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On Finsler spaces of Douglas type II. Projectively flat spaces

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Abstract. The notion of Douglas space may be regarded as a generalization of the notion of projectively flat space. On the basis of this viewpoint the characterization of the projective flatness is clearly established.

1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L(x,y))$ with the fundamental function $L(x,y)$. Let $g_{ij}(x,y) = \dot{\partial}_i \dot{\partial}_j F$, $F = L^2/2$, be the fundamental tensor and put

$$2g_{ir}G^r = (\dot{\partial}_i \partial_r F)y^r - \partial_i F. \tag{1.1}$$

The geodesics of F^n are given by the differential equations

$$\ddot{x}^i \dot{x}^j - \ddot{x}^j \dot{x}^i + 2D^{ij}(x, \dot{x}) = 0, \tag{1.2}$$

where $D^{ij}(x,y) = G^i(x,y)y^j - G^j(x,y)y^i$ are positively homogeneous in (y^i) of degree three.

F^n is said to be *of Douglas type* or a *Douglas space* [2], if $D^{ij}(x,y)$ are homogeneous polynomials in (y^i) of degree three. A Berwald space is of Douglas type, because its $G^i(x,y)$ are homogeneous polynomials in (y^i)

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of degree two. Next a projectively flat space is also of Douglas type, because such a space is covered by rectilinear coordinate neighborhoods in which $G^i(x,y)$ are proportional to y^i ([1], [8]).

The purpose of the second part of our papers is to characterize projectively flat Finsler spaces. We have had several papers ([3]–[5]) on this subject, but any of them is rather long and hard to understand. On the contrary, our method of characterization in the present paper is quite clear, on the basis of Theorem 5 of the first part [2]. To facilitate the understanding of our method, we are now concerned with a characterization of locally Minkowski spaces as follows.

Let F^n be a locally Minkowski space. It is covered by adapted coordinate neighborhoods ([2], [6]) in which $L(x,y)$ is independent of (x^i). Thus G^i vanish identically. Conversely, if $G^i = 0$ hold, then we have $L_{;i} = \partial_i L - (\dot{\partial}_r L) G^r{}_i = 0$ leads to $L = L(y)$.

We have the transformation law of connection coefficients for a coordinate change $(x^i) \to (\bar{x}^a)$:

$$G_j{}^i{}_k \bar{X}^a{}_i = \bar{G}_b{}^a{}_c \bar{X}^b{}_j \bar{X}^c{}_k + \partial_k \bar{X}^a{}_j, \qquad \bar{X}^a{}_i = \partial_i \bar{x}^a. \tag{1.3}$$

Let $(\bar{x}^a)$ be an adapted coordinate system. Then $\bar{G}_b{}^a{}_c$ vanish and we get the system of differential equations

$$\partial_i \bar{x}^a = \bar{X}^a{}_i, \qquad \partial_k \bar{X}^a{}_j = G_j{}^i{}_k(x) \bar{X}^a{}_i. \tag{1.4}$$

It should be remarked in (1.4) that $G_j{}^i{}_k$ are functions of (x^i) alone, because F^n is, of course, a Berwald space. The $(\bar{x}^a)$ of a solution $(\bar{x}^a, \bar{X}^a{}_i)$ of (1.4) is obviously an adapted coordinate system.

The integrability condition of (1.4) is

$$\partial_k G_i{}^h{}_j - \partial_j G_i{}^h{}_k + G_i{}^r{}_j G_r{}^h{}_k - G_i{}^r{}_k G_r{}^h{}_j = 0.$$

The left-hand side is nothing but the h-curvature tensor $H_i{}^h{}_{jk}$ of the Berwald connection of F^n. Consequently we have: A Berwald space is locally Minkowski, if and only if the curvature tensor H vanishes identically, as it is well-known ([1], [6]). The essential point of the procedure above is $G_j{}^i{}_k = G_j{}^i{}_k(x)$.

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2. Projective invariants

We are concerned with an n-dimensional Finsler space F^n with the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ ([1], [6]); the connection coefficients are derived from G^i given by (1.1), as $G^i{}_j = \dot\partial_j G^i$ and $G_j{}^i{}_k = \dot\partial_k G^i{}_j$. The h- and v-covariant differentiations with respect to $B\Gamma$ are denoted by $(;\,,\cdot)$, and we define the δ-differentiation as $\delta_i = \partial_i - G^r{}_i\dot\partial_r$. The surviving torsion and curvature tensors are

$$\begin{aligned}
&h\text{-curvature} && H_i{}^h{}_{jk} := \delta_k G_i{}^h{}_j + G_i{}^r{}_j G_r{}^h{}_k - [j,k],\\
&hv\text{-curvature} && G_i{}^h{}_{jk} := \dot\partial_k G_i{}^h{}_j,\\
&(v)h\text{-torsion} && R^h{}_{jk} := \delta_k G^h{}_j - [j,k],
\end{aligned}$$

where $[j,k]$ indicates the term(s), obtained from the preceding term(s) by interchanging the indices j,k.

