy the uniqueness property iff every non-neutral element has order 2 (then the group is of course abelian) (cf. BÖGE (1959)). For locally compact abelian groups, a sufficient condition for the uniqueness property is the request that the group have no non-trivial compact subgroup (cf. HEYER (1977), Theorem 3.5.15). In more general framework, some partial results have been obtained by HAZOD (1971). For stable and semi-stable semigroups on simply connected nilpotent Lie groups see DRISCH, GALLARDO (1984), NOBEL (1991), and HAZOD, SIEBERT (1997), 2.6. PAP (1994) proved the uniqueness property for the Gauss semigroups among all Gauss semigroups on simply connected nilpotent Lie groups, generalizing the corresponding result for simply connected step 2-nilpotent Lie groups by BALDI (1985), but he left open the question if Gaussian measures can also be embedded into non-Gaussian c.c.s. NEUENSCHWANDER (1996), Theorem 2.1 shows that for Gaussian measures on the (3-dimensional) Heisenberg group this is indeed not the case. For irreducible symmetric spaces G/K of noncompact type (i.e. G a semisimple noncompact Lie group with finite center and K a maximal compact subgroup) and K-biinvariant probability measures μ on G, GRACZYK (1994) used a method to associate to μ a bounded nonnegative measure $\tilde{\mu}$ on a Cartan subalgebra $(\boldsymbol{a}, +)$ such that $\mu_1 * \mu_2 = \mu_3$ iff $\tilde{\mu}_1 * \tilde{\mu}_2 = \tilde{\mu}_3$ and such that $\tilde{\mu}$ determines μ uniquely. This readily yields the uniqueness property for all c.c.s. of K-biinvariant probability measures on G by the uniqueness property on $(\boldsymbol{a}, +)$.

In this note, we will look at Poisson semigroups. We show that on a simply connected nilpotent Lie group Poisson semigroups $\{\mu_t^{(i)}\}_{t\geq 0}$ whose Lévy measures have bounded support and who satisfy $\mu_1^{(1)} = \mu_1^{(2)}$ have to coincide as a whole. As a consequence, we get e.g. that the aforementioned passage from convergence to functional convergence of rowwise i.i.d. triangular systems is possible provided all occurring measures are supported by some fixed circular annulus.

Our method, which is related to the idea of PAP (1994), consists of recursively calculating the moments and to make use of certain conditions guaranteeing the unicity in Hamburger's moment problem.

2. Poisson semigroups

Let G be a simply connected nilpotent Lie group, which will be identified with its Lie algebra $\mathcal{G} = \mathbb{R}^d$. Consider the adjoint repereserentation of the Lie algebra given by $\mathrm{ad}(x) : \mathcal{G} \to \mathcal{G}$, $\mathrm{ad}(x)(y) := [x, y]$ $(x, y \in \mathcal{G})$. The product on G is then given by the Campbell-Hausdorff formula, (cf. Serre (1965)), where only the terms up to order, say, $r \in \mathbb{N}_0$ (the step of nilpotency of G) arise:

$$x \cdot y = \sum_{n=1}^{r} z_n, \tag{1}$$

$$z_n = \frac{1}{n} \sum_{p+q=n} \left(z'_{p,q} + z''_{p,q} \right), \tag{2}$$

$$z'_{p,q} = \sum_{\substack{p_1+p_2+\ldots+p_m=p \\ q_1+q_2+\ldots+q_{m-1}=q-1 \\ p_i+q_i\geq 1 \\ p_m \geq 1}} \frac{(-1)^{m+1}}{m} \frac{\mathrm{ad}(x)^{p_1}\,\mathrm{ad}(y)^{q_1}\ldots\mathrm{ad}(x)^{p_m}(y)}{p_1!q_1!\ldots p_m!}, \tag{3}$$

$$z''_{p,q} = \sum_{\substack{p_1+p_2+\ldots+p_{m-1}=p-1 \\ q_1+q_2+\ldots+q_{m-1}=q \\ p_i+q_i\geq 1}} \frac{(-1)^{m+1}}{m} \frac{\mathrm{ad}(x)^{p_1}\,\mathrm{ad}(y)^{q_1}\ldots\mathrm{ad}(y)^{q_{m-1}}(x)}{p_1!q_1!\ldots q_{m-1}!}. \tag{4}$$

The first few terms are

$$x \cdot y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\{[[x, y], y] + [[y, x], x]\} + \ldots . \tag{5}$$

Clearly, $e = 0$ and $x^{-1} = -x$ $(x \in G)$. Consider an adapted vector space decomposition of $G = \mathcal{G}$, i.e.

