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On Nash stationary points

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Abstract. In this paper we introduce the notions of weak and strong Nash stationary points. It is shown that the Nash equilibrium points are always stationary points in both senses. Under convexity assumptions the converse can also be stated. Therefore, in numerical examples, equilibrium points can be determined after computing the stationary points.

One of the main results of the paper shows that weak stationary points always exist for a large class of functions.

1. Introduction

The notion of Nash equilibrium point has turned out to be very useful in game theoretical and economical applications. It has attracted much attention and it has been the subject of several investigations ([1], [3], [8]). In the most general context, this concept can be defined as follows. Let $K_1, \dots, K_n$ $(n \geq 2)$ be nonempty sets, and $f_i : K_1 \times \dots \times K_n \to \mathbb{R}$ $(i = 1, \dots, n)$ be given functions. A point $(x_1, \dots, x_n) \in K_1 \times \dots \times K_n$ is called a *Nash equilibrium point* if

$$f_i(x_1, \dots, x_i, \dots, x_n) \geq f_i(x_1, \dots, y_i, \dots, x_n), \qquad \forall\, y_i \in K_i,$$

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holds for $i = 1, \dots, n$.

We recall now the important existence theorem due to Nash ([4], [5]), (see also [1, p. 351, Thm. 23]).

Theorem 1 (Nash Equilibrium Theorem). *Let $K_1, \dots, K_n$ be nonempty compact convex subsets of Hausdorff topological vector spaces, and let $f_i : K_1 \times \dots \times K_n \to \mathbb{R}$ $(i = 1, \dots, n)$ be continuous functions such that*

$$y_i \longrightarrow f_i(x_1, \dots, y_i, \dots, x_n), \qquad (y_i \in K_i)$$

is quasiconcave for all fixed $x_j \in K_j$ $(j \neq i)$. Then there exists a Nash equilibrium point.

The above result offers a sufficient condition for the existence of an equilibrium point, however equilibrium points may exist even if the conditions of the theorem are not satisfied. Clearly, it is also important to find general methods that provide tools for determining these points.

In the theory of extremal problems it is usual to look for the extremum points among the stationary points. These latter points are defined by the property that a derivative vanishes or is nonnegative at the points in question. This approach motivates that the notion of Nash stationary point should be properly defined so that all equilibrium points are stationary points.

The first goal of this paper is to introduce two kinds of stationarity: weak and strong. In Section 2, we define a partial directional derivative and with its help the concept of strong stationary point. We also show that all equilibrium points are stationary in this sense. The assumptions in the above context are quite general, since we require neither the convexity and compactness of the underlying sets K_i, nor the concavity of the functions f_i in the ith variable. In this generality, the existence of strong stationary points cannot be stated. For this reason, in the next section, we introduce another notion: the weak stationarity. It will turn out that, assuming the convexity and the compactness of K_i, the existence of weak stationary points can be verified for a large class of functions f_i. The proof of this existence theorem is based upon the local Ky Fan inequality due to G. Kassay and Zs. Páles [6].

In the last section we consider a numerical example with one real parameter, where the Nash Equilibrium Theorem cannot be applied. We first determine the Nash stationary points, and then, we select those which are also Nash equilibrium points.

2. Strong stationary points

Let X be a real normed space, and let $K \subset X$ and $x \in X$ be fixed. We recall the notions of two tangent cones. The cone $S_K(x)$ consists of vectors $h \in X$ such that

$$\exists\, \varepsilon_k \to 0^+ : x + \varepsilon_k h \in K, \qquad \forall\, k \in \mathbb{N}.$$

Another important tangent cone is the *contingent cone* $T_K(x)$ whose elements are defined in the following way: $h \in T_K(x)$ if and only if

$$\exists\, \varepsilon_k \to 0^+, \qquad \exists\, h_k \to h : x + \varepsilon_k h_k \in K, \quad \forall\, k \in \mathbb{N}.$$

If K is a convex set, then one can easily see that

$$S_K(x) = \{\lambda(y-x) \mid y \in K\} = \mathrm{cone}(K-x)$$

and

$$T_K(x) = \overline{\mathrm{cone}}(K-x) = \mathrm{cl}(S_K(x)),$$

that is, $S_K(x)$ and $T_K(x)$ are the conical and closed conical hulls of $K-x$.

