

Semi-continuous mappings and fixed point theorems in quasi metric spaces

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Abstract. In this paper a class of selfmaps on quasi-metric spaces which satisfy the contractive definition (A), or (B), or (C) below are investigated and general common fixed and periodic point theorems are proved. These theorems generalize and extend the fixed point theorem of DOWING and Kirk [7] and a great number of known generalizations of CARISTI's Theorem [3]. Two examples are constructed to show the generality of the given theorems.

1. Introduction

Let X be a non-void set and $T : X \rightarrow X$ a selfmap. A point $x \in X$ is called a periodic point for T if there exists a positive integer k such that $T^k x = x$. If $k = 1$, then x is called a fixed point for T . Suppose X and Y are topological spaces and $S : X \rightarrow Y$ a mapping. S is said to be a closed mapping if for $\{x_n\} \subseteq X$ the conditions $x_n \rightarrow x$ and $Sx_n \rightarrow y$ imply $Sx = y$.

Let \mathbb{R}^+ denote the set of non-negative real numbers. D. DOWING and W. A. KIRK [7] proved and applied to non-linear mapping theory the following theorem which reduces to the theorem of J. CARISTI [3] in the case that $X = Y$, S is the identity mapping, and $c = 1$.

Theorem (Downing–Kirk [7]). *Let X and Y be complete metric spaces and $T : X \rightarrow X$ an arbitrary mapping. Suppose there exists a closed*

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mapping $S : X \rightarrow Y$, a lower semi-continuous mapping $\Phi : S(X) \rightarrow \mathbb{R}^+$, and a constant $c > 0$ such that for each $x \in X$,

$$(DK) \quad \max\{d(x, Tx), c \cdot d(Sx, STx)\} \leq \Phi Sx - \Phi STx.$$

Then T has a fixed point.

Caristi's or Caristi–Kirk–Browder theorem is known to be essentially equivalent to a theorem stated earlier by EKELAND [9]. On account of the generality and applicability of Caristi's fixed point theorem, many authors established results which generalize or revise that result ([1], [4–7], [9–19], [23–24], [26]).

The purpose of this paper is to investigate a class of selfmappings on quasi-metric spaces which satisfy the contractive condition (A) below, which is more general than the contractive condition (DK). We introduce a concept of weak lower semi-continuity (definition 1 below) and apply this concept to a governing function Φ , which appears in contractive definitions of Caristi and Kirk type. First we prove a Lemma, which then is used in fixed point and periodic point theorems. We prove common periodic point and fixed point theorems for mappings which satisfy the contractive definition (A), and a stationary theorem for set-valued (non-self) mappings which satisfy the contractive definition (C) below. The results of this paper generalize the results of DOWNING and KIRK [7] and a great number of known generalizations or modifications of Caristi's fixed point theorem ([6], [11–17], [19], [23]). Two examples are constructed to show that our results are proper generalizations of these results.

There are many papers which are related to the proof of Caristi's Theorem ([14], [18], [24], [26]). Our method used in this note is somewhat different from those.

2. Main results

Recall that a real valued function Φ , defined on a topological space X , is said to be lower semi-continuous (l.s.c.) at x in X iff $\{x_\lambda\}$ is a net in X and $\lim x_\lambda = x$ implies $\Phi x \leq \liminf \Phi x_\lambda$. Now we shall slightly generalize this concept.

Definition 1. A real-valued function Φ defined on a topological space X is said to be *weak lower semi-continuous* (w.l.s.c.) at $x \in X$, iff $\{x_\lambda\}$ is a net in X and

$$\lim x_\lambda = x \quad \text{implies} \quad \Phi x \leq \limsup \Phi x_\lambda.$$

A mapping Φ is said to be w.l.s.c. on X iff it is w.l.s.c. at every $x \in X$.

Clearly, l.s.c. functions are w.l.s.c., but the implication is not reversible.

Throughout this paper, \mathbb{N} denotes the set of positive integers and \mathbb{R}^+ the set of non-negative real numbers. Since many (pathological) quasi-metric spaces are not metrizable, all our main results are stated for quasi-metric spaces.

A pair (X, d) of a set X and a mapping d from $X \times X$ into real numbers is said to be a *quasi-metric space* iff for all $x, y, z \in X$:

$$1^\circ \quad d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \quad \text{iff} \quad x = y,$$

$$2^\circ \quad d(x, z) \leq d(x, y) + d(y, z).$$

Let $d_x : X \rightarrow [0, +\infty)$ be defined by $d_x(y) = d(x, y)$.

