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Starlikeness properties of meromorphic m-valent functions

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Abstract. We find conditions regarding the values of β such that $F(z)$ given by $(*)$ is nonvanishing, when $f \in \Sigma^*_{m,n}(\alpha)$ with $n \geq -m+1$ and employ a differential subordination technique, for certain conditions on β, to obtain the sharpest value of $\delta = \delta(\alpha,\beta,\gamma,m,n)$ such that

$$I^m_{\beta,\gamma}\left(\Sigma^*_{m,n}(\alpha)\right) \subset \Sigma^*_{m,n}(\delta) \subset \Sigma^*_{m,n}(\alpha).$$

1. Introduction

Let $\Sigma_{m,n}$ be the class of meromorphic m-valent functions f defined in $\dot{U} = \{z \in \mathbb{C},\ 0 < |z| < 1\}$ of the form

$$f(z) = \frac{1}{z^m} + a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

where m and n are integers such that $m > 0$ and $n \geq -m+1$.

A function $f \in \Sigma_{m,n}$ is called starlike of order α, $0 \leq \alpha < m$ if

$$\operatorname{Re}\left(\frac{-zf'(z)}{f(z)}\right) > \alpha.$$

Let $\Sigma^*_{m,n}(\alpha)$ be the class of all such functions.

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Let γ, β be real numbers, $\gamma > 0$ and let us define the integral operator $I^m_{\gamma,\beta} : \Sigma^*_{m,n}(\alpha) \to \Sigma_{m,n}$,

$$(*) \qquad I^m_{\gamma,\beta}(f)(z) = F(z) = \left(\frac{\gamma}{z^{\gamma+m\beta}} \int_0^z t^{\gamma+m\beta-1} f^\beta(t)dt\right)^{1/\beta}, \quad z \in \dot{U}.$$

In [1], [2] and [3] Hasson Al-Amiri and Petru T. Mocanu have investigated the properties of functions f belonging to $\Sigma^*_{1,n}(0) = \Sigma^*_n$ for $n > 0$. In [4] M.K. Aouf and M. Hossen obtained some interesting results concerning the function f from $\Sigma_{m,n}$ with $n \geq 0$.

In this paper we extend a part of results of H. Al-Amiri and P.T. Mocanu for f belonging to $\Sigma^*_{m,n}(0) = \Sigma^*_{m,n}$ with $n > 0$ and other results for f belonging to $\Sigma^*_{m,n}$ with $n \geq -m+1$.

Moreover, we find conditions regarding the values of β such that $F(z)$ given by $(*)$ is nonvanishing, when $f \in \Sigma^*_{m,n}(\alpha)$ with $n \geq -m+1$ and employ a differential subordination technique, for certain conditions on β, to obtain the sharpest value of $\delta = \delta(\alpha, \beta, \gamma, m, n)$ such that

$$I^m_{\beta,\gamma}(\Sigma^*_{m,n}(\alpha)) \subset \Sigma^*_{m,n}(\delta) \subset \Sigma^*_{m,n}(\alpha).$$

In the case of holomorphic m-valent functions similar results were obtained by Ponnusamy [5].

2. Preliminaries

We will need the following definitions and lemmas to prove our main results.

If f and g are holomorphic functions in the unit disc U and if g is univalent, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$. The holomorphic function h, with $h(0) = 0$ and $h'(0) \neq 0$ is starlike in U (i.e. h is univalent in U and $h(U)$ is starlike with respect to the origin) if and only if $\mathrm{Re}[zh'(z)/h(z)] > 0$ for $z \in U$.

The holomorphic function h in U is convex (i.e. h is univalent and $h(U)$ is convex domain) if $f'(0) \neq 0$ and

$$\mathrm{Re}\,\frac{zh''(z)}{h'(z)} + 1 > 0, \quad \text{for } z \in U.$$

Lemma A [2]. *Let n be a positive integer and α be real, with $0 < \alpha < n$. Let q be holomorphic in U, with $q(0) = 0$, $q'(0) \neq 0$ and suppose that*

i) $\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}$.