Let $K_1, \dots, K_n$ be nonempty subsets of real normed spaces $X_1, \dots, X_n$, respectively, and $f : K_1 \times \cdots \times K_n \to \mathbb{R}$. Define, for $(x_1, \dots, x_n) \in K_1 \times \cdots \times K_n$,

$$D_i^s f(x_1, \dots, x_n)(h_i) = \limsup_{\substack{\varepsilon \to 0^+ \\ x_i + \varepsilon h_i \in K_i}} \frac{f(x_1, \dots, x_i + \varepsilon h_i, \dots, x_n) - f(x_1, \dots, x_n)}{\varepsilon},$$

where $h_i \in S_{K_i}(x_i)$, $(i = 1, \dots, n)$. (We note that by the definition of $S_{K_i}(x_i)$, the *lim sup* is well defined).

If $f_i : K_1 \times \cdots \times K_n \to \mathbb{R}$, $(i = 1, \dots, n)$, then the point $(x_1, \dots, x_n) \in K_1 \times \cdots \times K_n$ is called a *strong Nash stationary point* for the functions $f_1, \dots, f_n$ if

$$D_i^s f_i(x_1, \dots, x_n)(h_i) \leq 0, \qquad \forall\, h_i \in S_{K_i}(x_i)$$

holds for $i = 1, \dots, n$.

Proposition 1. *If $K_1, \dots, K_n$ are nonempty subsets in normed spaces, $f_i : K_1 \times \dots \times K_n \to \mathbb{R}$, and $(x_1, \dots, x_n) \in K_1 \times \dots \times K_n$ is a Nash equilibrium point, then it is also a strong Nash stationary point.*

Proof. Let $i \in \{1, \dots, n\}$ and $h_i \in S_{K_i}(x_i)$ be fixed. There exists a sequence $\varepsilon_k \to 0^+$ such that

$$x_i + \varepsilon_k h_i \in K_i, \qquad \forall\, k \in \mathbb{N}$$

and

$$D_i^s f_i(x_1, \dots, x_n)(h_i) = \lim_{k\to\infty} \frac{f_i(x_1, \dots, x_i+\varepsilon_k h_i, \dots, x_n) - f_i(x_1, \dots, x_n)}{\varepsilon_k}.$$

Moreover, $(x_1, \dots, x_n)$ being an equilibrium point, we have

$$f_i(x_1, \dots, x_i + \varepsilon_k h_i, \dots, x_n) \le f_i(x_1, \dots, x_n)$$

for all $k \in \mathbb{N}$. Therefore,

$$D_i^s f_i(x_1, \dots, x_n)(h_i) \le 0. \qquad \square$$

Now we prove the converse of the above statement by taking convexity assumptions.

Proposition 2. *Let $K_1, \dots, K_n$ be convex sets and $f_i : K_1 \times \dots \times K_n \to \mathbb{R}$ $(i = 1, \dots, n)$ be functions such that*

$$y_i \to f_i(x_1, \dots, y_i, \dots, x_n) \qquad (y_i \in K_i)$$

is concave for all fixed $x_j \in K_j$, $(j \neq i)$. Then every strong Nash stationary point of the functions $f_1, \dots, f_n$ is also a Nash equilibrium point for the same system of functions.

Proof. Let $(x_1, \dots, x_n)$ be a strong Nash stationary point for the functions $f_1, \dots, f_n$, let $i \in \{1, \dots, n\}$ be fixed and let $h_i = y_i - x_i$. Clearly, $x_i + \varepsilon h_i \in K_i$ for all $0 \le \varepsilon \le 1$ by the convexity of K_i. Using the concavity of f_i in its ith variable, we have, for $0 < \varepsilon \le 1$

$$\frac{f_i(x_1, \dots, x_i + \varepsilon h_i, \dots, x_n) - f_i(x_1, \dots, x_n)}{\varepsilon}$$
$$\ge f_i(x_1, \dots, x_i + h_i, \dots, x_n) - f_i(x_1, \dots, x_n).$$

Therefore, taking the *limsup*, we get

$$0 \geq D_i^s f_i(x_1, \dots, x_n)(h_i) \geq f_i(x_1, \dots, x_i + h_i, \dots, x_n) - f_i(x_1, \dots, x_n),$$

which proves the statement. □

3. Weak stationary points

Let $X_1, \dots, X_n$ be normed spaces and $K_i \subseteq X_i$, $(i = 1, \dots, n)$ be nonempty compact and convex sets. Denote $K_1 \times \cdots \times K_n$ by K. A real valued function f will belong to the class $\mathcal{L}_i(K)$ if the following two properties hold:

(i) There exists a convex open set D_i such that $K_i \subseteq D_i \subseteq X_i$ and f is defined on $K_1 \times \cdots \times D_i \times \cdots \times K_n$.