$$G = \mathcal{G} = \mathbb{R}^d = \bigoplus_{i=1}^{r} V_i \tag{6}$$

such that

$$\bigoplus_{i=k}^{r} V_i = \mathcal{G}_{k-1},$$

where $\{\mathcal{G}_k\}_{0\leq k\leq r}$ is the *descending central series*:

$$\mathcal{G}_0 := \mathcal{G}, \quad \mathcal{G}_{k+1} := [\mathcal{G}, \mathcal{G}_k]$$

(and thus $\mathcal{G}_r = \{0\}$). In this case, one can take a *Jordan-Hölder basis* for $\mathcal{G} = \mathbb{R}^d$, i.e. a basis $E = \bigcup_{i=1}^r E_i$ where $E_i = \{e_{i,1}, e_{i,2}, \ldots, e_{i,d(i)}\}$ is a basis of V_i ($d(i)$ thus being the dimension of V_i).

A continuous convolution semigroup (c.c.s.) $\{\mu_t\}_{t\geq 0}$ of probability measures on G is a monoid homomorphism of $([0,\infty[,+)$ to $(M^1(G), *, \xrightarrow{w}, \delta_0)$, where the latter denotes the topological monoid of probability measures on G with the convolution $*$, the weak topology $\xrightarrow{w}$, and the unity δ_0 (= Dirac probability measure at 0) (cf. Heyer (1977), Theorem 1.2.2). For $f \in C_b^\infty(G)$ (= space of bounded C^∞-functions on G), the generating distribution of the c.c.s. $\{\mu_t\}_{t\geq 0}$ is defined as

$$\mathcal{A}(f) := \lim_{t\to 0+} \frac{1}{t} \int_G [f(x) - f(0)]\mu_t(dx)$$

(cf. Siebert (1981), p. 119). A *Poisson semigroup* on a simply connected nilpotent Lie group is a c.c.s.

$$\{\mu_t\}_{t\geq 0} = \{\exp t(\lambda - ||\lambda||\delta_0)\}_{t\geq 0}, \tag{7}$$

where λ is a bounded non-negative measure on $G\backslash\{0\}$ (the so-called *Lévy measure*), $||\,.\,||$ is the total variation norm of a bounded measure, and exp is the power series

$$\exp \eta := \delta_0 + \sum_{k=1}^{\infty} \frac{1}{k!}\eta^{*k},$$

which, for every bounded measure η on G, is convergent with respect to $||\,.\,||$ and thus also in the weak topology. The generating distribution of the Poisson semigroup (7) has the form

$$\mathcal{A}(f) = \int_{G\backslash\{0\}} [f(x) - f(0)]\lambda(dx)$$

(cf. Siebert (1981), p. 119 bottom). We will use the following estimation on the number of terms in the Campbell-Hausdorff formula:

Lemma 1. *In the Campbell-Hausdorff formula up to order r for m factors, the number of different summands may be estimated from above by m^{r+1}.*

Proof. By counting all k-tuples with m possible elements ($1 \le k \le r$), this follows at once from the estimation

$$\sum_{k=1}^{r} m^k = \frac{m^{r+1}-1}{m-1} - 1 \le m^{r+1} \quad (m \ge 2). \qquad \square$$

(For what follows, this trivial bound will suffice, we are not interested in better ones.)

Since the bound in Lemma 1 is polynomial in m, the following corollary follows immediately:

Corollary 1. *Let G be a simply connected nilpotent Lie group. If $\{\mu_t\}_{t\ge 0}$ is a Poisson semigroup on G whose Lévy measure has bounded support, then for every μ_t all moments exist.*

Now we formulate our main result:

Theorem 1. *Let $G = \mathbb{R}^d$ be a simply connected nilpotent Lie group. Assume $\{\mu_t^{(i)}\}_{t\ge 0}$ are Poisson semigroups whose Lévy measures $\lambda^{(i)}$ have bounded support (i.e. $\operatorname{supp}\lambda^{(i)} \subset \{x \in \mathbb{R}^d : ||x|| \le \rho\}$ for some $\rho > 0$ $(i = 1, 2)$). Then if $\mu_1^{(1)} = \mu_1^{(2)}$, it follows that $\mu_t^{(1)} = \mu_t^{(2)}$ for all $t \ge 0$.*

Consider on $\mathbb{N}_0^d$ the lexicographic ordering from behind defined by

$$\begin{aligned}(a_1, a_2, \dots, a_d) < (b_1, b_2, \dots, b_d) &\iff (a_d, a_{d-1}, \dots, a_{d-j+1}) \\ = (b_d, b_{d-1}, \dots, b_{d-j+1}),\ a_{d-j} < b_{d-j} &\quad \text{for some } j \in \mathbb{N}_0.\end{aligned}$$