A sequence $\{x_n\}$ in X is said to be a *left k -Cauchy sequence* if for each $k \in \mathbb{N}$ there is an N_k such that $d(x_n, x_m) < 1/k$ for all $m \geq n \geq N_k$. A quasi-metric space is *left k -sequentially complete* if each left k -Cauchy sequence is convergent (compare [20], [22], [25]).

We note that if a real-valued function G , defined on X by $G(x) = d(x, Tx)$, is (weak) lower semi-continuous, then any hypothesis of continuity of a function Φ in fixed point theorems of Caristi–Kirk type can be omitted (see [1], [5]).

Now we are able to state the following

Lemma. *Let (X, d) and (Y, ρ) be left k -sequentially complete quasi-metric spaces such that for each x in X the mapping $u \rightarrow d(x, u)$ is continuous on X and for each y in Y the mapping $v \rightarrow \rho(y, v)$ is continuous on Y . If there exists a closed mapping $S : X \rightarrow Y$ and a w.l.s.c. mapping $\Phi : S(X) \rightarrow \mathbb{R}^+$, then for each x in X the set*

$$P(x) = \{z \in X \mid \max\{d(x, z), \rho(Sx, Sz)\} \leq \Phi Sx - \Phi Sz\}$$

has the following properties:

$$1^\circ. \quad P(z) \subseteq P(x) \text{ for each } z \in P(x),$$

$$2^\circ. \quad P(p) = \{p\} \text{ for some } p \in P(x).$$

PROOF. The set $P(x)$ is nonempty, as $x \in P(x)$. Now we shall show
 1° . Let $z \in P(x)$, and let $y \in P(z)$. Then

$$(1) \quad \max\{d(x, z), \rho(Sx, Sz)\} \leq \Phi Sx - \Phi Sz,$$

$$(2) \quad \max\{d(z, y), \rho(Sz, Sy)\} \leq \Phi Sz - \Phi Sy,$$

Using the triangle inequality and (1) and (2) we get

$$(3) \quad d(x, y) \leq d(x, z) + d(z, y) \leq (\Phi Sx - \Phi Sz) + (\Phi Sz - \Phi Sy) = \Phi Sx - \Phi Sy$$

and similarly,

$$(4) \quad \rho(Sx, Sy) \leq \Phi Sx - \Phi Sy.$$

From (3) and (4) we have $y \in P(x)$. Therefore, we proved 1° .

Now we show 2° . For any x in X set

$$a(x) = \inf\{\Phi Sz : z \in P(x)\}.$$

Let x in X be arbitrary. We shall choose a sequence $\{x_n\}$ in $P(x)$ as follows: when $x = x_1, x_2, \dots, x_n$ have been chosen, choose $x_{n+1} \in P(x_n)$ such that $\Phi Sx_{n+1} \leq a(x_n) + 1/n$. Thus, one obtains a sequence $\{x_n\}$ such that

$$(5) \quad \max\{d(x_n, x_{n+1}), \rho(Sx_n, Sx_{n+1})\} \leq \Phi Sx_n - \Phi Sx_{n+1},$$

$$(6) \quad \Phi Sx_{n+1} - 1/n \leq a(x_n) \leq \Phi Sx_{n+1}.$$

By (5) $\{\Phi Sx_n\}$ is a non-increasing sequence of reals and so it converges. Therefore, by (6) there is some $a \geq 0$ such that

$$(7) \quad a = \lim_n a(x_n) = \lim_n \Phi Sx_n.$$

Let now $k \in \mathbb{N}$ be arbitrary. By (7) there exists some N_k such that $\Phi Sx_n < a + 1/k$ for all $n \geq N_k$. Thus, by monotony of $\{\Phi Sx_n\}$, for $m \geq n \geq N_k$ we have $a \leq \Phi Sx_m \leq \Phi Sx_n < a + 1/k$ and hence

$$(8) \quad \Phi Sx_n - \Phi Sx_m < 1/k \quad \text{for all } m \geq n \geq N_k.$$

From the triangle inequality, (5) and (8) we get (for all $m \geq n \geq N_k$)

$$(9) \quad d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \Phi Sx_n - \Phi Sx_m < 1/k.$$

Similarly,

$$(10) \quad \rho(Sx_n, Sx_m) \leq \sum_{i=1}^{m-1} \rho(Sx_i, Sx_{i+1}) \leq \Phi Sx_n - \Phi Sx_m < 1/k.$$