A projective change $F^n = (M^n, L(x,y)) \to \bar F^n = (M^n, \bar L(x,y))$ of the Finsler metric gives rise to various projective invariants. First we have

$$\begin{aligned}
&Q^0\textit{-invariants} && Q^h := G^h - \frac{1}{n+1} G y^h,\\
&Q^1\textit{-invariants} && Q^h{}_i := G^h{}_i - \frac{1}{n+1}(G_i y^h + G\delta^h{}_i),\\
&Q^2\textit{-invariants} && Q_i{}^h{}_j := G_i{}^h{}_j - \frac{1}{n+1}(G_{ij} y^h + G_i\delta^h{}_j + G_j\delta^h{}_i),
\end{aligned}$$

where $Q^h{}_i = \dot\partial_i Q^h$, $Q_i{}^h{}_j = \dot\partial_j Q^h{}_i$, $G = G^r{}_r$, $G_i = G_r{}^r{}_i$ and $G_{ij} = G_r{}^r{}_{ij}$ is the hv-Ricci tensor of $B\Gamma$. The Q^2-invariants satisfy the following important identities:

$$\text{(2.3)}\qquad \text{(a)}\quad Q_i{}^h{}_j = Q_j{}^h{}_i, \qquad \text{(b)}\quad Q_r{}^r{}_j = 0.$$

Secondly we have a projectively invariant tensor [2], the

$$\text{(2.4)}\qquad \textit{Douglas tensor}\quad D_i{}^h{}_{jk} := \dot\partial_k Q_i{}^h{}_j.$$

This can be written in terms of $B\Gamma$ as

$$D_i{}^h{}_{jk} = G_i{}^h{}_{jk} - \frac{1}{n+1} G_{ij\cdot k} y^h - \frac{1}{n+1}\{G_{ij}\delta^h{}_k + (i,j,k)\},$$

where (i, j, k) indicates the terms, obtained from the preceding term(s) by cyclic permutation of indices i, j, k.

Thirdly we have another invariant tensor ([1], [5]), the

$$\textit{Weyl tensor} \qquad W_i{}^h{}_{jk} := H_i{}^h{}_{jk} + \frac{1}{n+1}\{\delta^h{}_i H_{jk} + y^h H_{jk\cdot i} + \delta^h{}_j H_{k\cdot i} - [j,k]\},$$

where $H_{jk} = H_j{}^r{}_{kr}$ is the h-Ricci tensor of $B\Gamma$ and H_k is the

$$\textit{H-vector} \qquad H_k := \frac{1}{n-1}(nH_{0k} + H_{k0}),$$

with the subscript 0 denoting the transvection by y^i. Let us remark that the Weyl tensor vanishes identically in any two-dimensional Finsler space.

Finally the H-vector gives rise to the

$$\textit{K-tensor} \qquad K_{ij} := (n-1)\{H_{i;j} - [i,j]\}.$$

We shall turn our attention to the *projective connection* $P\Gamma = (P_j{}^i{}_k, G^i{}_j, 0)$ ([2], [7]), suggested by the Q^2-invariants:

$$P_j{}^i{}_k := G_j{}^i{}_k - \frac{1}{n+1}G_{jk}y^i. \tag{2.5}$$

The covariant differentiations with respect to $P\Gamma$ are denoted by $({}_{|}, \cdot)$. The surviving torsion and curvature tensors are

$$\begin{aligned}
&h\text{-curvature} && N_i{}^h{}_{jk} := \delta_k P_i{}^h{}_j + P_i{}^r{}_j P_r{}^h{}_k - [j,k],\\
&hv\text{-curvature} && U_i{}^h{}_{jk} := \dot\partial_k P_i{}^h{}_j,\\
&(v)h\text{-torsion} && N^h{}_{jk} := N_0{}^h{}_{jk},\\
&(v)hv\text{-torsion} && U^h{}_{jk} := U_0{}^h{}_{jk}.
\end{aligned} \tag{2.6}$$

We have the following relations among those tensors of $B\Gamma$ and $P\Gamma$:

$$\begin{aligned}
&N_i{}^h{}_{jk} = H_i{}^h{}_{jk} - \frac{1}{n+1}y^h\{G_{ij;k} - [j,k]\},\\
&U_i{}^h{}_{jk} = G_i{}^h{}_{jk} - \frac{1}{n+1}(G_{ij\cdot k}\, y^h + G_{ij}\delta^h{}_k\},\\
&N^h{}_{jk} = R^h{}_{jk}, \qquad U^h{}_{jk} = \frac{1}{n+1}y^h G_{jk}.
\end{aligned} \tag{2.7}$$

