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General determinantal representation of generalized inverses of matrices over integral domains

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Abstract. In this paper we derive a determinantal formula of $\{1,2\}$ generalized inverses, for matrices over an integral domain and over a commutative ring. The corresponding results are derived for the set of matrices which have rank factorizations as well as for the matrices which do not have rank factorizations. The determinantal formula of $\{1,2\}$ inverses for matrices which do not have rank factorizations, is derived using the characterizations of the class of reflexive g-inverses from [10] and [19]. For the set of matrices which have rank factorizations, the determinantal formula of $\{1,2\}$ inverses is derived using a general representation of $\{1,2\}$ inverses and the general determinantal representation from [20]. Also, we examine the existence of this determinantal formula. Representations and conditions for the existence of $\{1,2,3\}$ and $\{1,2,4\}$ inverses are introduced for the set of matrices which allow a rank factorization. Determinantal representations of the Moore–Penrose inverse, the weighted Moore–Penrose inverse and the group inverse are derived for arbitrary matrices. Moreover, we investigate representations of the minors from $A^{(1,2)}$, $A^{\dagger}$, $A^{\dagger}_{M,N}$ and $A^{(1,2)}$ by means of the expressions involving minors of A and the corresponding minors of randomly chosen matrices which satisfy specified conditions. If A allows a full-rank factorization, we obtain additional results for $\{1,2,3\}$ and $\{1,2,4\}$ inverses of A. Also, a determinantal representation of the corresponding solutions of a given linear system is investigated.

1. Introduction and preliminaries

Let us consider the set of matrices over an integral domain $\mathbb{I}$ with an involution $\lambda : a \mapsto \overline{a}$ and with unity 1. The totality of $m \times n$ matrices of rank r over $\mathbb{I}$ is denoted by $\mathbb{I}_r^{m\times n}$. The adjoint matrix of a square matrix B is denoted by $\operatorname{adj}(B)$, its determinant by $|B|$, and the trace of B is denoted

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by $\mathrm{Tr}(()B)$. The determinantal rank $\rho(A)$ of a matrix A is defined as the size of the largest nonvanishing minor of A. A $k \times k$ identity matrix is denoted by I_k.

For any $m \times n$ matrix A, consider the following Penrose equations in G, where the superscript $*$ denotes the transformation $(a_{ij}) \mapsto: (a_{ij})^* = \overline{a}_{ji}$:

$$(1) \quad AGA = A \qquad (2) \quad GAG = G$$

$$(3) \quad (AG)^* = AG \qquad (4) \quad (GA)^* = GA$$

In the case $m = n$, consider the equation

$$AG = GA. \tag{5}$$

A matrix G satisfying the equation (1) is called a *g-inverse* of A. If G satisfies (1) and (2), it is called a *reflexive g-inverse* of A. The *group inverse* of A, denoted by $A^{\#}$, is the unique solution of the equations (1), (2) and (5). The *Moore–Penrose inverse* of A, denoted by $A^{\dagger}$, is the unique solution of the equations (1)–(4). For a sequence $\mathcal{S}$ of elements from the set $\{1, 2, 3, 4, 5\}$, the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$.

The *weighted Moore–Penrose inverse* $A^{\dagger}_{M,N}$ is the unique solution of the matrix equations (1), (2) and the following matrix equations:

$$(6) \qquad (MAG)^* = MAG \qquad\qquad (7) \qquad (NGA)^* = NGA,$$

where M and N are nonsingular matrices of the order $m \times m$ and $n \times n$, respectively.

We follow the notation from [1–3], [10–14], [19]. Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$; let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ and $\beta = \{\beta_1, \ldots, \beta_r\}$ be subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. Then $\left|A^{\alpha}_{\beta}\right|$ denotes the minor of A determined by the rows indexed by α and the columns indexed by β. By $C_r(A)$ is denoted the rth compound matrix of A with rows indexed by r-element subsets of $\{1, \ldots, m\}$, columns indexed by r-element subsets of $\{1, \ldots, n\}$, and the (α, β) entry defined by $\left|A^{\alpha}_{\beta}\right|$.

For $1 \le k \le n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$, is denoted by

$$\mathcal{Q}_{k,n} = \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k),\ 1 \le \alpha_1 < \cdots < \alpha_k \le n\}.$$

Let $\mathcal{N} = \mathcal{Q}_{r,m} \times \mathcal{Q}_{r,n}$. For fixed $\alpha \in \mathcal{Q}_{k,m}$, $\beta \in \mathcal{Q}_{k,m}$, $1 \le k \le r$, let

$$\mathcal{I}(\alpha) = \{I : I \in \mathcal{Q}_{r,m},\ I \supseteq \alpha\}, \quad \mathcal{J}(\beta) = \{J : J \in \mathcal{Q}_{r,n},\ J \supseteq \beta\},$$
$$\mathcal{N}(\alpha,\beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta).$$

If A is a square matrix, then the coefficient of $\left|A^{\alpha}_{\beta}\right|$ in the Laplace expansion of $|A|$ is denoted by $\dfrac{\partial}{\partial \left|A^{\alpha}_{\beta}\right|}|A|$. For the special case $\alpha = \{i\}$, $\beta = \{j\}$, we give the cofactor of a_{ij}: $\dfrac{\partial}{\partial a_{ij}}\,|A|$.

Also, we use the following notation. Let A^{α}_{β} denote the submatrix of A determined by the rows contained in α and the columns contained in β, and ${}_{\alpha}z$ denote the vector $\{z_{\alpha_1}, \ldots, z_{\alpha_r}\}^T$. Let $A\,(i \to z)$, $i \in \{1, \ldots, n\}$ denote the matrix obtained from A replacing its column i by the vector z.

All the reflexive g-inverses of a matrix over an integral domain are characterized in [10].

Proposition 1.1 [10]. *An arbitrary matrix $A \in \mathbb{I}^{m\times n}_r$ has a reflexive g-inverse G if and only if there exist $\lambda_{\alpha,\beta} \in \mathbb{I}$, $(\alpha,\beta) \in \mathcal{N}$, satisfying*

$$\sum_{(\alpha,\beta)\in\mathcal{N}} \lambda_{\alpha,\beta} \left|A^{\alpha}_{\beta}\right| = 1, \tag{1.1}$$

where the $\binom{m}{r} \times \binom{n}{r}$ matrix $\Lambda = (\lambda_{\alpha,\beta})$ satisfies

$$\operatorname{rank}(\Lambda) = 1. \tag{1.2}$$

In this case $G = (g_{ij})$ is given by

$$g_{ij} = \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \lambda_{\alpha,\beta} \frac{\partial}{\partial a_{ji}} \left|A^{\alpha}_{\beta}\right|, \quad 1 \le i \le n,\ 1 \le j \le m. \tag{1.3}$$

For the sake of completeness, we restate the determinantal representations and conditions for the existence of the *Moore–Penrose inverse*, the *weighted Moore–Penrose inverse* and the *group inverse* over an integral domain, introduced in [1], [11], [12], [13].

Proposition 1.2 [1]**.** *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$, and let $A = PQ$ be a rank factorization of A. Then the following conditions are equivalent:*

(i) *A has a Moore–Penrose inverse.*

(ii) *P^*P and QQ^* are invertible over $\mathbb{I}$.*

(iii) *$C_r(A)$ has a Moore–Penrose inverse.*

(iv) $\sum_{(\alpha,\beta)\in\mathcal{N}} \left|\overline{A}_\beta^\alpha\right| \cdot \left|A_\beta^\alpha\right| = \mathrm{Tr}(C_r(A^*A))$ *is invertible in $\mathbb{I}$.*

Furthermore, the Moore–Penrose inverse $A^\dagger = (g_{ij})$, if it exists, is given by $A^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*$, and its determinantal representation is

$$g_{ij} = \left(\sum_{(\gamma,\delta)\in\mathcal{N}} \left|\overline{A}_\delta^\gamma\right| \, |A_\delta^\gamma| \right)^{-1} \cdot \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \left|\overline{A}_\beta^\alpha\right| \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right|.$$

Proposition 1.3 [12], [13]**.** *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$, M and N be invertible matrices of the order $m \times m$ and $n \times n$, respectively, and $A = PQ$ be a rank factorization of A. Then the following conditions are equivalent:*

(i) *A has a weighted Moore–Penrose inverse $A_{M,N}^\dagger$.*

(ii) *P^*MP and $QN^{-1}Q^*$ are hermitian and invertible over $\mathbb{I}$.*

(iii) *$C_r(A)$ has a weighted Moore–Penrose inverse with respect to $C_r(M)$ and $C_r(N)$.*

(iv) $\mathrm{Tr}(C_r(N^{-1}A^*MA))$ *is invertible in $\mathbb{I}$.*

The weighted Moore–Penrose inverse, if it exists, is given by

$$\begin{aligned} A_{M,N}^\dagger &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M \\ &= (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*, \end{aligned}$$

and the determinantal representation of its arbitrary (i,j)th element is

$$g_{ij} = \left(\mathrm{Tr}(C_r(N^{-1}A^*MA))\right)^{-1} \cdot \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \left|(N^{-1}A^*M)_\alpha^\beta\right| \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right|.$$

Proposition 1.4 [11]. *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$. Then the following conditions are equivalent:*

(i) *A has a group inverse.*

(ii) *$C_r(A)$ has a group inverse.*

(iii) *$\sum\limits_{\gamma} \left|A_{\gamma}^{\gamma}\right|$ is invertible in $\mathbb{I}$.*

(iv) *$\rho(A) = \rho(A^2)$, and A^2 is regular.*

Furthermore, the group inverse $G = (g_{ij})$, if it exists, is given by

$$g_{ij} = \left(\sum_{\gamma} \left|A_{\gamma}^{\gamma}\right| \right)^{-2} \cdot \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \left|A_{\alpha}^{\beta}\right| \frac{\partial}{\partial a_{ji}} \left|A_{\beta}^{\alpha}\right| .$$

The main results of this paper are as follows:

(1) Generalization of the concepts of *algebraic complement* and determinant. Incidentally, we derive a *general determinantal representation* of $\{1,2\}$ inverses and conditions for their existence, for matrices which allow full-rank factorizations over an integral domain. In certain cases, we introduce explicit determinantal representations and conditions for the existence of $\{1,2,3\}$, $\{1,2,4\}$ inverses, which contain known determinantal representations and conditions for the existence of the Moore–Penrose inverse. Also, we investigate determinantal representations and some more characterizations of the weighted Moore–Penrose and the group inverse.