(ii) For all $(x_1, \dots, x_n) \in K$, there exists $L > 0$ and a neighbourhood U of $(x_1, \dots, x_n)$ such that

$$\|f(y_1, \dots, y_i', \dots, y_n) - f(y_1, \dots, y_i'', \dots, y_n)\| \leq L\|y_i' - y_i''\|$$

if $(y_1, \dots, y_i', \dots, y_n), (y_1, \dots, y_i'', \dots, y_n) \in U \cap (K_1 \times \cdots \times D_i \times \cdots \cdots \times K_n)$.

(iii) f is continuous on $K_1 \times \cdots \times K_n$.

For a function f in $\mathcal{L}_i(K)$, we introduce the *weak partial directional derivative* $D_i^w f$ as follows

$$D_i^w f(x_1, \dots, x_n)(h_i) := \liminf_{\substack{\varepsilon \to 0^+ \\ y_j \to x_j (\forall j) \\ y_j \in K_j\ (\forall j \neq i)}} \frac{f(y_1, \dots, y_i + \varepsilon h_i, \dots, y_n) - f(y_1, \dots, y_n)}{\varepsilon}$$

for $h_i \in X_i$.

The following statement summarizes the most important properties of $D_i^w f$.

Lemma 1. *Let $f \in \mathcal{L}_i(K)$. Then*

(i) *For each $(x_1, \dots, x_n) \in K$, $h_i \to D_i^w f(x_1, \dots, x_n)(h_i)$ is a superlinear Lipschitz continuous function on X_i.*

(ii) *The function* $(x_1, \dots, x_n, h_i) \to D_i^w f(x_1, \dots, x_n)(h_i)$ *is lower semicontinuous on* $K \times X_i$.

The proof of this lemma is analogous to that of Lemma 1 in [6].

Let $f_1 \in \mathcal{L}_1(K), \dots, f_n \in \mathcal{L}_n(K)$ be given functions. A point $(x_1, \dots, x_n) \in K$ is called a *weak Nash stationary point of the functions* $f_1, \dots, f_n$ if, for $i = 1, \dots, n$,

$$D_i^w f_i(x_1, \dots, x_n)(h_i) \leq 0$$

holds for all $h_i \in T_{K_i}(x_i)$. It is obvious that

$$D_i^s f_i(x_1, \dots, x_n)(h_i) \geq D_i^w f_i(x_1, \dots, x_n)(h_i)$$

for all $h_i \in S_{K_i}(x_i)$. Therefore, if $(x_1, \dots, x_n)$ is a strong Nash stationary point, then

$$0 \geq D_i^w f_i(x_1, \dots, x_n)(h_i), \qquad \forall\, h_i \in S_{K_i}(x_i).$$

By the continuity in h_i, we obtain

$$0 \geq D_i^w f_i(x_1, \dots, x_n)(h_i) \qquad \forall\, h_i \in T_{K_i}(x_i).$$

Thus $(x_1, \dots, x_n)$ is also a weak stationary point.

It follows by Proposition 1 that each Nash equilibrium point is a weak Nash stationary point. The next proposition is analogous to Proposition 2.

Proposition 3. *Let* $f_i \in \mathcal{L}_i(K)$, $(i = 1, \dots, n)$ *such that, for all fixed* $x_j \in K_j$ $(j \neq i)$, *the function*

$$y_i \to f_i(x_1, \dots, y_i, \dots, x_n) \qquad (y_i \in D_i)$$

is concave. Then every weak Nash stationary point is a Nash equilibrium point.