The h-Ricci tensor $N_{ij} = N_i{}^r{}_{jr}$ and the hv-Ricci tensor $U_{ij} = U_i{}^r{}_{jr}$ of $P\Gamma$ are written as

$$(2.8) \qquad \text{(a)} \quad N_{ij} = H_{ij} - \frac{1}{n+1} G_{ij;0}, \qquad \text{(b)} \quad U_{ij} = \frac{2}{n+1} G_{ij}.$$

The Douglas tensor is written in terms of $P\Gamma$ as

$$D_i{}^h{}_{jk} = U_i{}^h{}_{jk} - \frac{1}{2}(\delta^h{}_i U_{jk} + \delta^h{}_j U_{ik}).$$

The Weyl tensor is also written in the form

$$(2.9) \qquad W_i{}^h{}_{jk} = N_i{}^h{}_{jk} + (\delta^h{}_i M_{jk} + \delta^h{}_j M_{ik} - [j,k]\},$$

where we defined the

$$(2.10) \qquad M^1\text{-}tensor \qquad M_{jk} := \frac{1}{n^2-1}(nN_{jk} + N_{kj}).$$

3. Q^3-invariants

Starting from the Q^2-invariants we shall introduce the following quantities in a way similar to constructing the h-curvature tensor:

$$(3.1) \qquad Q^3\text{-}invariants \qquad Q_i{}^h{}_{jk} := \delta_k Q_i{}^h{}_j + Q_i{}^r{}_j Q_r{}^h{}_k - [j,k].$$

These do not constitute a tensor field, but it is observed from (2.2) and (2.4) that

$$\begin{aligned}
\delta_k Q_i{}^h{}_j - [j,k] &= \partial_k Q_i{}^h{}_j - (\dot\partial_r Q_i{}^h{}_j) G^r{}_k - [j,k] \\
&= \partial_k Q_i{}^h{}_j - D_i{}^h{}_{jr} \left\{ Q^r{}_k + \frac{1}{n+1}(G_k y^r + G\delta^r{}_k) \right\} - [j,k] \\
&= \partial_k Q_i{}^h{}_j - (\dot\partial_r Q_i{}^h{}_j) Q^r{}_k - \frac{1}{n+1} G D_i{}^h{}_{jk} - [j,k].
\end{aligned}$$

Hence we have

$$(3.1') \qquad Q_i{}^h{}_{jk} = \partial_k Q_i{}^h{}_j - (\dot\partial_r Q_i{}^h{}_j) Q^r{}_k + Q_i{}^r{}_j Q_r{}^h{}_k - [j,k].$$

Thus the Q^3-invariants are in fact projective invariants.

Next (2.2) and (2.5) give

$$Q_i{}^h{}_j = P_i{}^h{}_j - \frac{1}{n+1}(G_i\delta^h{}_j + G_j\delta^h{}_i). \tag{3.2}$$

Hence the Q^3-invariants can be written in terms of $P\Gamma$ as

$$Q_i{}^h{}_{jk} = N_i{}^h{}_{jk} - \frac{1}{n+1}\{\delta^h{}_i J_{jk} + \delta^h{}_j J_{ik} - [j,k]\}, \tag{3.3}$$

where we put

$$J_{ij} := \delta_j G_i - G_r P_i{}^r{}_j + \frac{1}{n+1}G_i G_j. \tag{3.3a}$$

The identity (2.3) yields

$$\text{(a)} \quad Q_i{}^h{}_{jk} + (i,j,k) = 0, \qquad \text{(b)} \quad Q_r{}^r{}_{jk} = 0. \tag{3.4}$$

Hence we get the symmetric invariants

$$Q_{ij} := Q_i{}^r{}_{jr}. \tag{3.5}$$

From (3.3) it follows that

$$Q_{ij} = N_{ij} + \frac{1}{n+1}(nJ_{ij} - J_{ji}), \tag{3.6}$$

which implies $J_{ij} = \{(n+1)Q_{ij} - nN_{ij} - N_{ji}\}/(n-1)$. As a consequence, (3.3) shows that

$$\Pi_i{}^h{}_{jk} := Q_i{}^h{}_{jk} + \frac{1}{n-1}\{\delta^h{}_j Q_{ik} - [j,k]\} \tag{3.7}$$

is a tensor field with the components

$$\begin{aligned}\Pi_i{}^h{}_{jk} = N_i{}^h{}_{jk} &+ \frac{1}{n+1}\delta^h{}_i\{N_{jk} - [j,k]\} \\ &+ \frac{1}{n^2-1}\{\delta^h{}_j(nN_{ik} + N_{ki}) - [j,k]\}.\end{aligned} \tag{3.7'}$$

Since (2.10) gives $N_{ij} = nM_{ij} - M_{ji}$, it is easy to show that the right-hand side of (3.7') can be rewritten in the form of the right-hand side of (2.9). Therefore we have

Proposition 1. *The Weyl tensor W coincides with the Π-tensor the components of which are written in terms of Q-invariants as* (3.7).