(2) For matrices which do not have rank factorizations we obtain a determinantal formula for $\{1,2\}$ inverses and introduce a few necessary and sufficient conditions for the existence of this representation, using the results of Proposition 1.1. In exactly defined cases we derive determinantal representations and conditions for the existence of the Moore–Penrose, weighted Moore–Penrose and the group inverse.

(3) Also, we introduce a determinantal representation of the reflexive g-inverses, using their characterization introduced in [19]. For this purpose we introduce the notion of a general determinantal representation of a variable order.

(4) We investigate correlations between the minors of a given matrix A, minors of the matrix W which satisfy certain conditions, and the corresponding minors selected from $A^{(1,2)}$, $A^{\dagger}$, $A^{\dagger}_{M,N}$ and $A^{\#}$. If A allows a

rank factorization, we obtain additional results for $\{1,2,3\}$ and $\{1,2,4\}$ inverses of A.

(5) Furthermore, we obtain an explicit Cramer-type determinantal representation and conditions for the existence of the solution of a given system of linear equations $Ax = z$, by means of the derived determinantal representation for $\{1,2\}$ inverses of A. In partial cases we derive the determinantal formula for the least-squares solution [4], the minimum norm least-squares solution [5] and the weighted minimum norm least-squares solution [12], [13].

2. General determinantal representation

The concepts of *determinant, algebraic complement, adjoint matrix* and *determinantal representation* of generalized inverses are generalized in the following definition (see also [20]):

Definition 2.1. Let A, R be $m \times n$ matrices of rank r.

(i) *The generalized determinant* of A with respect to R, denoted by $N_{(R,r)}(A)$, is equal to

$$N_{(R,r)}(A) = \sum_{(\alpha,\beta)\in\mathcal{N}} \left|\overline{R}_{\beta}^{\alpha}\right| \left|A_{\beta}^{\alpha}\right| = \mathrm{Tr}(C_r(R^* A)).$$

(ii) *The generalized algebraic complement* of A corresponding to a_{ij} is

$$A_{ij}^{(\dagger,R)} = \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \left|\overline{R}_{\beta}^{\alpha}\right| \frac{\partial}{\partial a_{ji}} \left|A_{\beta}^{\alpha}\right|, \quad \begin{pmatrix} 1 \le i \le n \\ 1 \le j \le m \end{pmatrix}.$$

(iii) *The generalized adjoint matrix* of A with respect to R is denoted by $\mathrm{adj}^{(\dagger,R)}(A)$, and it is equal to the matrix whose elements are equal to $A_{ij}^{(\dagger,R)}$, $\begin{pmatrix} 1 \le i \le n \\ 1 \le j \le m \end{pmatrix}$.

(iv) *The general determinantal representation* for generalized inverses of A with respect to R is equal to

$$A^{(\dagger,R)} = \left(N_{(R,r)}(A)\right)^{-1} \cdot \mathrm{adj}^{(\dagger,R)}(A).$$

For two full-rank matrices A and R the following results can be verified:

Proposition 2.1 [20]**.** *If A is an $m \times n$ matrix of full-rank and the matrix R has the same dimensions and rank, then:*

(i) $N_{(R,r)}(A) = \begin{cases} |AR^*|, & r = m \\ |R^*A|, & r = n, \end{cases}$

(ii) $A_{ij}^{(\dagger,R)} = \begin{cases} \left(R^* \operatorname{adj}(AR^*)\right)_{ij}, & r = m \\ \left(\operatorname{adj}(R^*A)R^*\right)_{ij}, & r = n, \end{cases} \begin{pmatrix} 1 \le i \le n \\ 1 \le j \le m \end{pmatrix},$

(iii) $A^{(\dagger,R)} = \begin{cases} R^*(AR^*)^{-1}, & r = m \\ (R^*A)^{-1}R^*, & r = n, \end{cases}$

(iv) $\operatorname{adj}^{(\dagger,R)}(A) = \begin{cases} R^* \operatorname{adj}(AR^*), & r = m \\ \operatorname{adj}(R^*A)R^*, & r = n. \end{cases}$

The main properties of the *generalized adjoint matrix*, the *generalized algebraic complement* and the *generalized determinant* are investigated in [20].

Proposition 2.2 [20]**.** *Let $A = PQ$ be a full-rank factorization of an $m \times n$ matrix A of rank r, R_1 be an $n \times r$ matrix A of rank r and R_2 be an $r \times m$ matrix A of rank r. Then:*

(i) $\operatorname{adj}^{(\dagger,R_1)}(Q) \cdot \operatorname{adj}^{(\dagger,R_2)}(P) = \operatorname{adj}^{(\dagger,R_2R_1)}(A)$;

(ii) $N_{(R_1,r)}(Q) \cdot N_{(R_2,r)}(P) = N_{(R_2,r)}(P) \cdot N_{(R_1,r)}(Q) = N_{(R_2R_1,r)}(A)$;

(iii) $Q^{(\dagger,R_1)} \cdot P^{(\dagger,R_2)} = A^{(\dagger,R_2R_1)}$.

Remark 2.1. In the special case $R = A$ the function $N_{(R,r)}(A)$ reduces to the function $\Delta_r^2(A)$ examined in [3]. Also, in the case $R_1 = Q$, $R_2 = P$, the statement (ii) from Proposition 2.2 produces a well-known property of the function $\Delta_r^2(A)$ [3]:

$$\Delta_r^2(PQ) = \Delta_r^2(P) \cdot \Delta_r^2(Q).$$

Also, we propose the following extensions of the presented notations, which are based on the minors of the order $s \le r = \operatorname{rank}(A)$:

$$\mathcal{N}_s = \mathcal{Q}_{s,m} \times \mathcal{Q}_{s,n}, \text{ where } s \le r = \operatorname{rank}(A);$$

for fixed $\alpha, \beta \in \mathcal{Q}_{p,n}$, $1 \le p \le s$, let

$$\mathcal{I}_s(\alpha) = \{I : I \in \mathcal{Q}_{s,m},\ I \supseteq \alpha\}, \quad \mathcal{J}_s(\beta) = \{J : J \in \mathcal{Q}_{s,n},\ J \supseteq \beta\},$$
$$\mathcal{N}_s(\alpha,\beta) = \mathcal{I}_s(\alpha) \times \mathcal{J}_s(\beta).$$

We are now in a position to define the notions of *generalized determinant, algebraic complement, adjoint matrix* and *general determinantal representation* of a variable order.

Definition 2.2. Let A, R be $m \times n$ matrices which satisfy $\operatorname{rank}(A) = r$, $\operatorname{rank}(R) = s \leq r$, where s is an arbitrary integer.

(i) *The generalized determinant of the order* s of A, denoted by $N_{(R,s)}(A)$, is equal to

$$N_{(R,s)}(A) = \sum_{(\alpha,\beta)\in\mathcal{N}_s} \left|\overline{R}_\beta^\alpha\right| \left|A_\beta^\alpha\right|.$$

(ii) *The generalized algebraic complement of the order* s of A corresponding to a_{ij} is the following expression:

$$A_{ij}^{(\dagger,R,s)} = \sum_{(\alpha,\beta)\in\mathcal{N}_s(j,i)} \left|\overline{R}_\beta^\alpha\right| \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right|, \quad \begin{pmatrix} 1 \leq i \leq n \\ 1 \leq j \leq m \end{pmatrix}.$$

(iii) *The generalized adjoint matrix of the order* s of A with respect to R, denoted by $\operatorname{adj}^{(\dagger,R,s)}(A)$, is the matrix whose elements are defined by $A_{ij}^{(\dagger,R,s)}$, $1 \leq i \leq n$, $1 \leq j \leq m$.

(iv) *The general determinantal representation of the order* s with respect to R is equal to

$$A^{(\dagger,R,s)} = \left(N_{(R,s)}(A)\right)^{-1} \cdot \operatorname{adj}^{(\dagger,R,s)}(A).$$

3. General determinantal representation with rank factorization

In the following theorem we investigate a determinantal formula for the class $A\{1,2\}$ as well as a few necessary and sufficient conditions for its existence, under the supposition that the matrix A allows a full-rank factorization.