Proof. Let $(x_1, \dots, x_n)$ be a weak Nash stationary point, let $i \in \{1, \dots, n\}$ be fixed $z_i \in K_i$ and $h_i = z_i - x_i$. Then, by the concavity and continuity properties of f_i,

$$\begin{aligned} 0 &\geq D_i^w f(x_1,\dots,x_n)(h_i) \\ &= \liminf_{\substack{\varepsilon\to 0^+ \\ y_j\to x_j\,(\forall j) \\ y_j\in K_j\ (\forall j\neq i)}} \frac{f_i(y_1,\dots,y_i+\varepsilon h_i,\dots,y_n)-f_i(y_1,\dots,y_n)}{\varepsilon} \\ &\geq \liminf_{\substack{y_j\to x_j\ (\forall j) \\ y_j\in K_j\ (\forall j\neq i)}} \left(f_i(y_1,\dots,y_i+h_i,\dots,y_n)-f_i(y_1,\dots,y_n)\right) \\ &= f_i(x_1,\dots,x_i+h_i,\dots,x_n)-f_i(x_1,\dots,x_n). \end{aligned}$$

Hence $(x_1,\dots,x_n)$ is a Nash equilibrium point. □

In order to state the result concerning the existence of weak stationary points, we need the following two lemmas.

Lemma 2. $T_K(x_1,\dots,x_n)=T_{K_1}(x_1)\times\cdots\times T_{K_n}(x_n)$.

Proof. See Aubin–Frankowska [2] for the properties of contingent cones. □

Lemma 3 (Local Ky Fan inequality [6, Theorem 1])**.** *Let X be a normed space, $K\subseteq X$ convex compact and $F\in\mathcal{L}_1(K\times K)$. Then there exists $x\in K$ such that*

$$D_1^w F(x,x)(h)\leq 0, \qquad \forall\, h\in T_K(x).$$

For the sake of completeness, we reproduce the proof done in [6].

Proof. Let $g: K\times K\to\mathbb{R}$ be defined by

$$g(u,x)=D_1^w F(x,x)(u-x).$$

Then, it follows from Lemma 1, that

(1) For each fixed $x\in K$, $u\mapsto g(u,x)$ is concave;

(2) For each $u\in K$, $x\mapsto g(u,x)$ is lsc on K.

It is also obvious that $g(u,u)=0$ for $u\in K$.

Therefore, by Ky Fan's minimax inequality (see [7]), there exists an element $x \in K$ such that

$$g(u,x) \le 0 \qquad\qquad \text{for all } u \in K,$$

i.e.

$$D_1^w F(x,x)(u-x) \le 0 \quad \text{for all } u \in K.$$

Hence, by the positive homogeneity,

$$D_1^w F(x,x)(\lambda(u-x)) \le 0 \quad \text{for all } u \in K,\ \lambda \ge 0.$$

Applying (ii) of Lemma 1 again, we obtain the statement. □

Theorem 2 (Existence of weak Nash stationary points). *Let $K_1,\dots,K_n$ be compact convex subsets of normed spaces, $K := K_1 \times \cdots \times K_n$ and $f_1 \in \mathcal{L}_1(K),\dots,f_n \in \mathcal{L}_n(K)$.*
Then there exists a weak Nash stationary point for the functions $f_1,\dots,f_n$.

Proof. Define $F : K \times K \to \mathbb{R}$ by

$$F((y_1,\dots,y_n),(z_1,\dots,z_n)) := \sum_{i=1}^{n} f_i(z_1,\dots,y_i,\dots,z_n).$$

It can be easily seen that $F \in \mathcal{L}_1(K \times K)$. By Lemma 3, there exists $x = (x_1,\dots,x_n) \in K$ such that

$$D_1^w F(x,x)(h) \le 0, \qquad \forall\, h \in T_K(x).$$

Let $i \in \{1,\dots,n\}$ be fixed and $h = (0,\dots,h_i,\dots,0)$, where $h_i \in T_{K_i}(x_i)$. Then $h \in T_K(x)$ (by Lemma 2), therefore

$$\begin{aligned} 0 &\ge D_1^w F(x,x)(h) \\ &= \liminf_{\substack{\varepsilon\to 0^+\\ (y,z)\to(x,x)\\ z\in K}} \frac{f_i(z_1,\dots,y_i+\varepsilon h_i,\dots,z_n) - f_i(z_1,\dots,y_i,\dots,z_n)}{\varepsilon} \\ &= D_i^w f_i(x_1,\dots,x_n)(h_i). \end{aligned}$$