Therefore, $\{x_n\}$ is a left k -Cauchy sequence in X , and $\{Sx_n\}$ is a left k -Cauchy sequence in Y . By completeness there exist $p \in X$ and $q \in Y$ such that $x_n \rightarrow p$ and $Sx_n \rightarrow q$. Since S is a closed mapping, $Sp = q$. Since Φ is w.l.s.c., by (7) we have

$$(11) \quad \Phi Sp \leq \lim_n \sup \Phi Sx_n = a.$$

From (9) and (10),

$$\Phi Sx_m \leq \Phi Sx_n - \max\{d(x_n, x_m), \rho(Sx_n, Sx_m)\}.$$

Since Φ is w.l.s.c. on SX and $u \rightarrow d(x, u)$ on X and $v \rightarrow d(y, v)$ on Y are continuous, we have

$$\begin{aligned} \Phi Sp &\leq \lim_m \sup \Phi Sx_m \\ &\leq \Phi Sx_n + \lim_m \sup [-\max\{d(x_n, x_m), \rho(Sx_n, Sx_m)\}] \\ &\leq \Phi Sx_n - \lim_m \inf [\max\{d(x_n, x_m), \rho(Sx_n, Sx_m)\}] \\ &\leq \Phi Sx_n - \max\{d(x_n, p), \rho(Sx_n, Sp)\}. \end{aligned}$$

Hence

$$(12) \quad \max\{d(x_n, p), \rho(Sx_n, Sp)\} \leq \Phi Sx_n - \Phi Sp.$$

From (12) it follows that $p \in P(x_n)$ and hence

$$a(x_n) \leq \Phi Sp \quad \text{for every } n \in \mathbb{N}.$$

Taking the limit when n tends to infinity we have

$$(13) \quad \lim_{n \rightarrow \infty} a(x_n) \leq \Phi Sp.$$

From (7), (11) and (13),

$$(14) \quad \Phi Sp = a.$$

Since $p \in P(x_n)$ and $x_n \in P(x)$ for each $n \in \mathbb{N}$, by property 1° we have $p \in P(x)$.

Suppose now that $p_1 \in P(p)$ and $p_1 \neq p$. Then $\Phi Sp_1 < \Phi Sp$, or by (14), $\Phi Sp_1 < a$. Since $p \in P(x_n)$, by 1° we have $P(p) \subseteq P(x_n)$. Hence $p_1 \in P(x_n)$. Thus

$$a(x_n) \leq \Phi Sp_1 \quad \text{for every } n \in \mathbb{N}.$$

Taking the limit when n tends to infinity we get

$$a \leq \Phi Sp_1.$$

This is in contradiction with $\Phi Sp_1 < p$. Therefore, $p_1 = p$ which proves 2°.

Now we shall use the Lemma to prove the following periodic point and fixed point theorems.

Theorem 1. *Let (X, d) and (Y, ρ) be complete left k -sequentially quasi-metric spaces and F a family of arbitrary mappings $T : X \rightarrow X$. Suppose there exist a closed mapping $S : X \rightarrow Y$ and a w.l.s.c. mapping $\Phi : S(X) \rightarrow \mathbb{R}^+$ such that for each $x \in X$ and each $T \in F$:*

$$(A) \quad \max\{d(x, T^n x), \rho(Sx, ST^n x)\} \leq \Phi Sx - \Phi ST^n x,$$

where $n = n(x, T)$ is a positive integer. If for each x in X the mapping $u \rightarrow d(x, u)$ on X is continuous, and for each y in Y the mapping $v \rightarrow d(y, v)$ on Y is continuous, then F has a common periodic point.

PROOF. Condition (A) means that for each x in X and T in F , $T^n x$ is in $P(x)$. By the Lemma there exists some p in X such that $P(p) = \{p\}$. So it follows from (A) with $x = p$ that $T^{n(p, T)} p \in P(p) = \{p\}$ for every $T \in F$. Therefore, $T^{n(p, T)} p = p$ for each $T \in F$, that is p is a common periodic point of F . \square

Corollary 1. *Theorem 1 holds with inequality (A') below in the place of inequality (A):*

$$(A') \quad \max\{d(x, T^n x), \rho(Sx, ST^n x)\} \leq \Phi Sx - \Phi STx.$$

PROOF. From (A'), $\Phi STx \leq \Phi Sx$ for each $x \in X$. Hence $\Phi ST^2x \leq \Phi STx$, and so on. Therefore, $\{\Phi ST^n x\}_{n=0}^{\infty}$ is a non-increasing sequence of reals. Hence

$$\Phi Sx - \Phi STx \leq \Phi Sx - \Phi ST^{n(x, Tx)}x.$$

Therefore, (A') implies (A). □

Remark 1. Example 1 below shows that a periodic point in Theorem 1 need not be a fixed point. Therefore, one must add some hypotheses in order to ensure that F possesses a common fixed point.