Let G be as in Theorem 1. Let $E = \{e_1, e_2, \dots, e_d\}=\{e_{1,1}, e_{1,2}, \dots, e_{1,d(1)}, e \dots, e_{r,1}, e_{r,2}, \dots, e_{r,d(r)}\}$ $(d(i) = \dim V_i$ in (6)) be a Jordan-Hölder basis of $G = \mathbb{R}^d$, and put $G \ni x =: \sum_{j=1}^{d} x_j e_j$. For $\mu \in M^1(G)$, $\ell = (\ell_1, \ell_2, \dots, \ell_d) \in \mathbb{N}_0^d$, define the "mixed moments"

$$M_\ell(\mu) := \int_G \prod_{j=1}^{d} x_j^{\ell_j}\, \mu(dx)$$

(if they exist).

Lemma 2. *Assume μ, ν are probability measures on G satisfying $\mu = \nu^{*2}$ and such that all moments $M_\ell(\nu)$ ($\ell \in \mathbb{N}_0^d$) exist. Then all $M_\ell(\mu)$ exist and the $M_\ell(\nu)$ ($\ell \in \mathbb{N}_0^d$) may be calculated out of the $M_\ell(\mu)$ recursively with respect to ℓ.*

Proof. The existence of $M_\ell(\mu)$ follows at once from the existence of the $M_\ell(\nu)$ and the nilpotency with the aid of the Campbell-Hausdorff formula. Assume X, Y are i.i.d. G- valued random variables with $\mathcal{L}(X) = \nu$. Write

$$\begin{aligned} M_\ell(\mu) &= E\Big(\prod_{j=1}^{d} (X \cdot Y)_j^{\ell_j}\Big) \\ &= E\Big(\prod_{j=1}^{d} (X + Y + \frac{1}{2}[X, Y] + \dots)_j^{\ell_j}\Big) \end{aligned} \tag{8}$$

By the adaptedness, we get, by multiplying out and considering the coordinates (with respect to (6))

$$\Big(X + Y + \frac{1}{2}[X, Y] + \dots\Big)_j^{\ell_j} = X_j^{\ell_j} + Y_j^{\ell_j} + P_j, \tag{9}$$

P_j being a polynomial in $X_1, Y_1, X_2, Y_2, \dots, X_j, Y_j$, where in every monomial the exponents of X_j and Y_j are strictly smaller than ℓ_j. Now, by multiplying out the product $\prod_{j=1}^{d} (\dots)^{\ell_j}$ in (8), we get by (9)

$$\prod_{j=1}^{d} (\dots)^{\ell_j} = \prod_{j=1}^{d} X_j^{\ell_j} + \prod_{j=1}^{d} Y_j^{\ell_j} + P,$$

where P is a polynomial in $X_1, Y_1, X_2, Y_2, \dots, X_d, Y_d$ such that for every monomial $\gamma \prod_{j=1}^{d} (X_j^{r_j} Y_j^{s_j})$ we have $(r_1, r_2, \dots r_d), (s_1, s_2, \dots, s_d) < \ell$. Now the assertion follows from the independence of X and Y and the fact that $E\big(\prod_{j=1}^{d} X_j^{\ell_j}\big) = E\big(\prod_{j=1}^{d} Y_j^{\ell_j}\big)$. □

A bounded measure μ on $\mathbb{R}^d$ is called determinate, if it is uniquely determined by its ℓ-th moments $M_\ell(\mu)$ ($\ell \in \mathbb{N}_0^d$), i.e. if there is no other bounded measure on $\mathbb{R}^d$ with the same ℓ-th moments.

Lemma 3. *If $\mathbb{C}[x_1, x_2, \ldots, x_d]$ is dense in $L^p(\mu)$ for some $p > 2$ (μ some bounded non-negative measure on $\mathbb{R}^d$), then μ is determinate.*

(Cf. Fuglede (1983), Berg (1995), p. 3.)

Now we are ready to prove Theorem 1:

Proof of Theorem 1. Since $\lambda^{(i)}$ have compact support, $\mathbb{C}[x_1, x_2, \ldots, x_d]$ is dense in, say, $L^3(\lambda^{(i)})$ by Weierstrass' Theorem (cf. Berg (1995), p. 3). Hence, with the aid of Corollary 1, it follows that $\mathbb{C}[x_1, x_2, \ldots, x_d]$ is dense in $L^3(\mu^{(i)}_{1/2}) = L^3(\exp(-(1/2)||\lambda^{(i)}||)\exp(\lambda^{(i)})) \subset L^3(\lambda^{(i)})$, thus by Lemma 3 the measures $\mu^{(i)}_{1/2}$ are determinate. So by Lemma 2 we get that $\mu^{(1)}_t = \mu^{(2)}_t$ for all dyadic t. By continuity, Theorem 1 follows. □