Theorem 3.1. *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$ and $A = PQ$ be a full-rank factorization of A. Then the following conditions are equivalent:*

(i) *A is regular.*

(ii) *There exist matrices* $W_1 \in \mathbb{I}^{n\times r}$ *and* $W_2 \in \mathbb{I}^{r\times m}$, *such that* QW_1 *and* W_2P *are invertible matrices over* $\mathbb{I}$.

(iii) $N_{(W_1^*,r)}(Q)$ *and* $N_{(W_2^*,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{((W_1W_2)^*,r)}(A) = \mathrm{Tr}\,(C_r(W_1W_2\cdot A))$ *is invertible in* $\mathbb{I}$.

(v) $C_r(A)$ *is regular over* $\mathbb{I}$.

In the case when A *is regular, the set of* $\{1,2\}$*-inverses of* A *is determined by the following general representation*

$$A^{(1,2)} = W_1(QW_1)^{-1}(W_2P)^{-1}W_2 = W_1(W_2AW_1)^{-1}W_2$$

and by the following determinantal formula

$$A^{(1,2)} = Q^{(\dagger,W_1^*)}P^{(\dagger,W_2^*)} = A^{(\dagger,(W_1W_2)^*)}.$$

Proof. (i) $\Rightarrow$ (ii): If $A^{(1)}$ exists then $A^{(1,2)}$ exists, and according to [1], its general form is $A^{(1,2)} = Q_R^{-1}P_L^{-1}$, where Q_R^{-1} and P_L^{-1} denote the right inverse of Q and the left inverse of P, respectively. In order to develop an effective determinantal representation for $A^{(1,2)}$, we derive the general representations of the right and left inverses. Applying the principles from [17, p. 20], it is easy to conclude that Q_R^{-1} and P_L^{-1} exist if and only if there exist $m\times m$ and $n\times n$ matrices U and V, respectively, such that QVQ^* and P^*UP are invertible. In that case, their general representations are

$$Q_R^{-1} = VQ^*(QVQ^*)^{-1}, \quad P_L^{-1} = (P^*UP)^{-1}P^*U.$$

Consequently, when A is regular, there exist appropriate nonsingular matrices QVQ^* and P^*UP, such that

$$A^{(1,2)} = VQ^*(QVQ^*)^{-1}(P^*UP)^{-1}P^*U.$$

Using the substitutions $W_1 = VQ^*$ and $W_2 = P^*U$, we conclude that there exist matrices W_1 and W_2 of the order $n\times r$ and $r\times m$, respectively, such that QW_1 and W_2P are invertible matrices in $\mathbb{I}$, and

$$A^{(1,2)} = W_1(QW_1)^{-1}\,(W_2P)^{-1}W_2.$$

(ii) $\Rightarrow$ (i): If there exist matrices W_1 and W_2 of the order $n\times r$ and $r\times m$, respectively, such that QW_1 and W_2P are invertible, then it is not difficult to verify that $W_1(QW_1)^{-1}\ (W_2P)^{-1}W_2$ is an $\{1,2\}$ inverse of A. This implies that A is a regular matrix.

(ii) $\Leftrightarrow$ (iii): A square matrix over a ring $\mathbb{R}$ is invertible if and only if its determinant is invertible in $\mathbb{R}$ [8], [9]. Hence, QW_1 and W_2P are invertible matrices over $\mathbb{I}$ if and only if $|QW_1|$ and $|W_2P|$ are invertible in $\mathbb{I}$. From Proposition 2.1, we obtain

$$|QW_1| = N_{(W_1^*,r)}(Q), \qquad |W_2P| = N_{(W_2^*,r)}(P),$$

completing this part of the proof.

(iii) $\Leftrightarrow$ (iv): The statement (ii) of Proposition 2.2 implies

$$N_{(W_2^*,r)}(P)\cdot N_{(W_1^*,r)}(Q) = N_{((W_1W_2)^*,r)}(A).$$

Therefore, $N_{((W_1W_2)^*,r)}(A)$ is invertible if and only if both $N_{(W_2^*,r)}(P)$ and $N_{(W_1^*,r)}(Q)$ are invertible.

(i) $\Leftrightarrow$ (v): Follows from the results of the paper [18].

Also, using the general representation of the reflexive g-inverses and the results from Proposition 2.1 and Proposition 2.2, we obtain

$$\begin{aligned}
A^{(1,2)} &= W_1(QW_1)^{-1}(W_2P)^{-1}W_2 = \\
&= (|QW_1|)^{-1}\, W_1\operatorname{adj}(W_1Q)\cdot(|W_2P|)^{-1}\, W_2\operatorname{adj}(W_2P) = \\
&= \left(N_{(W_1^*,r)}(Q)\right)^{-1}\operatorname{adj}^{(\dagger,W_1^*)}(Q)\cdot\left(N_{(W_2^*,r)}(P)\right)^{-1}\operatorname{adj}^{(\dagger,W_2^*)}(P) = \\
&= Q^{(\dagger,W_1^*)}P^{(\dagger,W_2^*)} = A^{(\dagger,(W_1W_2)^*)}.
\end{aligned}$$
□

The determinantal representations and the conditions for existence of $\{1,2,3\}$ and $\{1,2,4\}$ generalized inverses are introduced in the following two statements:

Theorem 3.2. *Let A be an $m\times n$ matrix of rank r and $A=PQ$ be a full-rank factorization of A. Then the following conditions are equivalent:*

(i) $A^{(1,2,3)}$ *exists.*

(ii) P^*P *is an invertible matrix over* $\mathbb{I}$ *and there exists* $W_1\in\mathbb{I}^{n\times r}$, *such that* QW_1 *is invertible over* $\mathbb{I}$.

(iii) $N_{(W_1^*,r)}(Q)$ *and* $N_{(P,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{(PW_1^*,r)}(A)$ *is invertible in* $\mathbb{I}$.

(v) $C_r(A)$ *has a* $\{1,2,3\}$*-inverse.*

In the case when $A^{(1,2,3)}$ *exists, it is given by the general representation*

$$A^{(1,2,3)} = W_1(QW_1)^{-1}(P^*P)^{-1}P^* = W_1(P^*AW_1)^{-1}P^*$$

and by the following determinantal representation

$$A^{(1,2,3)} = Q^{(\dagger,W_1^*)}P^{(\dagger,P)} = A^{(\dagger,PW_1^*)}.$$

Proof. (i) $\Rightarrow$ (ii): If $G = A^{(1,2,3)}$ exists, then according to [1], it is equal to $G = Q_R^{-1}P_L^{-1}$. But, $G = A^{(1,2,3)}$ also satisfies the equation (3), which implies $AGG^*A^*A = A$. Hence,

$$(QGG^*Q^*)(P^*P) = I_r,$$

which means that P^*P is invertible. From the equation (3), using $G = Q_R^{-1}P_L^{-1}$, we get

$$(P_L^{-1})^*P^* = PP_L^{-1}.$$

Multiplying this equation by P from the right, we obtain

$$(P_L^{-1})^*P^*P = P.$$

Using invertibility of the matrix P^*P, we conclude

$$P_L^{-1} = (P^*P)^{-1}P^*.$$

The right inverse of Q is defined as in Theorem 3.1: Q_R^{-1} exists if there exists a $W_1 \in \mathbb{I}^{n\times r}$, such that QW_1 is invertible and $Q_R^{-1} = W_1(QW_1)^{-1}$.

(ii) $\Rightarrow$ (i): If QW_1 and P^*P are invertible, it is not difficult to verify that $W_1(QW_1)^{-1}(P^*P)^{-1}P \in A\{1,2,3\}$.

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Proposition 3.1. *Assume that* A *is an* $m \times n$ *matrix of rank* r *and* $A = PQ$ *is a full-rank factorization of* A*. Then the following conditions are equivalent:*

(i) $A^{(1,2,4)}$ *exists.*

(ii) QQ^* *is invertible over* $\mathbb{I}$ *and there exists* $W_2 \in \mathbb{I}^{r\times m}$*, such that* W_2P *is invertible over* $\mathbb{I}$*.*

(iii) $N_{(Q,r)}(Q)$ *and* $N_{(W_2^*,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{(W_2^* Q,r)}(A)$ *is invertible in* $\mathbb{I}$.

(v) $C_r(A)$ *has a* $\{1,2,4\}$*-inverse.*

In the cases when $A^{(1,2,4)}$ *exists, it possesses the following representations:*

$$\begin{aligned} A^{(1,2,4)} &= Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2 = Q^*(W_2AQ^*)^{-1}W_2 \\ &= Q^{(\dagger,Q)}P^{(\dagger,W_2^*)} = A^{(\dagger,W_2^*Q)}. \end{aligned}$$

Representations and characterizations of the Moore–Penrose inverse follow from Theorem 3.2 and Proposition 3.1.

Corollary 3.1. *For an* $m \times n$ *matrix* A *of rank* r*, the full-rank factorization of which is given by* $A = PQ$*, the following conditions are equivalent:*

(i) $A^\dagger$ *exists.*

(ii) QQ^* *and* P^*P *are invertible matrices in* $\mathbb{I}$.

(iii) $N_{(Q,r)}(Q)$ *and* $N_{(P,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{(A,r)}(A)$ *is invertible in* $\mathbb{I}$.