Next (2.10) and (3.6) yield

$$M_{ij} = \frac{1}{n-1} Q_{ij} - \frac{1}{n+1} J_{ij}. \tag{3.8}$$

Let us define in terms of $P\Gamma$ the

$$M^2\text{-}tensor \qquad M_{ijk} := \{M_{ij|k} - [j,k]\} - \frac{1}{2} U_{ir} N^r{}_{jk}.$$

On account of (2.7) and (2.8) we have

$$M_{ijk} = \left\{ \frac{1}{n-1} Q_{ij|k} - \frac{1}{n+1} J_{ij|k} - [j,k] \right\} - \frac{1}{n+1} G_{ir} R^r{}_{jk}.$$

We examine this expression of the M^2-tensor. On account of (3.2), we have

$$Q_{ij|k} - [j,k] = \Pi_{ijk} - \frac{1}{n+1} \{Q_{ij} G_k - [j,k]\},$$

where we define

$$\Pi_{ijk} := \delta_k Q_{ij} + Q_i{}^r{}_j Q_{rk} - [j,k]. \tag{3.9}$$

Next, we recall the commutation formula ([6], (3.11))

$$\delta_k \delta_j - [j,k] = -R^r{}_{jk} \partial_r.$$

Then (3.3a) together with (2.6) leads to

$$\begin{aligned} J_{ij|k} - [j,k] = & -R^r{}_{jk} G_{ri} - G_r N_i{}^r{}_{jk} \\ & + \frac{1}{n+1} \{G_j J_{ik} + G_i J_{jk} - [j,k]\}. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} M_{ijk} = & \frac{1}{n-1} \Pi_{ijk} + \frac{1}{n+1} G_r N_i{}^r{}_{jk} \\ & + \frac{1}{n+1} \left\{ G_j \left(\frac{1}{n-1} Q_{ik} - \frac{1}{n+1} J_{ik} \right) - \frac{1}{n+1} G_i J_{ik} - [j,k] \right\}. \end{aligned}$$

Finally (2.9) and (3.8) lead to the conclusive expression of the M^2-tensor as follows:

$$M_{ijk} = \frac{1}{n-1}\Pi_{ijk} + \frac{1}{n+1}G_r W_i{}^r{}_{jk}. \tag{3.10}$$

It should be remarked that M_{ijk} and $W_i{}^r{}_{jk}$ are components of the tensors, but Π_{ijk} are not components of any tensor, nor are they projective invariants, because δ_k appears in (3.9).

In the paper [7] we showed the following relations between the M^2-tensor and the K-tensor:

$$M_{ijk} = \frac{1}{n^2-1}K_{jk\cdot i} - U_{ir}W_0{}^r{}_{jk}, \qquad K_{ij} = (n^2-1)M_{0ij}.$$

Since the Weyl tensor vanishes in the two-dimensional case, we have

Proposition 2. *In the two-dimensional case*

$$\text{(a)} \quad 3M_{ijk} = K_{jk\cdot i}, \qquad \text{(b)} \quad K_{ij} = 3M_{0ij}.$$

4. Douglas spaces

A Finsler space F^n is said to be of Douglas type or a Douglas space [2], if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. F^n is a Douglas space, if and only if

(1) the Douglas tensor D vanishes identically or, as shown by (2.4),

(2) the Q^2-invariants $Q_i{}^h{}_j$ are functions of position (x^i) alone.

Then (2.2) gives $2Q^h = Q_i{}^h{}_j(x)y^i y^j$ and

$$2D^{ij} = \{Q_r{}^i{}_s(x)y^r y^s\}y^j - \{Q_r{}^j{}_s(x)y^r y^s\}y^i.$$

Hence the equations of geodesics (1.2) are written in the form

$$\ddot{x}^i\dot{x}^j - \ddot{x}^j\dot{x}^i + \{Q_r{}^i{}_s(x)\dot{x}^r\dot{x}^s\dot{x}^j - [i,j]\} = 0. \tag{4.1}$$

We shall, in particular, deal with the two-dimensional case. Denote (x^1, x^2) by (x, y) and put $y' = dy/dx$. Then (4.1) is written in the form

$$y'' - Q_2{}^1{}_2(y')^3 + (Q_2{}^2{}_2 - 2Q_1{}^1{}_2)(y')^2 + (2Q_1{}^2{}_2 - Q_1{}^1{}_1)y' + Q_1{}^2{}_1 = 0. \tag{4.2}$$