The following corollary says that a convergent triangular system of rowwise i.i.d. probability measures which are all supported by some fixed circular annulus automatically converges in a functional sense:

Corollary 2. *Let $\{\nu_n\}_{n\geq 1}$ be a sequence of probability measures on the simply connected nilpotent Lie group $G = \mathbb{R}^d$ such that $\operatorname{supp}\nu_n \subset A_{\varepsilon,\rho} = \{x \in \mathbb{R}^d : \varepsilon \leq ||x|| \leq \rho\}$ $(n \geq 1)$ for some $0 < \varepsilon < \rho < \infty$. Suppose $\{k(n)\}_{n\geq 1}$ is an increasing sequence of natural numbers. Then if $\nu_n^{*k(n)} \xrightarrow{w} \mu$ $(n \to \infty)$ for some probability measure μ on G, we have that $\mu = \mu_1$ for some uniquely determined Poisson semigroup $\{\mu_t\}_{t\geq 0}$ and $\nu_n^{*\lfloor k(n)t\rfloor} \xrightarrow{w} \mu_t$ $(n \to \infty)$.*

Proof. By Nobel (1991), Theorem 1, it follows that for every subsequence $\{n'\} \subset \{n\}_{n\geq 1}$ there is another subsequence $\{n''\} \subset \{n'\}$ and a c.c.s. $\{\mu_t\}_{t\geq 0}$ such that $\mu_1 = \mu$ and $\nu_{n''}^{*\lfloor k(n'')\rfloor} \xrightarrow{w} \mu_t$ $(n'' \to \infty)$ for every $t \geq 0$. By Hazod, Scheffler (1993), Theorem 2.1, (i)$\Longrightarrow$(iii) it follows that all such limiting c.c.s. $\{\mu_t\}_{t\geq 0}$ are Poisson semigroups with Lévy measures supported by $A_{\varepsilon,\rho}$. Now the assertion follows from Theorem 1. □

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References

[1] P. Baldi, Unicité du plongement d'une mesure de probabilité dans un semi-groupe de convolution gaussien. Cas non-abélien, *Math. Z.* **188** (1985), 411–417.

[2] Chr. Berg, Recent results about moment problems, In: Probability measures on groups and related structures (H. Heyer, ed.), *World Scientific*, 1995, 1–13.

[3] W. Böge, Ueber die Charakterisierung unendlich teilbarer Wahrscheinlichkeitsverteilungen, *J. Reine Angew. Math.* **201** (1959), 150–156.

[4] Q. L. Burrell and M. McCrudden, Infinitely divisible distributions on connected nilpotent Lie groups, *J. London Math. Soc.* (2) **7** (1974), 584–588.

[5] T. Drisch and L. Gallardo, Stable laws on the Heisenberg groups, In: Probability measures on groups VII (H. Heyer, ed.), Lecture Notes in Mathematics 1064, *Springer*, *Berlin*, 1984, 56–79.

[6] B. Fuglede, The multidimensional moment problem, *Expo. Math* **1** (1983), 47–65.

[7] P. Graczyk, Cramér theorem on symmetric spaces of noncompact type, *J. Theoret. Probab.* **7** (3) (1994), 609–613.

[8] W. Hazod, Ueber Wurzeln und Logarithmen beschränkter Masse, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **20** (1971), 259–270.

[9] W. Hazod and H. P. Scheffler, The domains of partial attraction of probabilities on groups and on vector spaces, *J. Theoret. Probab.* **6** (1) (1993), 175–186.

[10] W. Hazod and E. Siebert, Stable probability measures on euclidean spaces and on locally compact groups: Structural probabilities and limit theorems, (1997), (*in preparation*).

[11] H. Heyer, Probability measures on locally compact groups, *Berlin*, 1977.

[12] S. Nobel, Limit theorems for probability measures on simply connected nilpotent Lie groups, *J. Theoret. Probab.* **4** (2) (1991), 261–284.

[13] D. Neuenschwander, Probabilities on the Heisenberg group: Limit theorems and Brownian motion, Lecture Notes in Mathematics 1630, *Springer*, *Berlin*.

[14] G. Pap, Uniqueness of embedding into a gaussian semigroup on a nilpotent Lie group, *Arch. Math. (Basel)* **62** (1994), 282–288.

[15] J.-P. Serre, Lie algebras and Lie groups, *Benjamin*, *New York*, 1965.

[16] E. Siebert, Fourier analysis and limit theorems for convolution semigroups on a locally compact group, *Adv. Math.* **39** (1981), 111–154.

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