(v) $C_r(A)$ *has a Moore–Penrose inverse.*

When $A^\dagger$ *exists, it is given by*

$$\begin{aligned} A^\dagger &= Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^* \\ &= Q^{(\dagger,Q)}P^{(\dagger,P)} = A^{(\dagger,A)}. \end{aligned}$$

Applying the principles of [12], [13] and the proof of Theorem 3.1, we obtain known results from Proposition 1.3 and some complementary results.

Proposition 3.2. *For an* $m\times n$ *matrix* A *of rank* r *the full-rank factorization of which is given by* $A = PQ$ *and for invertible matrices* M *and* N *of the order* m *and* n*, respectively, the following conditions are equivalent:*

(i) $A^\dagger_{M,N}$ *exists.*

(ii) P^*MP *and* $QN^{-1}Q^*$ *are hermitian and invertible matrices over* $\mathbb{I}$.

(iii) $N_{(Q(N^{-1})^*,r)}(Q) = N_{(QN^{-1},r)}(Q)$ *and* $N_{(M^*P,r)}(P) = N_{(MP,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{(M^*A(N^{-1})^*,r)}(A) = N_{(MAN^{-1},r)}(A)$ *is invertible in* $\mathbb{I}$.

(v) $C_r(A)$ *has a weighted Moore–Penrose inverse with respect to* $C_r(M)$ *and* $C_r(N)$.

Moreover,

$$\begin{aligned} A^{\dagger}_{M,N} &= (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* \\ &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M \\ &= Q^{(\dagger,Q(N^{-1})^*)}P^{(\dagger,M^*P)} = Q^{(\dagger,QN^{-1})}P^{(\dagger,MP)} \\ &= A^{(\dagger,M^*A(N^{-1})^*)} = A^{(\dagger,MAN^{-1})}. \end{aligned}$$

Proof. (i) $\Leftrightarrow$ (ii) is a known result from [12] and [13]. Also,

$$\begin{aligned} A^{\dagger}_{M,N} &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M \\ &= (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* \end{aligned}$$

can be proved using principles from [13]. □

In the following theorem we introduce a few complementary conditions for the existence of the group inverse with respect to Proposition 1.4.

Theorem 3.3. *For a square matrix* $A = PQ$ *of order* n *the following conditions are equivalent:*

(i) $A^{\#}$ *exists.*

(ii) QP *is an invertible matrix over* $\mathbb{I}$.

(iii) $N_{(P^*,r)}(Q)$ *and* $N_{(Q^*,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{(A^*,r)}(A)$ *is invertible in* $\mathbb{I}$.

(v) $\sum_{\gamma} \left|A^{\gamma}_{\gamma}\right|$ *is invertible in* $\mathbb{I}$.

(vi) $C_r(A)$ *has a group inverse*

When $A^{\#}$ *exists, it is given by*

$$\begin{aligned} A^{\#} &= P(QP)^{-2}Q = P(QAP)^{-1}Q \\ &= Q^{(\dagger,Q^*)}P^{(\dagger,P^*)} = A^{(\dagger,A^*)}. \end{aligned}$$

Proof. (i) $\Rightarrow$ (ii): The group inverse of A, when it exists, satisfies the equation $A^2A^{\#} = A$, which implies $A^2(A^{\#})^2A = A$. Using the full-rank factorization $A = PQ$, we conclude that

$$(QP)(Q(A^{\#})^2P) = I_r$$

so that QP is a nonsingular matrix.

In this case, starting from $A^{\#} = Q_R^{-1}P_L^{-1}$ and using the equation (5) we get $PP_L^{-1} = Q_R^{-1}Q$. In view of the invertibility of QP, we derive

$$Q_R^{-1} = P(QP)^{-1}, \quad P_L^{-1} = (QP)^{-1}Q,$$

which yields $A^{\#} = P(QP)^{-2}Q = P(QAP)^{-1}Q$.

(ii) $\Rightarrow$ (i): If QP is invertible, it is not difficult to verify

$$A^{\#} = P(QP)^{-2}Q = P(QAP)^{-1}Q.$$

(iv) $\Leftrightarrow$ (v): Follows from the following result [11]:

$$\left(N_{(A^*,r)}(A)\right)^{-1} = \left(\sum_{\gamma} \left|A_{\gamma}^{\gamma}\right|\right)^{-2}. \qquad \square$$

Remark 3.1. In Theorem 3.1, Theorem 3.2 and Proposition 3.1 we present an elegant proof and generalize known results for the set of complex matrices, introduced in [16]. Also, the result (i) $\Leftrightarrow$ (ii) of Theorem 3.3 and the general representation of the group inverse represent a transfer of the known results from [6], concerning the group inverse of complex matrices, to the set of matrices over an integral domain.

In the following theorem we examine the existence of the general determinantal representation of generalized inverses, under the hypothesis that full-rank factorization is allowed.

Theorem 3.4. *Let A, R be $m \times n$ matrices of rank r over $\mathbb{I}$, $A = PQ$ be a full-rank factorization of A and $R = ST$ be a full-rank factorization of R. Then the following conditions are equivalent:*

(i) $A^{(\dagger,R)}$ *exists.*

(ii) QT^* *and* S^*P *are invertible matrices in* $\mathbb{I}$.

(iii) $N_{(T,r)}(Q)$ *and* $N_{(S,r)}(P)$ *are invertible in* $\mathbb{I}$.

(iv) $N_{(R,r)}(A)$ *is invertible in* $\mathbb{I}$.

(v) *The reflexive g-inverse* $(C_r(A))^{(\dagger,R_1)}$ *of* $C_r(A)$ *exists, where* R_1 *is an arbitrary* $\binom{n}{r} \times \binom{m}{r}$ *matrix of rank 1.*

Proof. From Proposition 2.2 and part (iii) of Proposition 2.1 we get

$$A^{(\dagger,R)} = Q^{(\dagger,T)} \cdot P^{(\dagger,S)} = T^*(QT^*)^{-1}(S^*P)^{-1}S^*.$$

Now the proof is implied by Theorem 3.1. □

4. Determinantal representation without rank factorization

In the following theorem we derive a determinantal formula and conditions for the existence of $\{1,2\}$-inverses, using the characterization of $\{1,2\}$-inverses from [10] (restated in Proposition 1.1.) The results are valid for arbitrary matrices over an integral domain.

Theorem 4.1. *For a given matrix* $A \in \mathbb{I}_r^{m\times n}$ *the following conditions are equivalent:*

(i) A *is regular.*

(ii) *There exist* $W \in \mathbb{I}_r^{n\times m}$, *such that* $N_{(W^*,r)}(A) = \mathrm{Tr}\,(C_r(W \cdot A))$ *is invertible in* $\mathbb{I}$.

(iii) $C_r(A)$ *is regular over* $\mathbb{I}$.

In the case when $A^{(1)}$ *exists, the corresponding reflexive g-inverse* $G = (g_{ij})$ *of* A *possesses the following determinantal representation:*

$$G = A^{(\dagger,W^*)}.$$

Proof. (i) $\Rightarrow$ (ii): If A is regular, then A has a reflexive g-inverse. According to Proposition 1.1, an arbitrary reflexive g-inverse $G = (g_{ij})$ of A is of the form (1.3), where the matrix $\Lambda = (\lambda_{\alpha,\beta})$ satisfies the conditions (1.1) and (1.2). We give an explicit representation of the matrix Λ and conditions for its existence. The matrix Λ is of the order $\binom{m}{r} \times \binom{n}{r}$ and $\mathrm{rank}(\Lambda) = 1$, so that it can be generated in the form

$$\Lambda = c \cdot C_r(W^T),$$

where W is an $n \times m$ matrix of rank r and c is a constant from $\mathbb{I}$. In this way, the existence of the matrix Λ is defined by the existence of the constant c. In the case when such a constant c exists, we can write

$$\lambda_{\gamma,\delta} = c \cdot \left|W_\gamma^\delta\right|, \quad \text{for all } (\gamma,\delta) \in \mathcal{N}. \tag{4.1}$$

Now, from (4.1) and (1.1) we get

$$c \cdot \left(\sum_{(\gamma,\delta)\in\mathcal{N}} \left|W_\gamma^\delta\right| \left|A_\delta^\gamma\right| \right) = \left(\sum_{(\gamma,\delta)\in\mathcal{N}} \left|W_\gamma^\delta\right| \left|A_\delta^\gamma\right| \right) \cdot c = 1.$$

It is evident that $c \in \mathbb{I}$ exists if and only if

$$\sum_{(\gamma,\delta)\in\mathcal{N}} \left|W_\gamma^\delta\right| \left|A_\delta^\gamma\right| = N_{(W^*,r)}(A) = \operatorname{Tr}\left(C_r(W \cdot A)\right)$$

is an invertible element in $\mathbb{I}$. According to Proposition 1.1, this condition for the existence of the matrix Λ is also a condition for the existence of an arbitrary reflexive g-inverse of A. In this case, we obtain

$$c = \left(N_{(W^*,r)}(A)\right)^{-1},$$

which implies, together with (4.1), the following:

$$\lambda_{\gamma,\delta} = \left(N_{(W^*,r)}(A)\right)^{-1} \left|W_\gamma^\delta\right|, \quad \text{for all } (\gamma,\delta) \in \mathcal{N}. \tag{4.2}$$

According to Proposition 1.1, when an arbitrary $G = A^{(1,2)}$ exists, it is given by (1.3). Substituting (4.2) in (1.3), for arbitrary $1 \le i \le n$, $1 \le j \le m$ we get

$$\begin{aligned} g_{ij} &= \left(N_{(W^*,r)}(A)\right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \left|W_\alpha^\beta\right| \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right| \\ &= \left(N_{(W^*,r)}(A)\right)^{-1} A_{ij}^{(\dagger,W^*)} \end{aligned}$$

which means that $G = A^{(\dagger,W^*)}$.