Let

ii) $h(z) = nzq'(z) - \alpha q(z)$.

If $p(z) = p_n z^n + \cdots$ is holomorphic in U and $zp'(z) - \alpha p(z) \prec h(z)$, then $p(z) \prec q(z)$ and this result is sharp.

Lemma B [3]. *Let n be a positive integer, let $\lambda > 0$ and let $\beta^* = \beta^*(\lambda, n)$ be the solution of the equation $\beta\pi - 2\operatorname{arctg}(n\lambda\beta) = 0$. Let $\alpha = \alpha(\beta, \lambda, n) = \frac{2}{\pi}\operatorname{arctg}(n\lambda\beta) - \beta$ for $0 < \beta \leq \beta^*$.*

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in U, then

$$p(z) - \lambda z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

Lemma C [3]. *Let n be a positive integer, $\lambda > 0$, $\beta^* = \beta^*(\lambda, n)$ be the solution of the equation $\beta\pi = \frac{3\pi}{2} - \operatorname{arctg}(n\lambda\beta)$. Let $\alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi}\operatorname{arctg}(n\lambda\beta)$ for $0 < \beta < \beta^*$. If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in U, then*

$$p(z) + \lambda z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

Lemma D [6]. *Let h be starlike in U and let $p(z) = 1 + p_n z^n + \cdots$ be holomorphic in U. If $\frac{zp'(z)}{p(z)} \prec h(z)$, then $p \prec q$, where $q(z) = \exp \frac{1}{n} \int_0^z \frac{h(t)}{t} dt$.*

Lemma E [7]. *Let $\mu(t)$ be a positive measure on the unit interval $I=[0,1]$. Let $g(t,z)$ be a complex valued function defined on $U\times[0,1]$ and integrable in t for each $z\in U$ and for almost all $t\in[0,1]$, and suppose that* $\mathrm{Re}\{g(t,z)\}>0$ *on U and $g(z)=\int\limits_I g(t,z)d\mu(t)$. If for fixed λ $(0\le\lambda<2\pi)$ $g(t,re^{i\lambda})$ is real for r real and*

$$\mathrm{Re}\left\{\frac{1}{g(t,z)}\right\}\ge\frac{1}{g(t,re^{i\lambda})},$$

for $|z|\le r$ and $t\in[0,1]$, then

$$\mathrm{Re}\left\{\frac{1}{g(z)}\right\}\ge\frac{1}{g(re^{i\lambda})}$$

for $|z|\le r$ and $0\le\lambda<2\pi$.

This Lemma can be proved in a similar manner as that of Lemma 2 of Wilken and Feng in [7].

Theorem A [8]. *Let β', γ' be complex numbers with $\beta'\ne 0$ and let k be a positive integer. Let h be analytic in U with $h(0)=a\ne 0$,* $\mathrm{Re}[\beta' a+\gamma']>0$, *and*

i) $\mathrm{Re}(\beta' h(z)+\gamma')>0$.

If q is the analytic solution of the Briot-Bouquet differential equation

$$q(z)+\frac{kzq'(z)}{\beta' q(z)+\gamma'}=h(z)$$

given by

$$q(z)=\left(\frac{\beta'}{k}\int_0^z[H(tz)/H(z)]^{\frac{\beta' a}{k}}t^{\frac{\gamma'}{k}-1}dt\right)^{-1}-\frac{\gamma'}{\beta'}$$

with

$$H(z)=z\exp\int_0^t[h(t)-a]/at\,dt$$

and if

ii) *h is convex or $Q(z)=\frac{zq'(z)}{\beta' q(z)+\gamma'}$ is starlike then q and h are univalent. Furthermore, if the holomorphic function $p(z)=a+a_kz^k+\cdots$ satisfies*

$$p(z)+\frac{zp'(z)}{\beta' p(z)+\gamma'}\prec h(z)$$

then $p\prec q\prec h$ and q is the best (a,k) dominant.

More general form of this theorem may be found in [8].

We also need the following well known formula.

For a, b, c real numbers with $c \neq 0, -1, -2, \dots$ the function

$$(**) \qquad {}_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, \quad z \in U,$$

is called the (Gaussian) hypergeometric function.