Theorem 2. *A family F in Theorem 1 possesses a common fixed point if in addition F satisfies the following condition: for any $T \in F$:*

$$(15) \quad x \neq Tx \quad \text{implies} \quad \Phi ST^k x < \Phi Sx,$$

where $k = k(x, T)$ is a positive integer.

PROOF. By Theorem 1 there is p in X such that $T^{n(p, T)}p = p$. Then for any fixed $T \in F$ the orbit $O(p) = O(p, T)$ is a finite set of points in X . Let $y \in O(p)$ be such that

$$(16) \quad \Phi Sy = \min\{\Phi Sz : z \in O(p)\}.$$

Assume that $y \neq Ty$. Since $T^k y \in O(p)$ for all $k \in N$, from (16) we have that $\Phi ST^k y \geq \Phi Sy$ for all $k \in N$, which contradicts (15). Therefore, $Ty = y$. Hence $p = y$ as $p \in O(y, T)$. □

Corollary 2. *Let (X, d) and (Y, ρ) be left k -complete metric spaces and F a family of arbitrary mappings $T : X \rightarrow X$. Suppose there exist a closed mapping $S : X \rightarrow Y$, and a w.l.s.c. mapping $\Phi : S(X) \rightarrow \mathbb{R}^+$, such that for each $x \in X$ and $T \in F$ the conditions (A) and (15) hold. Then F has a common fixed point.*

Remark 2. We pointed out that if $F = \{T\}$ is a singleton and T satisfies (A) with $n(x, T) = 1$ for all $x \in X$, then (A) implies the condition (15) with $k(x, T) = 1$. Therefore DOWNING–KIRK's Theorem [7] is a special case of Corollary 2, if in that theorem Φy is replaced by $\max\{c, 1/c\}\Phi y$.

Remark 3. Example 2 below shows that Corollary 2 is a proper generalization of DOWNING–KIRK's Theorem [7].

Corollary 3. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping. Suppose there exist a w.l.s.c. mapping $\Phi : X \rightarrow \mathbb{R}^+$ and mappings $n, k : X \rightarrow \mathbb{N}$ such that for each $x \in X$ the conditions*

$$(B) \quad d(x, T^{n(x)}x) \leq \Phi x - \Phi T^{n(x)}x$$

and 15) hold. Then T has a fixed point.

PROOF. Corollary 3 is a special case of Corollary 2 in the case that $Y = X$, S is the identity mapping and $F = \{T\}$ is a singleton. \square

Corollary 4. *Suppose that $T : X \rightarrow X$ and $\Phi : X \rightarrow \mathbb{R}^+$, where X is a left k -complete quasi-metric space and Φ is w.l.s.c. If for each $x \in X$,*

$$(B') \quad d(x, Tx) \leq \Phi x - \Phi Tx$$

and a mapping $u \rightarrow d(x, u)$ on X is continuous, then T has a fixed point.

PROOF. Corollary 4 is a special case of Theorem 2 in the case that $F = \{T\}$ is a singleton, $Y = X$, S is the identity mapping, $n(x)=k(x) = 1$. \square

The following theorem is an extension of Downing–Kirk's Theorem to multi-valued mappings and contains the main theorems of [13], [17] and [23] as corollaries.

Theorem 3. *Let (X, d) and (Y, ρ) be left k -complete quasi-metric spaces, A a closed subset of X and $F : A \rightarrow 2^X$ a set-valued mappings such that Fx is non-empty for all $x \in X$. Suppose that there exist a closed mapping $S : X \rightarrow Y$ and a w.l.s.c. mapping $\Phi : SX \rightarrow \mathbb{R}^+$ such that for each $x \in A$ with $\{x\} \neq Fx$, there exists $z = z(x) \in A \setminus \{x\}$ such that*

$$(C) \quad \max\{d(x, z), d(Sx, Sz)\} \leq \Phi Sx - \Phi Sz.$$

If for each $x \in X$ and $y \in Y$ the mappings $u \rightarrow d(x, u)$ on X and $v \rightarrow d(y, v)$ on Y are continuous, then F has a stationary point $\xi \in A$, that is, $F\xi = \{\xi\}$.