(ii) $\Rightarrow$ (i): On the other hand, assume the existence of a matrix $W \in \mathbb{I}_r^{n\times m}$, such that $N_{(W^*,r)}$ is invertible. Consider the matrix $\Lambda = (\lambda_{\alpha,\beta})$, defined by

$$\Lambda = \left(N_{(W^*,r)}(A)\right)^{-1} \cdot C_r(W^T).$$

This implies

$$\operatorname{rank}(\Lambda) = \operatorname{rank}(C_r(W^T)) = 1$$

and

$$\sum_{(\alpha,\beta)\in\mathcal{N}} \lambda_{\alpha,\beta} \left|A^{\alpha}_{\beta}\right| = 1.$$

In this way, the conditions of Proposition 1.1 are satisfied, and $A^{(1,2)}$ exists, which implies that $A^{(1)}$ exists. □

Problems 4.1. (i) In Theorem 4.1 it is shown that the matrix

$$\left(N_{(W^*,r)}(A)\right)^{-1} C_r(W^T), \quad W \in \mathbb{I}^{n\times m}_r$$

satisfies the conditions imposed on the matrix Λ, defined in Proposition 1.1. In the light of this result, it seems interesting to state the following problem: To find alternative representations of the matrix Λ, if possible.

(ii) The results from [14] provide a reason to state the following problem: develop an effective determinantal representation of an arbitrary $\{1\}$-inverse $G = (g_{ij})$ of $A = (a_{ij})$ in the form

$$g_{ij} = \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \lambda_{\alpha,\beta} \frac{\partial}{\partial a_{ji}} \left|A^{\alpha}_{\beta}\right|,$$

finding the constants $\lambda_{\alpha,\beta}$, which solve the following system [14]:

$$a_{ij} \left(\sum_{(\alpha,\beta)\in\mathcal{N}} \lambda_{\alpha,\beta} \left|A^{\alpha}_{\beta}\right| \right) = a_{ij}, \quad \text{for all } i, j.$$

A partial solution of this problem is given in Theorem 3.1 and Theorem 4.1, appropriate for the class of reflexive g-inverses.

(iii) Using the known result from [18], it is possible to derive a determinantal representation of $A\{1\}$, solving the system of equations (1.1).

Remark 4.1. (i) For $A \in \mathbb{C}^{m\times n}_r$, if the matrix $W \in \mathbb{C}^{n\times m}_r$ satisfies $\left|W^{\beta}_{\alpha}\right| = 1$ for all $(\alpha,\beta) \in \mathcal{N}$, then the determinantal representation from Theorem 4.1 reduces to the determinant and generalized inverses introduced in [7].

(ii) If the matrix $W \in \mathbb{C}_r^{n\times m}$ is selected according to the condition

$$\left|W_\beta^\alpha\right| = (-1)^{\alpha_1+\cdots+\alpha_r+\beta_1+\cdots+\beta_r} \quad \text{for all } (\alpha,\beta)\in\mathcal{N},$$

then the general determinantal representation from Theorem 4.1 yields the definition of determinant and generalized inverses introduced in [15].

(iii) In the cases $W = A^*$, $W = N^{-1}A^*M$ (or $W = (MAN^{-1})^*$) and $W = A$, from Theorem 4.1 we obtain the determinantal representations and characterizations of the Moore–Penrose, the weighted Moore–Penrose and the group inverse, respectively (introduced in [1], [12], [13], [11]).

In the case $\operatorname{rank}(A) = 1$ we obtain the following representation and characterization of the reflexive g-inverses of A:

Lemma 4.1. *Assume that A is a given $m\times n$ matrix of rank 1 over $\mathbb{I}$. Then A is regular if and only if there exists an $n\times m$ matrix W of rank 1, such that $\operatorname{Tr}(WA)$ is invertible.*

In this case, the corresponding reflexive g-inverse G of A is given by

$$G = (\operatorname{Tr}(WA))^{-1}\, W.$$

Proof. Using the result of Theorem 4.1, we conclude that A is regular if and only if $\operatorname{Tr}(C_1(WA)) = \operatorname{Tr}(WA)$ is invertible (which also means that $\operatorname{rank}(WA) = 1$).

Furthermore, in this case we show that the corresponding reflexive g-inverse of A is given by $G = (\operatorname{Tr}(WA))^{-1}\, W$. Since $\operatorname{rank}(A) = \operatorname{rank}(W) = 1$, it is well-known (see for example [11]) that A and W can be represented by $A = xy^T$, $W = uv^T$, where x, v are arbitrary matrices of the order $m\times 1$ over $\mathbb{I}$, and y, u are arbitrary $n\times 1$ matrices over $\mathbb{I}$. Also, y^T and v^T denote the transpose of the matrices y and v, respectively.

Now, for arbitrary $1\le k\le m$, $1\le l\le n$, we get the following:

$$\begin{aligned}
(AWA)_{kl} &= \sum_{i,j} a_{ki}w_{ij}a_{jl} = \sum_{i,j} x_k y_i^T u_i v_j^T x_j y_l^T \\
&= \Big(\sum_{i,j} u_i v_j^T x_j y_i^T\Big) x_k y_l^T = \Big(\sum_i (WA)_{ii}\Big) a_{kl} = \operatorname{Tr}(WA)\cdot a_{kl};\\
(WAW)_{lk} &= \sum_{i,j} u_l v_i^T x_i y_j^T u_j v_k^T = \Big(\sum_{i,j} x_i y_j^T u_j v_i^T\Big) u_l v_k^T \\
&= \Big(\sum_j (WA)_{jj}\Big) a_{lk} = \operatorname{Tr}(WA)\cdot a_{lk}.
\end{aligned}$$

Consequently, we get

$$AWA = \mathrm{Tr}(WA)A, \qquad WAW = \mathrm{Tr}(WA)W.$$

Now, it is not difficult to verify that the matrix $G = (\mathrm{Tr}(WA))^{-1} \cdot W$ satisfies the equations (1) and (2).

$$\begin{aligned} AGA &= (\mathrm{Tr}(WA))^{-1} \cdot AWA = (\mathrm{Tr}(WA))^{-1}\, \mathrm{Tr}(WA)A = A \\ GAG &= (\mathrm{Tr}(WA))^{-2}\, WAW = (\mathrm{Tr}(WA))^{-2}\, \mathrm{Tr}(WA)W \\ &= (\mathrm{Tr}(WA))^{-1}\, W = G. \end{aligned}$$

□

In the following theorem we derive a determinantal representation of the reflexive g-inverses in the category of finite matrices over a commutative ring by means of their characterization introduced in [19], and using the result of Theorem 4.1 and Lemma 4.1. For this purpose, suppose that $\mathbb{R}$ is a commutative ring with 1 and with involution $\lambda : a \mapsto \overline{a}$. For a finite matrix A from $\mathbb{R}$, let $\rho(A)$ denote the determinantal rank of A, and let $\mathcal{C}_s(A)$ be the ideal of $\mathbb{R}$ generated by the $s \times s$ minors of A.

Theorem 4.2. *Let A be an $m \times n$ matrix over $\mathbb{R}$ with Rao index t, idempotents $i_A = (e_1, \ldots, e_t)$ and ranks $\rho_A = (r_1, \ldots, r_t)$. If A is regular, then an arbitrary reflexive g-inverse G of A is equal to*

$$G = \sum_{s=1}^{t-1} A^{(\dagger,(e_s W)^*, r_s)} = \sum_{s=1}^{t-1} (e_s A)^{(\dagger,(e_s W)^*, r_s)},$$

where W is an arbitrary $n \times m$ matrix of rank r.