If $c > a > 0$, then

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt$$

and

$${}_2F_1(a,b,c;z) = (1-z)^{-b}{}_2F_1\left(b, c-a, c, \frac{z}{z-1}\right).$$

3. Main results

Theorem 1. *Let q be holomorphic, with $q(0) = 0$, $q'(0) \neq 0$ and suppose that* $\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{m}{n+m}$. *Let $h(z) = (m+n)zq'(z) - mq(z)$.*

If n is positive integer, $f \in \Sigma_{m,n}$ and

$$z^{m+1}f'(z) + m \prec h(z),$$

then $z^m f(z) - 1 \prec q(z)$ and the result is sharp.

Proof. If we denote $p(z) = z^m f(z) - 1$, then $p(z) = a_n z^{m+n} + \cdots$ is holomorphic in U and

$$(1) \qquad zp'(z) - mp(z) = z^{m+1}f'(z) + m \prec h(z).$$

Applying Lemma A with $\alpha = m$, $n = m+n$ from (1) we obtain that $z^m f(z) - 1 \prec q(z)$.

Corollary 1. *Let M be a positive real number and n, m be positive integers. If $f \in \Sigma_{m,n}$ and*

$$(2) \qquad |z^{m+1}f'(z) + m| < M,$$

then

$$|z^m f(z) - 1| < \frac{M}{n}. \tag{3}$$

Proof. If we take $q(z) = \frac{Mz}{n}$, then $h(z) = Mz$ and the result follows from Theorem 1.

Corollary 1. *Let $f \in \Sigma_{m,n}$ with m, n positive integers and suppose that*

$$|z^{m+1} f'(z) + m| < \frac{mn}{\sqrt{m^2+n^2}}, \tag{4}$$

then $f \in \Sigma^*_{m,n}$.

Proof. Let $0 < M \leq \frac{mn}{\sqrt{m^2+n^2}}$. If we denote $P(z) = z^m f(z)$, then by Corollary 1 we obtain $|P(z) - 1| < \frac{M}{n}$, which implies that $P(z) \neq 0$ in U. Hence the function $p(z) = -\frac{zf'(z)}{f(z)} = m + p_n z^{m+n} + \cdots$ is holomorphic in U and we have $-z^{m+1} f'(z) = P(z)p(z)$. Hence the inequality (4) becomes

$$|P(z)p(z) - m| < M, \quad z \in U. \tag{5}$$

We have to show that this inequality implies $\operatorname{Re} p(z) > 0$ in U. Suppose $\operatorname{Re} p(z) \not> 0$ in U. Then there exists a point $z_0 \in U$ such that $p(z_0) = is$ where s is real number.

If $M \leq \frac{mn}{\sqrt{m^2+n^2}}$, then $|\operatorname{Im} P(z)| < \frac{\sqrt{m^2-M^2}}{m}|P(z)|$ and

$$|P(z)is - m|^2 - M^2 = |P(z)|^2 s^2 + 2ms \operatorname{Im} P(z) + m^2 - M^2 \geq 0$$

for all real s, then we deduce that

$$|P(z)is - m|^2 \geq M^2$$

for $z \in U$ and s real.

In particular $|P(z_0)p(z_0) - m| \geq M$ which contradicts (5).

Hence we must have $\operatorname{Re} p(z) > 0$ in U, which shows that $f \in \Sigma^*_{m,n}$.

Remark. For $m = 1$ Theorem 1, Corollary 1, Corollary 2 were proved in [2].