PROOF. Suppose $\{x\} \neq Fx$ for each $x \in A$. Consider a choice function $T : A \rightarrow A$ such that $Tx = z$, where $z = z(x)$ satisfies (C). Then T satisfies all assumptions of Theorem 2 in case F is a singleton with $n(x) = k(x) = 1$. Consequently, there is some $\xi \in A$ such that $T\xi = \xi$. This implies that $\{\xi\} = F\xi$, for otherwise by hypothesis there is $z = z(\xi) \in A \setminus \{\xi\}$, such that $T\xi = z(\xi) \neq \xi$, contradicting $T\xi = \xi$. Therefore $F\xi = \{\xi\}$, which completes the proof. \square

Remark 4. In [25] an example is presented to show that there are quasi-metric spaces which are not metrizable, although $y \rightarrow d(x, y)$ is continuous for each $x \in X$.

3. Examples

Example 1. Let $X = [-3, -1] \cup [1, 3]$ and $Y = [1, 2]$ have the usual metric and $F = \{T\}$ be a singleton. Define $T : X \rightarrow X$ by $Tx = -x$ and $S : X \rightarrow Y$ by $Sx = \frac{1}{2} + \frac{x}{2}$ if $x > 0$ and $Sx = \frac{1}{2} - \frac{x}{2}$ if $x < 0$. Then T satisfies (A) with $n(x) = 2$ for any (continuous) function $\Phi : Y \rightarrow \mathbb{R}^+$ but T has no fixed point. Each point in X is a periodic point for T .

Example 2. Let $X = \{0\} \cup \{\pm 1/n : n = 1, 2, \dots\}$ and $Y = \mathbb{R}$ have the usual metric. Let $F = \{T\}$ be a singleton and define $T : X \rightarrow X$ by $T(1/n) = -1/(n + 1)$, $T(-1/n) = 1/(n + 1)$ and $T(0) = 0$. Define $S : X \rightarrow Y$ by $Sx = \frac{x}{1+2|x|}$ and $\Phi : Y \rightarrow \mathbb{R}^+$ by $\Phi(y) = 4 \left(|y| + \frac{|y|}{1+|y|} \right)$. Then for $x = \pm \frac{1}{n}$ we have:

$$d(x, Tx) = \frac{1}{n} + \frac{1}{n+1}; \quad d(x, T^2x) = \frac{1}{n} - \frac{1}{n+2},$$

$$d(Sx, STx) = \frac{1}{n+2} + \frac{1}{n+3}; \quad d(Sx, ST^2x) = \frac{1}{n+2} - \frac{1}{n+4}.$$

Hence

$$\max\{d(x, T^2x)d(Sx, ST^2x)\} = d(x, T^2x),$$

$$\begin{aligned} d(x, T^2x) &= \frac{1}{n} - \frac{1}{n+2} < 4 \left(\frac{1}{n+2} + \frac{1}{n+3} \right) - 4 \left(\frac{1}{n+4} + \frac{1}{n+5} \right) \\ &= \Phi(Sx) - \Phi(ST^2x). \end{aligned}$$

Since $\Phi ST^2x < \Phi Sx$ for each $x \neq 0$, we conclude that for every $x \in X$, T satisfies (A) with $n(x) = 2$ and (15) with $k(x) = 2$. As X and Y are complete metric spaces and $\Phi(y) = 4 \left(|y| + \frac{|y|}{1+|y|} \right)$ is continuous on Y , we conclude that Corollary 2 can be applied and $x = 0$ is a fixed point.

To show that Downing–Kirk’s theorem is not applicable, we establish that there does not exist a function $\Phi : Y \rightarrow \mathbb{R}^+$ such that T satisfies (A) with $n(x) = 1$ for all $x \in X$. Similarly as in [8] it can be shown that such a function exists if and only if the series $\sum_{n=0}^{\infty} \max\{d(T^n x, T^{n+1} x), d(ST^n x, ST^{n+1} x)\}$ converges for all $x \in X$. Since in our example for any fixed $x = \pm 1/m_0$ we have

$$\begin{aligned} \max\{d(T^n x, T^{n+1} x), d(ST^n x, ST^{n+1} x)\} &= d(T^n x, T^{n+1} x), \\ d(T^n x, T^{n+1} x) &= \frac{1}{n+m_0} + \frac{1}{n+3+m_0} > \frac{2}{n+3+m_0}, \end{aligned}$$

we conclude that the series diverges and hence there is no function $\Phi : Y \rightarrow \mathbb{R}^+$ such that (A) holds with $n(x) = 1$ for any $x \neq 0$ in X .

We note that there are examples in which $\{n(x) : x \in X\}$ must be unbounded.

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