Proof. According to Theorem 2 from [19], an arbitrary reflexive g-inverse $G = (g_{ij})$ of A can be represented in the form

$$g_{ij} = \sum_{s=1}^{t-1} (A_{B_s})_{ij} = \sum_{s=1}^{t-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_s}(j,i)} (B_s)_{\beta,\alpha} \cdot \frac{\partial}{\partial a_{ji}} \left|A^{\alpha}_{\beta}\right| \tag{4.3}$$

where B_s is a reflexive g-inverse of $C_{r_s}(e_s A)$, $s = 1, \ldots, t-1$. In view of Lemma 4.1 we can use

$$B_s = \left[\mathrm{Tr}\left(H_s \cdot C_{r_s}(e_s A)\right)\right]^{-1} H_s, \quad s = 1, \ldots, t-1$$

where H_s is an arbitrary $\binom{n}{r_s} \times \binom{m}{r_s}$ matrix of rank 1. We can use $H_s = C_{r_s}(W_s)$, where $W_s = e_s W$ are $n \times m$ matrices of rank r_s, $s = 1, \dots, t-1$, and W is an $n \times m$ matrix of rank r. The element e_s is an idempotent, which produces

$$\begin{aligned}
\operatorname{Tr}(H_s \cdot C_{r_s}(e_s A)) = \operatorname{Tr}(C_{r_s}(e_s W) C_{r_s}(e_s A)) &= \sum_{(\gamma,\delta)\in\mathcal{N}_{r_s}} \left|(e_s W)_\gamma^\delta\right| \left|(e_s A)_\delta^\gamma\right| \\
&= e_s \sum_{(\gamma,\delta)\in\mathcal{N}_{r_s}} \left|(e_s W)_\gamma^\delta\right| \left|A_\delta^\gamma\right| = e_s \operatorname{Tr}(C_{r_s}(e_s W A)) \\
&= e_s \sum_{(\gamma,\delta)\in\mathcal{N}_{r_s}} \left|W_\gamma^\delta\right| \left|A_\delta^\gamma\right| = \sum_{(\gamma,\delta)\in\mathcal{N}_{r_s}} \left|(e_s W)_\gamma^\delta\right| \left|A_\delta^\gamma\right| \\
&= \operatorname{Tr}(C_{r_s}(e_s W A)).
\end{aligned}$$

Hence

$$\begin{aligned}
B_s &= \left[\operatorname{Tr}(C_{r_s}(e_s W A))\right]^{-1} C_{r_s}(e_s W) \\
&= \left[e_s \operatorname{Tr}(C_{r_s}(e_s W A))\right]^{-1} C_{r_s}(e_s W).
\end{aligned} \tag{4.4}$$

Now, from (4.3) and (4.4), using that e_s is an idempotent, we get

$$\begin{aligned}
g_{ij} &= \sum_{s=1}^{t-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_s}(j,i)} \left[\operatorname{Tr}(C_{r_s}(e_s W A))\right]^{-1} C_{r_s}(e_s W)_{\beta,\alpha} \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right| \\
&= \sum_{s=1}^{t-1} \left[N_{((e_s W)^*, r_s)}(A)\right]^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_s}(j,i)} \left|(e_s W)_\alpha^\beta\right| e_s \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right| \\
&= \sum_{s=1}^{t-1} \left[N_{((e_s W)^*, r_s)}(A)\right]^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_s}(j,i)} \left|(e_s W)_\alpha^\beta\right| \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right| \\
&= \sum_{s=1}^{t-1} \left[N_{((e_s W)^*, r_s)}(A)\right]^{-1} A_{ij}^{(\dagger,(e_s W)^*, r_s)}.
\end{aligned}$$

This implies

$$G = \sum_{s=1}^{t-1} A^{(\dagger, \quad (e_s W)^*, r_s)}.$$

Finally, it is not difficult to verify the following:

$$
\begin{aligned}
g_{ij} &= \sum_{s=1}^{t-1} [\mathrm{Tr}\,(C_{r_s}(e_sWe_sA))]^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_s}(j,i)} \left|(e_sW)_\alpha^\beta\right| \frac{\partial}{\partial(e_sA)_{ji}} \left|(e_sA)_\beta^\alpha\right| \\
&= \sum_{s=1}^{t-1} \left[N_{((e_sW)^*,r_s)}(e_sA)\right]^{-1} (e_sA)_{ij}^{(\dagger,(e_sW)^*,r_s)}
\end{aligned}
$$

which is equivalent to

$$
G = \sum_{s=1}^{t-1} (e_sA)^{(\dagger,(e_sW)^*,r_s)}. \qquad \square
$$

Remark 4.2. (i) As a motivation for the results of Theorem 4.2 we use the following facts. From Theorem 4.2, in view of Theorem 4.1, we get

$$
A^{(1,2)} = \sum_{s=1}^{t-1} (e_sA)^{(1,2)},
$$

which is a known result from [19].

(ii) If $\mathbb{R}$ does not contain zero divisors, then in the case $i_A = (1,0)$, $\rho_A = (r,0)$ the result of Theorem 4.2 reduces to the result of Theorem 4.1.

For the sake of underpinning of Theorem 4.2 we prove the following

Lemma 4.2. *Let the matrix A satisfy the conditions as in Theorem 4.2. Also, suppose that W is an $n \times m$ matrix of rank $r = \mathrm{rank}(A)$ and that the $W_s = e_sW$ are $n \times m$ matrices of rank r_s, $s = 1, \dots, t-1$. Then the matrices B_s, $s = 1, \dots, t-1$, defined by*

$$
B_s = [\mathrm{Tr}\,(C_{r_s}(e_sWA))]^{-1}\, C_{r_s}(e_sW), \quad s = 1, \dots, t-1
$$

satisfy the following conditions which are imposed in Theorem 3 from [19]:

(i) *B_s is an $\binom{n}{r_s} \times \binom{m}{r_s}$ matrix of rank 1 with elements in $e_s\mathbb{R}$;*

(ii) *$e_sB_s = B_s$;*

(iii) *$e_s = \mathrm{Tr}\,(B_s \cdot C_{r_s}(e_sA))$.*

Proof. (i): It is easy to see that B_s is an $\binom{n}{r_s} \times \binom{m}{r_s}$ matrix of rank 1. Also, using that e_s is idempotent, we get $C_{r_s}(e_sW) = e_sC_{r_s}(W)$, and

$$
B_s = e_s \cdot [e_s \cdot \mathrm{Tr}\,(C_{r_s}(WA))]^{-1}\, C_{r_s}(W).
$$

This means that the elements of the matrix B_s are in $e_s\mathbb{R}$.

(ii): Follows from the idempotency of e_s.

(iii): It is easy to verify the following:

$$\begin{aligned}\operatorname{Tr}(B_s \cdot C_{r_s}(e_sA)) &= \operatorname{Tr}\Big([\operatorname{Tr}(C_{r_s}(e_sWA))]^{-1} C_{r_s}(e_sW)\cdot C_{r_s}(e_sA)\Big)\\ &= [\operatorname{Tr}(C_{r_s}(e_sWA))]^{-1}\cdot e_s \operatorname{Tr}(C_{r_s}(e_sWA)) = e_s. \qquad \square\end{aligned}$$

Remark 4.3. In Lemma 4.2 it is shown that in Theorem 4.2 there is obtained an effective representation of the matrices B_s, which are used in generating the class of reflexive g-inverses in [19]. Also, in the cases $W = A$ and $W = A^*$, from (4.4) we obtain the well-known determinantal representations of the group inverse and of the Moore–Penrose inverse of A, respectively, introduced in Theorem 3 in [19].

5. Correlations between the minors

In this section, minors of generalized inverses of a given matrix A over $\mathbb{I}$ are expressed in terms of minors of the matrix A and minors of arbitrary selected matrices which satisfy exactly defined conditions.

Theorem 5.1. *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$, and $G = (g_{ij})$ be a reflexive g-inverse of A. Then there exists an $n \times m$ matrix W of rank r, such that*

$$\left|G^{\alpha}_{\beta}\right| = \left(N_{(W^*,r)}(A)\right)^{-1}\left|W^{\alpha}_{\beta}\right|, \quad \text{for all } (\alpha,\beta)\in\mathcal{N}.$$

Proof. If $G = (g_{ij})$ is a reflexive g-inverse of A, then, according to Theorem 4.1, we get

$$g_{ij} = \left(N_{(W^*,r)}(A)\right)^{-1}\sum_{(\alpha,\beta)\in\mathcal{N}(j,i)}\left|W^{\beta}_{\alpha}\right|\frac{\partial}{\partial a_{ji}}\left|A^{\alpha}_{\beta}\right|, \quad \begin{pmatrix}1\le i\le n\\ 1\le j\le m\end{pmatrix}.$$

Now the proof can be completed by comparison of this result with the following relation, proved in [1]:

$$g_{ij} = \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)}\left|G^{\beta}_{\alpha}\right|\frac{\partial}{\partial a_{ji}}\left|A^{\alpha}_{\beta}\right|, \quad \begin{pmatrix}1\le i\le n\\ 1\le j\le m\end{pmatrix}. \qquad \square$$

Corollary 5.1. *If A is a matrix of type $m \times n$ and rank r over $\mathbb{I}$, then for all $(\alpha, \beta) \in \mathcal{N}$ we have:*

$$\left|A^{\dagger}{}_{\beta}^{\alpha}\right| = \left(N_{(A,r)}(A)\right)^{-1} \left|A^{*}{}_{\beta}^{\alpha}\right|;$$
$$\left|A^{\#}{}_{\beta}^{\alpha}\right| = \left(\sum_{\gamma} \left|A_{\gamma}^{\gamma}\right|\right)^{-2} \left|A_{\beta}^{\alpha}\right| = \left(N_{(A^*,r)}(A)\right)^{-1} \left|A_{\beta}^{\alpha}\right|.$$

If M and N are invertible matrices of the order $m \times m$ and $n \times n$, respectively, then

$$\begin{aligned}\left|(A_{M,N}^{\dagger})_{\beta}^{\alpha}\right| &= \left(N_{(M^*A(N^{-1})^*,r)}(A)\right)^{-1} \left|(N^{-1}A^*M)_{\beta}^{\alpha}\right| \\ &= \left(N_{(MAN^{-1},r)}(A)\right)^{-1} \left|(MAN^{-1})^{*}{}_{\beta}^{\alpha}\right|.\end{aligned}$$