Paying attention to (2.3), the above can be rewritten as

$$y'' := f(x,y,y') = Y_3(y')^3 + Y_2(y')^2 + Y_1 y' + Y_0 \tag{4.2'}$$

where Y's are given by

$$\begin{aligned} Y_3 &= Q_2{}^1{}_2(x,y), \qquad Y_2 = 3Q_1{}^1{}_2(x,y) = -3Q_2{}^2{}_2(x,y),\\ Y_1 &= 3Q_1{}^1{}_1(x,y) = -3Q_1{}^2{}_2(x,y), \qquad Y_0 = -Q_1{}^2{}_1(x,y). \end{aligned} \tag{4.2a}$$

Since the Q^2-invariants $Q_i{}^h{}_j(x)$ will play various essential roles in the theory of Douglas spaces, we state the following

Definition. The set $\{Q_i{}^h{}_j(x)\}$ is called the *characteristic of a Douglas space*.

For a Douglas space the Q^3-invariants, given by (3.1), are functions of position alone and written in the form

$$Q_i{}^h{}_{jk}(x) = \partial_k Q_i{}^h{}_j + Q_i{}^r{}_j Q_r{}^h{}_k - [j,k], \tag{4.3}$$

and Proposition 1 yields the

Proposition 3. *For a Douglas space the components of the Weyl tensor are functions of position alone.*

Further, (3.9) is reduced for a Douglas space to

$$\Pi_{ijk}(x) = \partial_k Q_{ij} + Q_i{}^r{}_j Q_{rk} - [j,k]. \tag{4.4}$$

Thus, for a Douglas space the Π_{ijk} are projective invariants.

Let us find the transformation law of the Q^2-invariants for a coordinate change $(x^i) \to (\bar{x}^a)$. First (1.3) is rewritten in the

$$G_j{}^i{}_k = \bar{G}_b{}^a{}_c \underline{X}^i{}_a \bar{X}^b{}_j \bar{X}^c{}_k + \bar{X}^a{}_{jk} \underline{X}^i{}_a,$$

where $\underline{X}^i{}_a = \partial x^i/\partial \bar{x}^a$ and $\bar{X}^a{}_{jk} = \partial_k \bar{X}^a{}_j$. Then we get

$$G_j = \bar{G}_b \bar{X}^b{}_j + (n+1)Y_j, \qquad Y_j := \frac{1}{n+1}\bar{X}^a{}_{jr}\underline{X}^r{}_a.$$

Since the $P_j{}^i{}_k$ of $P\Gamma$ satisfy the same transformation law as the $G_j{}^i{}_k$, (3.2) yields

$$Q_j{}^i{}_k \bar{X}^a{}_i = \bar{Q}_b{}^a{}_c \bar{X}^b{}_j \bar{X}^c{}_k + \bar{X}^a{}_{jk} - Y_j \bar{X}^a{}_k - Y_k \bar{X}^a{}_j. \tag{4.5}$$

Proposition 4. *The Q^2-invariants ${Q_j}^i{}_k$ obey the transformation law (4.5) for a change of coordinates $(x^i) \to (\bar{x}^a)$, where $Y_i = \bar{X}^a{}_{ir}\underline{X}^r{}_a/(n+1)$.*

We quote from the paper [7] the equation

$$W_m{}^i{}_{hj|k} + D_m{}^i{}_{hr}W_0{}^r{}_{jk} - \delta^i{}_h M_{mjk} + (h,j,k) = 0.$$

Contract this in $i = k$. Since W satisfies $W_m{}^r{}_{rj} = W_m{}^r{}_{jr} = 0$ [7], the above leads to

Proposition 5. *For a Douglas space F^n, $n > 2$, the M^2-tensor can be written as $M_{mhj} = W_m{}^r{}_{hj|r}/(n-2)$.*

5. Projectively flat Finsler spaces

Let us consider a projectively flat Finsler space F^n ([5]; [1], §3.3). As has been mentioned in the Introduction, F^n has a covering by rectilinear coordinate neighborhoods in which there exists a function $P(x,y)$ satisfying $G^i = Py^i$, that is, $D^{ij} = G^i y^j - G^j y^i = 0$. Consequently F^n is a kind of Douglas space. We have

$$G^i{}_j = P_{\cdot j}y^i + P\delta^i{}_j, \qquad G^r{}_r (= G) = (n+1)P,$$

which implies $Q^h{=}0$ by (2.2). Conversely $Q^h{=}0$ leads to $G^h{=}Gy^h/(n{+}1)$, so that the space is projectively flat. Therefore we have

Proposition 6. (1) *A projectively flat Finsler space is a Douglas space.* (2) *A coordinate system (x^i) of a projectively flat space is rectilinear, if and only if the characteristic vanishes identically in (x^i).*

We are concerned with the transformation law (4.5) in a projectively flat space, where (x^i) is an arbitrary coordinate system, while $(\bar{x}^a)$ is assumed to be rectilinear. Then Proposition 6 leads to the differential equations

$$\partial_i \bar{x}^a = \bar{X}^a{}_i, \tag{5.1}$$

$$\partial_j \bar{X}^a{}_i = Q_i{}^r{}_j \bar{X}^a{}_r + Y_i \bar{X}^a{}_j + Y_j \bar{X}^a{}_i. \tag{5.2}$$

It should be remarked that $Q_i{}^r{}_j$ are functions of (x^i) alone in (5.2).