The proof is completed. □

Remark. In what follows, we point out that, in fact, the statement of the theorem is equivalent to that of Lemma 3. Indeed, if $F : K \times K \to \mathbb{R}$ satisfies the condition of Lemma 3, define

$$f_1(x_1, x_2) = F(x_1, x_2) \quad \text{and} \quad f_2(x_1, x_2) = -\|x_1 - x_2\| \quad \text{for } x_1, x_2 \in K.$$

By Theorem 2, there exists a point $(x_1, x_2) \in K \times K$ such that

$$D_1^w F(x_1, x_2)(h_1) \leq 0 \quad \text{for } h_1 \in T_K(x_1)$$

and, for all $h_2 \in T_K(x_2)$,

$$\begin{aligned} 0 &\geq \liminf_{\substack{\varepsilon \to 0^+ \\ y_j \to x_j\, (j=1,2) \\ y_1 \in K_1}} \frac{-\|y_1 - (y_2 + \varepsilon h_2)\| + \|y_1 - y_2\|}{\varepsilon} \\ &\geq \liminf_{\substack{y_j \to x_j\, (j=1,2) \\ y_1 \in K_1}} -\|y_1 - (y_2 + h_2)\| + \|y_1 - y_2\| \\ &= -\|x_1 - (x_2 + h_2)\| + \|x_1 - x_2\|. \end{aligned}$$

Putting $h_2 = x_1 - x_2$, we get that $\|x_1 - x_2\| \leq 0$. Hence, $x_1 = x_2$. Therefore the statement of Lemma 3 follows from the first inequality above.

According to [6], the classical fixed point theorems and Ky Fan's minimax inequality are equivalent to the statement of Lemma 3. Hence, Theorem 2 itself is equivalent to these results.

Now we list two immediate consequences of Theorem 2.

Corollary 1. *Let $K_1, \dots, K_n$ be compact convex sets in normed spaces, $D_1, \dots, D_n$ open convex sets such that $K_i \subset D_i$, $(i = 1, \dots, n)$ and $f_i : K_1 \times \dots \times D_i \times \dots \times K_n \to \mathbb{R}$ be functions such that the partial derivatives $\partial_i f_i$ exist and are continuous on $K_1 \times \dots \times D_i \times \dots \times K_n$ for each $i \in \{1, \dots, n\}$. Then there exists an element $x = (x_1, \dots, x_n) \in K$ such that, for $i = 1, \dots, n$,*

$$\partial_i f_i(x_1, \dots, x_n) h_i \leq 0, \qquad \forall\, h_i \in T_{K_i}(x_i).$$

Proof. Let $(x_1, \dots, x_n) \in K$ and $h_i \in T_{K_i}(x_i)$ be arbitrarily fixed. If $y_j \in K_j$ $(j \neq i)$, $\varepsilon > 0$ and $y_i \in D_i$, then by the Mean Value Theorem, there exists $\theta \in [0,1]$ such that

$$\frac{f_i(y_1,\dots,y_i+\varepsilon h_i,\dots,y_n)-f_i(y_1,\dots,y_n)}{\varepsilon} = \partial_i f_i(y_1,\dots,y_i+\theta\varepsilon h_i,\dots,y_n)h_i.$$

By the continuity property of $\partial_i f_i$, this tends to $\partial_i f_i(x_1,\dots,x_n)(h_i)$ if $y_j \to x_j$, $\forall j$, $y_j \in K_j$ $(j \neq i)$ and $\varepsilon \to 0$. Therefore,

$$D_i^w f_i(x_1,\dots,x_n)(h_i) = \partial_i f_i(x_1,\dots,x_n)(h_i).$$

By Theorem 2, there exists a weak Nash stationary point, which, in this case, satisfies the statement of this Corollary. □

The following special case of Corollary 1 is a local analogue of the result concerning the existence of saddle points (cf. [1]), that is, it can be considered as a local minimax theorem.