If A admits a full-rank factorization $A = PQ$, P^*P is an invertible matrix over $\mathbb{I}$ and W_1 is an arbitrary $n \times r$ matrix over $\mathbb{I}$, such that QW_1 is invertible over $\mathbb{I}$, then we obtain the following additional result:

$$\left|A^{(1,2,3)}{}_{\beta}^{\alpha}\right| = \left(N_{(PW_1^*,r)}(A)\right)^{-1} \left|(W_1P^*)_{\alpha}^{\beta}\right|.$$

If $A = PQ$ is a full-rank factorization of A, $W_2 \in \mathbb{I}^{r \times m}$ and QQ^*, W_2P are invertible matrices in $\mathbb{I}$, then we get

$$\left|A^{(1,2,4)}{}_{\beta}^{\alpha}\right| = \left(N_{(W_2^*Q,r)}(A)\right)^{-1} \left|(Q^*W_2)_{\alpha}^{\beta}\right|.$$

Remark 5.1. (i) It is a known result from [2] that the matrix $W \in \mathbb{I}^{n \times m}$, chosen so that $N_{(W^*,r)}(A) = \mathrm{Tr}\,(C_r(WA))$ is invertible, has minors proportional with an arbitrary g-inverse G of A. In Theorem 5.1 we show that in the case when G is a reflexive g-inverse of A, the minors $\left|G_{\beta}^{\alpha}\right|$ and $\left|W_{\beta}^{\alpha}\right|$ are proportional with the coefficient $\left(N_{(W^*,r)}(A)\right)^{-1}$ for all $(\alpha, \beta) \in \mathcal{N}$.

(ii) Correlations between the minors of A and the corresponding minors of the Moore–Penrose inverse and the group inverse of A are introduced (in another way) in [1] and [11].

Using the results of Theorem 5.1, Theorem 4.1 and Lemma 1.1 from [19], we give the following

Theorem 5.2. *Let A be an $m\times n$ matrix of rank r over a commutative ring $\mathbb{R}$. If A is regular then there exists an $n\times m$ matrix W over $\mathbb{R}$, such that $N_{(W^*,r)}(A)$ is an invertible element in $\mathbb{R}$ and $\left(N_{(W^*,r)}(A)\right)^2$ is an identity element in the ideal $\mathcal{C}_r(A)$.*

Proof. Suppose that G is a g-inverse of A. According to Theorem 4.1, there exists an $n\times m$ matrix W over $\mathbb{R}$, such that $N_{(W^*,r)}(A)$ is invertible. Applying the result of Theorem 5.1, we get

$$\begin{aligned} N_{(W^*,r)}(A) &= \sum_{(\alpha,\beta)\in\mathcal{N}} \left|W_\alpha^\beta\right| \left|A_\beta^\alpha\right| = \left(N_{(W^*,r)}(A)\right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}} \left|G_\alpha^\beta\right| \left|A_\beta^\alpha\right| \\ &= \left(N_{(W^*,r)}(A)\right)^{-1} \operatorname{Tr}\left(C_r(GA)\right) \end{aligned}$$

which means

$$\left(N_{(W^*,r)}(A)\right)^2 = \operatorname{Tr}\left(C_r(GA)\right).$$

According to Lemma 1.1 from [19], $\operatorname{Tr}\left(C_r(GA)\right)$ is an identity element in $\mathcal{C}_r(A)$, and the proof is completed. $\square$

In the category of finite matrices over a commutative ring $\mathbb{R}$ with 1 and involution we get the following

Corollary 5.1. *Let A be an $m\times n$ matrix over $\mathbb{R}$ with Rao index t, idempotents $i_A=(e_1,\dots,e_t)$ and ranks $\rho_A=(r_1,\dots,r_t)$. If A is regular, then the minors of generalized inverses of A satisfy the following relations:*

$$\begin{aligned} \left|A^{(1,2)}{}_\beta^{\alpha}\right| &= \sum_{s=1}^{t-1} \left(N_{((e_sW)^*,r_s)}(A)\right)^{-1} \left|(e_sW)_\beta^\alpha\right|; \\ \left|A^\dagger{}_\beta^\alpha\right| &= \sum_{s=1}^{t-1} \left(N_{(e_sA,r_s)}(A)\right)^{-1} \left|(e_sA)^*{}_\beta^\alpha\right|; \\ \left|A^\#{}_\beta^\alpha\right| &= \sum_{s=1}^{t-1} \left(N_{((e_sA)^*,r_s)}(A)\right)^{-1} \left|(e_sA)_\beta^\alpha\right|. \end{aligned}$$

$$\begin{aligned} \left|(e_sA)^{(1,2)}{}_\beta^\alpha\right| &= \left(N_{(e_sW)^*,r_s)}(A)\right)^{-1} \left|(e_sW)_\beta^\alpha\right|; \\ \left|(e_sA)^\dagger{}_\beta^\alpha\right| &= \left(N_{e_sA,r_s)}(A)\right)^{-1} \left|(e_sA)^*{}_\beta^\alpha\right|; \\ \left|(e_sA)^\#{}_\beta^\alpha\right| &= \left(N_{(e_sA)^*,r_s)}(A)\right)^{-1} \left|(e_sA)_\beta^\alpha\right|. \end{aligned}$$

PROOF. Follows from Theorem 4.2 and $\left|A^{(1,2)}{}^{\alpha}_{\beta}\right| = \sum_{s=1}^{t-1}\left|(e_sA)^{(1,2)}{}^{\alpha}_{\beta}\right|$. $\square$

6. Cramer-type solutions of linear systems

An explicit determinantal representation of the *Moore–Penrose solution* of an arbitrary linear system is derived in [5]. Using this representation, it is proved that the *Moore–Penrose solution* is a *convex combination* of the solutions of all uniquely solvable partial subsystems. In [4] there is derived an equivalent determinantal representation for the *least-squares solution* of an overdetermined linear system. From this formula, it is proved that the *least-squares solution* lies in the *convex hull* of the solutions to the square subsystems of the original system. Also, in [4] this result is extended, and it is proved that this geometric property holds for a more general class of problems which includes the *weighted least-squares* and the l_p norm minimization problems. In [12] and [13] there is derived a determinantal representation of the *weighted Moore–Penrose solution* of a given system of linear equations, for matrices over an integral domain.

In this paper we obtain an explicit determinantal representation for the solution of a linear system, by means of the general determinantal representation for generalized inverses.

Theorem 6.1. *Let be given a system of linear equations $Ax = z$, where $A \in \mathbb{I}_r^{m\times n}$ and x, z are vectors of the order n and m, respectively. Then a solution $x^{(1,2)} = A^{(1,2)}z$ of the linear system $Ax = z$ exists if and only if there exist constants $\lambda_{\alpha,\beta} \in \mathbb{I}$, $(\alpha,\beta) \in \mathcal{N}$, which satisfy the conditions* (1.1) *and* (1.2).

In this case an arbitrary ith component of the solution $x^{(1,2)}$ can be represented by the following determinantal formula:

$$x_i^{(1,2)} = \sum_{(\alpha,\beta):i\in\beta} \lambda_{\alpha,\beta}\left|A^{\alpha}_{\beta}\left(i \to {}_{\alpha}z\right)\right|.$$

PROOF. Clearly, $x^{(1,2)}$ exists if and only if $A^{(1,2)}$ exists, i.e. there exist constants $\lambda_{\alpha,\beta} \in \mathbb{I}$, $(\alpha,\beta) \in \mathcal{N}$, which satisfy the conditions (1.1) and (1.2). Starting from $x_i^{(1,2)} = \sum\limits_{k=1}^{m} (A^{(1,2)})_{ik} \cdot z_k$, we obtain

$$\begin{aligned} x_i^{(1,2)} &= \sum_{k=1}^{m} \sum_{(\alpha,\beta)\in\mathcal{N}(k,i)} \lambda_{\alpha,\beta} \frac{\partial}{\partial a_{ki}} \left|A_\beta^\alpha\right| z_k = \sum_{(\alpha,\beta)\in\mathcal{N}(k,i)} \lambda_{\alpha,\beta} \sum_{k=1}^{m} \frac{\partial}{\partial a_{ki}} \left|A_\beta^\alpha\right| z_k \\ &= \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} \lambda_{\alpha,\beta} \left|A\left(i \to z\right)_\beta^\alpha\right| = \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} \lambda_{\alpha,\beta} \left|A_\beta^\alpha\left(i \to {}_\alpha z\right)\right|. \quad \square \end{aligned}$$

Corollary 6.1. *Let A be a given $m \times n$ matrix of rank r, and let x, z be vectors of the order n and m, respectively. Then the solution $x^{(\dagger,W^*)} = A^{(\dagger,W^*)}z$ of a linear system $Ax = z$ exists if and only if there exists an $n \times m$ matrix W of rank r, such that $N_{(W^*,r)}(A)$ is invertible. In this case, an arbitrary ith component of the vector $x^{(\dagger,W^*)}$ can be represented by the following determinantal formula:*

$$x_i^{(\dagger,W^*)} = \left(N_{(W^*,r)}(A)\right)^{-1} \cdot \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} \left|W_\alpha^\beta\right| \left|A_\beta^\alpha\left(i \to {}_\alpha z\right)\right|.$$

Proof. The proof immediately follows from Theorem 6.1 and the result (4.2), derived in Theorem 4.1. □

Remark 6.1. In the case $W = N^{-1}A^*M$ or $W = (MAN^{-1})^*$, where M and N are appropriate nonsingular matrices, we obtain from Corollary 6.1 a result known from [12] and [13]. Similarly, for $W = A^*$ we get a known result from [5]. Also, in the case $W = A^*$, where A is a matrix of full column rank, the result of Corollary 6.1 yields a representation of the least square solution [4].