Since $Q_i{}^r{}_j$ are symmetric, we have $\partial_j(\partial_i \bar{x}^a) - [i,j] = 0$. Next we consider $\partial_k(\partial_j \bar{X}^a{}_i) - [j,k] = 0$. We put $Y_{jk} = \partial_k Y_j$ and $Z_{jk} = Y_{jk} - Y_j Y_k - Y_r Q_j{}^r{}_k$. Paying attention to (4.3) we get

$$\underline{X}^h{}_a\{\partial_k(\partial_j \bar{X}^a{}_i) - [j,k]\} = Q_i{}^h{}_{jk} + \{\delta^h{}_i Z_{jk} + \delta^h{}_j Z_{ik} - [j,k]\} = 0.$$

Let us contract this in $h = i$. Then (3.4a) leads to $Z_{jk} = Z_{kj}$. Next, contracting in $h = k$, we get $Z_{ij} = Q_{ij}/(n-1)$, that is,

$$\partial_k Y_j = Y_j Y_k + Y_r Q_j{}^r{}_k + \frac{1}{n-1} Q_{jk}, \tag{5.3}$$

and finally the above is written as $\Pi_i{}^h{}_{jk} = 0$ by (3.7).

Therefore we obtain the complete system of differential equations (5.1), (5.2) and (5.3) for the functions $(\bar{x}^a, \bar{X}^a{}_i, Y_i)$ and $\Pi_i{}^h{}_{jk} = 0$ as the integrability condition of (5.2).

Now we must deal with the integrability condition of (5.3). Paying attention to (3.7), (3.9) and (4.3), it follows from (5.3) that

$$\partial_k(\partial_j Y_i) - [j,k] = \frac{1}{n-1}\Pi_{ijk} + Y_r \Pi_i{}^r{}_{jk}.$$

Consequently we get another condition $\Pi_{ijk} = 0$.

Now we can consider the system (5.1), (5.2) and (5.3) of differential equations in a Douglas space, because of $Q_j{}^i{}_k = Q_j{}^i{}_k(x)$. Then it can be stated that the system is completely integrable, if and only if both $\Pi_i{}^h{}_{jk}$ and Π_{ijk} vanish identically. Thus we obtain a set of solutions $(\bar{x}^a, \bar{X}^a{}_i, Y_i)$ for given initial values at a point x_0, where $\det(\bar{X}^a{}_i) \neq 0$ at x_0 is to be assumed.

The following observation is necessary: For a set of solutions $(\bar{x}^a, \bar{X}^a{}_i, Y_i)$ (5.2) gives

$$\bar{X}^a{}_{ij}\underline{X}^h{}_a = Q_i{}^h{}_j + Y_i \delta^h{}_j + Y_j \delta^h{}_i.$$

Contracting in $h = i$, we have $Y_j = \bar{X}^a{}_{jr}\underline{X}^r{}_a/(n+1)$ on account of (2.5b). Hence we have $\bar{Q}_b{}^a{}_c = 0$ from (4.5), so that $(\bar{x}^a)$ is certainly a rectilinear coordinate system.

In the case $n > 2$, $\Pi_i{}^h{}_{jk} = 0$ leads to $\Pi_{ijk} = 0$ from Propositions 1, 5 and (3.10). In the case $n = 2$, $\Pi_i{}^h{}_{jk} = 0$ is only an identity from Proposition 1.

Summarizing all the above, we have

Theorem. *A Finsler space F^n is projectively flat if and only if F^n is a Douglas space and its characteristic satisfies*

(1) $n > 2 : \Pi_i{}^h{}_{jk} = 0$,
(2) $n = 2 : \Pi_{ijk} = 0$,

where these Π*'s are given by* (3.7), (4.3), (3.5) *and* (4.4). *Then a rectilinear coordinate system* $(\bar{x}^a)$ *is obtained by solving the system of differential equations* (5.1), (5.2) *and* (5.3).