Corollary 2. *Let K_1, K_2 be compact convex sets in normed spaces, D_1, D_2 open convex sets such that $K_i \subset D_i$, $(i=1,2)$, and $f : D_1 \times D_2 \to \mathbb{R}$ be a continuously differentiable function. Then there exists an element $(x_1,x_2) \in K_1 \times K_2$ such that*

$$\partial_1 f(x_1,x_2)(h_1) \le 0, \quad \forall\, h_1 \in T_{K_1}(x_1) \quad \textit{and}$$
$$\partial_2 f(x_1,x_2)(h_2) \ge 0, \quad \forall\, h_2 \in T_{K_2}(x_2).$$

Proof. Let $n = 2$, $f_1 = f$ and $f_2 = -f$ in Corollary 1. Then the result follows immediately. □

Remark. It follows from Proposition 1 that all the saddle points (x_1, x_2) of a given function f (if they exist at all) satisfy the above inequalities from Corollary 2. Therefore, these inequalities can be used to determine the saddle points of f.

4. Numerical example

In [3], F. Forgó considered numerical examples for which the Nash Equilibrium Theorem (Theorem 1) cannot be applied. However, these examples are covered by Theorem 3 of Forgó (see [3]), thus the existence of a Nash equilibrium point can be stated.

In this section we present an example that is more general than that of Forgó [3]. The problem depends on a real parameter t. This method presented below, can be applied for many examples satisfying the assumptions of Corollary 1.

Let $K_1 = K_2 = [-1, 1]$, $D_1 = D_2 = \mathbb{R}$ and $f, g : D_1 \times D_2 \to \mathbb{R}$ given by

$$f(x, y) = x^2y + x, \qquad g(x, y) = tx^2y^2 - y,$$

where t is a real parameter.

Clearly, $y \to g(1, y)$ is not quasiconcave for $t > 0$, therefore, Theorem 1 cannot be applied. It is easy to see that, (x, y) is a Nash stationary point for the functions f, g, if and only if

$$2xy + 1 \begin{cases} = 0 & \text{if } -1 < x < 1, \\ \geq 0 & \text{if } x = 1, \\ \leq 0 & \text{if } x = -1, \end{cases}$$

and

$$2tx^2y - 1 \begin{cases} = 0 & \text{if } -1 < y < 1, \\ \geq 0 & \text{if } y = 1, \\ \leq 0 & \text{if } y = -1, \end{cases}$$

In addition, (x, y) is a Nash equilibrium point if and only if

$$x^2y + x \geq z^2y + z \qquad \forall z \in [-1, 1],$$

and

$$tx^2y^2 - y \geq tx^2z^2 - z \quad \forall z \in [-1, 1].$$

After an elementary but lengthy computation, we can first determine the Nash stationary points for the problem (which exist in virtue of Corollary 1). Then, one can easily select those that are Nash equilibrium points as well.

The results can be summarized in the following way.

For each $t \in \mathbb{R}$, there exists at least one Nash equilibrium point, more precisely,

- if $t < -2$, then $(1, 1/2t)$ is a unique Nash equilibrium point;
- if $t = -2$, then $(1, -1/4)$ and $(1/2, -1)$ are Nash equilibrium points;
- if $-2 < t < -1$, then $(1, 1/2t)$, $(-1/t, t/2)$ and $(1/2, -1)$ are Nash equilibrium points;
- if $t = -1$, then $(1, -1/2)$ and $(1/2, -1)$ are Nash equilibrium points;
- if $-1 < t$, then $(1/2, -1)$ is a unique Nash equilibrium point;

Clearly, the above points are Nash stationary points. The additional stationary points (that are not equilibrium points) are the following:

- if $t < 1/2$, then there is no additional stationary point;
- if $t = 1/2$, then $(1, 1)$ and $(-1, 1)$ are additional Nash stationary points;
- if $1/2 < t \leq 1$, then $(1, 1)$, $(-1, 1)$, $(1, 1/2t)$, and $(-1, 1/2t)$ are additional Nash stationary points;
- if $1 < t \leq 2$, then $(1, 1)$, $(-1, 1)$, $(1, 1/2t)$, and $(-1/t, t/2)$, are additional Nash stationary points;
- if $2 < t$, then $(1, 1)$, $(-1, 1)$, $(1, 1/2t)$, and $(-1/2, 1)$, are additional Nash stationary points.

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