If a full-rank factorization of the matrix A is allowed, we obtain the following additional result:

Theorem 6.2. *Let $A = PQ$ be a rank factorization of $A \in \mathbb{I}_r^{m\times n}$ and let W_1, W_2 be matrices of the order $n\times r$ and $r\times m$, respectively. Consider a system $Ax = z$ of linear equations. Then the following conditions are equivalent:*

(i) *$x^{(\dagger,(W_1W_2)^*)} = A^{(\dagger,(W_1W_2)^*)}z$ exists.*

(ii) *QW_1 and W_2P are invertible matrices over $\mathbb{I}$.*

(iii) $N_{(W_1^*,r)}(Q)$ *and* $N_{(W_2^*,r)}(P)$ *are invertible matrices over* $\mathbb{I}$.

(iv) $N_{((W_1W_2)^*,r)}(A)$ *is an invertible matrix over* $\mathbb{I}$.

In the case when $x^{(\dagger,(W_1W_2)^*)}$ *exists, it can be represented by the following linear combination of the solutions of all uniquely solvable* $r \times r$ *subsystems of the system* $Ax = z$*:*

$$x^{(\dagger,(W_1W_2)^*)} = \sum_{(\alpha,\beta)\in\mathcal{N}:\, i\in\beta} p_\alpha q_\beta x^{\alpha,\beta},$$

where

$$p_\alpha = \left(N_{(W_2^*,r)}(P)\right)^{-1} |(W_2)_\alpha|\, |P^\alpha|\,,$$
$$q_\beta = \left(N_{(W_1^*,r)}(Q)\right)^{-1} \left|(W_1)^\beta\right| |Q_\beta|$$

and $x^{\alpha,\beta}$ *is the unique solution of the subsystem*

$$A_\beta^\alpha \cdot {}_\alpha x = {}_\alpha z.$$

Proof. It is clear that $x^{(\dagger,(W_1W_2)^*)}$ exists if and only if $A^{(\dagger,(W_1W_2)^*)}$ exists, which implies equivalence of the presented conditions (i)–(iv). It is not difficult to derive the following representation for $x^{(\dagger,(W_1W_2)^*)}$:

$$x_i^{(\dagger,(W_1W_2)^*)} = \left(N_{((W_1W_2)^*,r)}(A)\right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} \left|(W_1W_2)_\alpha^\beta\right| \left|A_\beta^\alpha\left(i \to {}_\alpha z\right)\right|.$$

Applying the results of Proposition 2.1 and Proposition 2.2, we get

$$\begin{aligned}
x_i^{(\dagger,(W_1W_2)^*)} &= \left(N_{(W_1^*,r)}(Q)\right)^{-1} \left(N_{(W_2^*,r)}(P)\right)^{-1} \\
&\quad\times \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} \left|(W_1)^\beta\right| \left|(W_2)_\alpha\right| \left|A_\beta^\alpha\left(i \to {}_\alpha z\right)\right| \\
&= \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} \left(N_{(W_1^*,r)}(Q)\right)^{-1} \left|(W_1)^\beta\right| \left|Q_\beta\right| \left(N_{(W_2^*,r)}(P)\right)^{-1} \\
&\quad\times \left|(W_2)_\alpha\right| \left|P^\alpha\right| \left|A_\beta^\alpha\right|^{-1} \left|A_\beta^\alpha\left(i \to {}_\alpha z\right)\right| \\
&= \sum_{(\alpha,\beta)\in\mathcal{N}:i\in\beta} p_\alpha q_\beta x_i^{\alpha,\beta}.
\end{aligned}$$

In the case $\left|A^{\alpha}_{\beta}\right| \neq 0$, it is easy to verify that

$$x_i^{\alpha,\beta} = \left|A^{\alpha}_{\beta}\right|^{-1} \left|A^{\alpha}_{\beta}\left(i \rightarrow {}_{\alpha}z\right)\right|$$

represents the unique solution of the $r \times r$ subsystem

$$A^{\alpha}_{\beta} \cdot {}_{\alpha}x = {}_{\alpha}z.$$

In the singular case $\left|A^{\alpha}_{\beta}\right| = 0$ we define $x^{\alpha,\beta}$ to be the zero vector. □

Remark 6.2. The identities $\sum\limits_{\alpha} p_{\alpha} = 1$ and $\sum\limits_{\beta} q_{\beta} = 1$ can easily be verified. In the case $W = A^*$, we conclude $p_{\alpha} \geq 0$ and $q_{\beta} \geq 0$, which implies that the Moore–Penrose solution of a linear system over an integral domain can be represented as the convex combination of the solutions of all uniquely solvable $r \times r$ subsystems.

Corollary 6.2. *Let be given a system of linear equations $Ax = z$, where A is an $m \times n$ matrix over a commutative ring $\mathbb{R}$ with Rao index t, idempotents $i_A = (e_1, \dots, e_t)$ and ranks $\rho_A = (r_1, \dots, r_t)$. Then an arbitrary solution $x^{(1,2)} = A^{(1,2)}z$ of the linear system $Ax = z$ exists if and only if $e_t A = 0$, and*

$$x^{(1,2)} = A^{(1,2)}z = \left(\sum_{s=1}^{t-1} A^{(\dagger,(e_sW)^*,r_s)}\right) z = \left(\sum_{s=1}^{t-1} (e_sA)^{(\dagger,(e_sW)^*,r_s)}\right) z$$

$$x^{\dagger} = A^{\dagger}z = \left(\sum_{s=1}^{t-1} A^{(\dagger,e_sA,r_s)}\right) z = \left(\sum_{s=1}^{t-1} (e_sA)^{(\dagger,e_sA,r_s)}\right) z,$$

where W is an arbitrary $n \times m$ matrix of rank r.

References

[1] R. B. Bapat, K. P. S. B. Rao and K. Manjunatha Prasad, Generalized inverses over integral domains, *Linear Algebra Appl.* **140** (1990), 181–196.

[2] R. B. Bapat, Generalized inverses with proportional minors, *Linear Algebra Appl.* **211** (1994), 27–35.

[3] A. Ben-Israel, A volume associated with $m \times n$ matrices, *Linear Algebra Appl.* **167** (1992), 87–111.

[4] A. Ben-Tal, A geometric property of the least squares solution of linear equations, *Linear Algebra Appl.* **139** (1990), 165–170.

[5] L. Berg, Three results in connection with inverse matrices, *Linear Algebra Appl.* **84** (1986), 63–77.

[6] R. E. Cline, Inverses of rank invariant powers of a matrix, *SIAM J. Numer. Anal.* **5 No 1** (1968), 182–197.

[7] V. N. Joshi, A determinant for rectangular matrices, *Bull. Austral. Math. Soc.* **21** (1980), 137–146.

[8] S. S. Kesler and N. J. Krupnik, On invertibility matrices with elements from the ring, *Uchennie Zapiski, Kishinjev* **91** (1967), 51–54. (in *Russian*)

[9] N. J. Krupnik, On normal solvability problem and singular and integral equations index, *Uchennie Zapiski, Kishinjev* **82** (1965), 3–7. (in *Russian*)

[10] J. Miao, Reflexive generalized inverses and their minors, *Linear and Multilinear Algebra* **35** (1993), 153–163.

[11] K. M. Prasad, K. P. S. B. Rao and R. B. Bapat, Generalized inverses over integral domains. II. Group inverses and Drazin inverses, *Linear Algebra Appl.* **146** (1991), 31–47.

[12] K. M. Prasad and R. B. Bapat, The Generalized Moore–Penrose inverse, *Linear Algebra Appl.* **165** (1992), 59–69.

[13] K. M. Prasad and R. B. Bapat, A note of the Khatri inverse, *Sankhya: Indian J. Stat.* **54** (1992), 291–295.

[14] K. M. Prasad, Generalized inverses of matrices over commutative ring, *Linear Algebra Appl.* **211** (1994), 35–53.

[15] M. Radić, A definition of the determinant of a rectangular matrix, *Glasnik Matematički* **1(21)-No. 1** (1966), 17–22.

[16] M. Radić, Some Contributions to the inversions of rectangular matrices, *Glasnik Matematički* **1** (21) no. 1 (1966), 23–37.

[17] C. R. Rao and S. K. Mitra, Generalized inverse of matrices and its applications, *John Wiley & Sons Inc.*, *New York, London, Sydney, Toronto*, 1971.

[18] K. P. S. B. Rao, On generalized inverses of matrices over integral domains, *Linear Algebra Appl.* **49** (1983), 179–189.

[19] D. W. Robinson, The determinantal rank idempotents of a matrix, *Linear Algebra Appl.* **237/238** (1996), 83–96.

[20] P. Stanimirović and M. Stanković, Determinantal representation of generalized inverses over integral domains, *Filomat* **9:3** (1995), 931–940.

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