As has been shown in Proposition 1, $\Pi_i{}^h{}_{jk}$ is the Weyl tensor W. In the two-dimensional case, $\Pi_{ijk} = 0$ is equivalent to $M_{ijk} = 0$ from (3.10) and the latter is also equivalent to $K_{ij} = 0$ from Proposition 2. Therefore the Theorem may also be given in another formulation as follows:

Corollary. *A Finsler space F^n is projectively flat, if and only if its Douglas tensor D vanishes identically and*

(1) $n > 2$*: the Weyl tensor* $W = 0$,
(2) $n = 2$*: the* K*-tensor* $K = 0$.

The statement of the Corollary is well known and is the same as Theorem 1 of the paper [5]. However, to find the components of W or K requires tiresome and complicated calculations in most cases.

On the contrary, we may safely say that the Π-quantities in the Theorem are comparatively easy to find. Moreover, the method to find a rectilinear coordinate system is given concretely.

In particular, we shall be concerned with the two-dimensional case. In each coordinate neighborhood the geodesics are given by a second order differential equation of the normal form $y'' = f(x, y, y')$. Conversely it is known [9] that with such a differential equation $y'' = f(x, y, y')$ a two-dimensional Finsler space, defined on a domain of the (x, y)-plane, is connected so that $y'' = f(x, y, y')$ is the equation of the geodesics.

Example 1. Assume that a two-dimensional Finsler space F^2 on a domain of the (x, y)-plane has the geodesics given by the equation $y'' = f(x, y)$ where y' is not contained. (4.2') with (4.2a) shows that $Q_1{}^2{}_1 = -f(x, y)$ and the other $Q_j{}^i{}_k = 0$. Then (4.3) gives $Q_1{}^2{}_{12} = -f_y$ and the other $Q_i{}^h{}_{jk} = 0$. Thus (3.5) gives $(Q_{11}, Q_{12}, Q_{22}) = (-f_y, 0, 0)$ and (4.4) leads to $(\Pi_{112}, \Pi_{212}) = (-f_{yy}, 0)$. Consequently the Theorem leads to the conclusion that F^2 is projectively flat, if and only if $f(x, y)$ *is linear in* $y : A(x)y + B(x)$.

Example 2. We consider a differential equation

$$y'' + P(x)y' + Q(x)y = R(x), \tag{5.4}$$

called linear in y. As has been shown in the paper [9], this is the equation of geodesics of the two-dimensional Finsler space, defined on the (x,y)-plane, with the fundamental function of the Kropina type

$$L(x,y;p,q) = \frac{1}{p}\exp\left(\int P dx\right)\{(2R - Qy)yp^2 + q^2\}.$$

Then we obtain successively:

$$(Q_1{}^1{}_1, Q_1{}^1{}_2, Q_2{}^1{}_2, Q_1{}^2{}_1, Q_1{}^2{}_2, Q_2{}^2{}_2) = \left(-\frac{P}{3}, 0, 0, Qy - R, \frac{P}{3}, 0\right),$$

$$(Q_1{}^1{}_{12}, Q_2{}^1{}_{12}, Q_1{}^2{}_{12}, Q_2{}^2{}_{12}) = \left(0, 0, -\frac{2P^2}{9} - \frac{P'}{3}, 0\right),$$

$$(Q_{11}, Q_{12}, Q_{22}) = \left(Q - \frac{2P^2}{9} - \frac{P'}{3}, 0, 0\right).$$

Consequently we obtain $\Pi_{112} = \Pi_{212} = 0$, and hence the space is projectively flat.

We are concerned with the differential equation (5.1), (5.2) and (5.3). The last of these is written as

$$\partial_x Y_1 = (Y_1)^2 - \frac{1}{3}PY_1 + (Qy - R)Y_2 + Q - \frac{2P^2}{9} - \frac{P'}{3}, \tag{1}$$

$$\partial_y Y_1 = Y_2\left(Y_1 + \frac{P}{3}\right) = \partial_x Y_2, \tag{2}$$

$$\partial_y Y_2 = (Y_2)^2. \tag{3}$$

From (3) we get $Y_2 = 1/\{g(x) - y\}$ with some function $g(x)$. (2) shows that there exists a function $Y(x,y)$ such that $\partial_x Y = Y_1$ and $\partial_y Y = Y_2$. Hence we have

$$Y = -\log|g(x) - y| + h(x), \qquad Y_1 = -\frac{g'}{g-y} + h'(x),$$

with some function $h(x)$. Then (2) leads to $h' + P/3 = 0$, and $Y_1 = -g'/(g-y) - P/3$. Substituting these Y_1 and Y_2 in (1), we get

$$g'' + P(x)g' + Q(x)g = R(x).$$

Consequently $g(x)$ must be a solution of the geodesics equation (5.4). We take a solution $w(x)$ of (5.4), and then

$$Y_1 = -\frac{w'}{w-y} - \frac{P}{3}, \qquad Y_2 = \frac{1}{w-y}.$$

Next we consider (5.2), which is now written as

$$\partial_x \bar{X}^a{}_1 = -\left\{P + \frac{2w'}{w-y}\right\}\bar{X}^a{}_1 + (Qy - R)\bar{X}^a{}_2, \tag{4}$$

$$\partial_y \bar{X}^a{}_1 = \frac{\bar{X}^2{}_1 - w'\bar{X}^a{}_2}{w-y} = \partial_x \bar{X}^a{}_2, \tag{5}$$

$$\partial_y \bar{X}^a{}_2 = \frac{2\bar{X}^a{}_2}{w-y}. \tag{6}$$

First (6) gives $\bar{X}^a{}_2 = f(x)/(w-y)^2$ with some $f(x)$, and (5) gives $\bar{X}^a{}_1 = f'/(w-y) - fw'/(w-y)^2$ immediately. Then (4) leads to

$$f'' + P(x)f' + Q(x)f = 0.$$

Hence $f(x)$ must be a solution of the homogenized equation of (5.4). Thus we shall choose a solution $u(x)$ for $a = 1$ and $v(x)$ for $= 2$, where u and v should be chosen as independent of each other, i.e., $J = u'v - uv' \neq 0$. Therefore we obtain

$$\bar{X}^1{}_1 = \frac{u'}{w-y} - \frac{uw'}{(w-y)^2}, \qquad \bar{X}^1{}_2 = \frac{u}{(w-y)^2},$$

$$\bar{X}^2{}_1 = \frac{v'}{w-y} - \frac{vw'}{(w-y)^2}, \qquad \bar{X}^2{}_2 = \frac{v}{(w-y)^2}.$$

Then we have the $\det(\bar{X}^a{}_i) = J/(w-y)^3$.

Consequently (5.1) gives easily

$$\bar{x}^1 = \frac{u}{w-y} + c^1, \qquad \bar{x}^2 = \frac{v}{w-y} + c^2,$$

where c's are arbitrary constants. Therefore we could find a rectilinear coordinates system $(\bar{x}^a)$.

Finally we shall show concretely that, in the two-dimensional case, (5.1), (5.2) and (5.3) certainly give a rectilinear coordinate system. We denote $(\bar{x}^a)$ by $(\bar{x}, \bar{y})$. In our notation we have

$$d\bar{x} = \bar{X}^1{}_i dx^i, \qquad d\bar{y} = \bar{X}^2{}_i dx^i,$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{X}^2{}_1 + \bar{X}^2{}_2 y'}{\bar{X}^1{}_1 + \bar{X}^1{}_2 y'} := D(x, y, y'),$$

$$\frac{d}{d\bar{x}}\left(\frac{d\bar{y}}{d\bar{x}}\right) = \frac{D_x dx + D_y dy + D_{y'} dy'}{\bar{X}^1{}_1 dx + \bar{X}^1{}_2 dy} = \frac{D_x + D_y y' + D_{y'} y''}{\bar{X}^1{}_1 + \bar{X}^1{}_2 y'} := \bar{y}''.$$

Putting $X_{ijk} := \bar{X}^2{}_{ij}\bar{X}^1{}_k - \bar{X}^1{}_{ij}\bar{X}^2{}_k$ and $X := \bar{X}^1{}_1 + \bar{X}^1{}_2 y'$, we get

$$D_x = \frac{1}{X^2}\{X_{111} + (X_{112} + X_{121})y' + X_{122}(y')^2\},$$

$$D_y = \frac{1}{X^2}\{X_{121} + (X_{122} + X_{221})y' + X_{222}(y')^2\},$$

$$D_{y'} = \frac{J}{X^2}, \qquad J := X^1{}_1 X^2{}_2 - X^1{}_2 X^2{}_1.$$

It follows from (5.2) that

$$X_{111} = JQ_1{}^2{}_1, \quad X_{121} = J(Q_1{}^2{}_2 + Y_1), \quad X_{221} = J(Q_2{}^2{}_2 + 2Y_2),$$

$$X_{112} = -J(Q_1{}^1{}_1 + 2Y_1), \quad X_{122} = -J(Q_1{}^1{}_2 + Y_2), \quad X_{222} = -JQ_2{}^1{}_2.$$

Thus the numerator of $\bar{y}''$ is equal to

$$J\{y'' - Q_2{}^1{}_2(y')^3 + (Q_2{}^2{}_2 - 2Q_1{}^1{}_2)(y')^2 + (2Q_1{}^2{}_2 - Q_1{}^1{}_2)y + Q_1{}^2{}_1\}.$$

As has been shown by (4.2), the terms inside parentheses vanish along geodesics. Therefore $\bar{y}'' = 0$ is nothing but the equation of geodesics in terms of the rectilinear coordinate system $(\bar{x}, \bar{y})$.

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