Theorem 2. *Let m, n be integers such that $m > 0$ and $n \geq -m+1$.*
Let

$$\beta^* = \beta^*(m,n) \tag{6}$$

be the root of the equation $\beta\pi - 2\operatorname{arctg}\left(\frac{n+m}{m}\beta\right) = 0$.
Let

$$\alpha = \alpha(m,n,\beta) = \frac{2}{\pi}\operatorname{arctg}\frac{(n+m)\beta}{n} - \beta \tag{7}$$

for $0 < \beta < \beta^*$.
If $f \in \Sigma_{m,n}$ and

$$|\arg[-z^{m+1}f'(z)]| < \alpha\frac{\pi}{2}, \quad z \in U, \tag{8}$$

then

$$|\arg[z^m f(z)]| < \beta\frac{\pi}{2}. \tag{9}$$

Proof. If we let $p(z) = z^m f(z) = 1 + a_n z^{m+n} + \cdots$, then

$$\frac{-z^{m+1}f'(z)}{m} = p(z) - \frac{1}{m}zp'(z)$$

and the inequality (8) becomes

$$p(z) - \frac{1}{m}zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Hence by Lemma B with $\lambda = \frac{1}{m}$ and $m+n \geq 1$ we deduce

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}$$

where α and β satisfy (6) and (7).

Theorem 3. *Let m, n be integers such that $m > 0$ and $n \geq -m+1$ and suppose that α, β satisfy (6) and (8).*

If $f \in \Sigma_{m,n}$ and $|arg[-z^{m+1}f'(z)]| < \alpha\frac{\pi}{2}$, then

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right| < \operatorname{arctg}\frac{m+n}{m}\beta, \quad z \in U,$$

hence $f \in \Sigma^*_{m,n}$.

Proof. Applying Theorem 1 from (9) and (10) we deduce

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right| \leq |\arg[-z^{m+1}f'(z)]| + |\arg[z^m f(z)]| < \gamma\frac{\pi}{2},$$

where

$$\gamma = \beta + \alpha = \frac{2}{\pi}\operatorname{arctg}\frac{m+n}{m}\beta.$$

Consequently

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right| < \operatorname{arctg}\frac{m+n}{n}\beta < \frac{\pi}{2}$$

from which it follows that

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > 0.$$

Theorem 4. *Let m, n be integers such that $m > 0$ and $n \geq -m+1$. Suppose that the positive real numbers α, β, γ, c satisfy*

$$\alpha = \beta + \frac{2}{\pi}\operatorname{arctg}\frac{m+n}{c}\beta \tag{10}$$

$$\beta = \frac{2}{\pi}\operatorname{arctg}\frac{m+n}{m}\gamma - \gamma. \tag{11}$$

If $f \in \Sigma_{m,n}$ and

$$|\arg[-z^{m+1}f'(z)]| < \alpha\frac{\pi}{2}, \quad z \in U, \tag{12}$$

then

$$\left|\arg\left(-\frac{zF'(z)}{F(z)}\right)\right| < \operatorname{arctg}\left(\frac{m+n}{m}\gamma\right), \quad z \in \dot{U},$$

consequently $F \in \Sigma^*_{m,n}$, *where*

$$F(z) = \frac{c}{z^{c+m}} \int_0^z t^{c+m-1} f(t)dt, \quad z \in \dot{U}. \tag{13}$$

Proof. From (13) we obtain

$$(c+m)F(z) + zF'(z) = cf(z),$$

and if we denote $p(z) = \frac{-z^{m+1}F'(z)}{m} = 1 + p_m z^{m+n} + \dots$, then

$$p(z) + \frac{1}{c} zp'(z) = -\frac{1}{m} z^{m+1} f'(z),$$

and (12) becomes

$$p(z) + \frac{1}{c} zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Applying Lemma C we obtain

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}$$

that means

$$|\arg[-z^{m+1}F'(z)]| < \beta \frac{\pi}{2}, \tag{14}$$

where α and β satisfy

$$\alpha = \beta + \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{c} \beta.$$

Now by Theorem 2, the inequality (14) implies

$$|\arg[z^m F(z)]| < \gamma \frac{\pi}{2},$$

where

$$\beta + \gamma = \frac{2}{\pi} \operatorname{arctg} \frac{m+n}{m} \gamma$$

and we deduce

$$\left|\arg\left(-\frac{zF'(z)}{F(z)}\right)\right| \leq |\arg[-z^{m+1}F'(z)]| + |\arg[z^m F(z)]|$$
$$< (\beta+\gamma)\frac{\pi}{2} = \operatorname{arctg}\frac{m+n}{m}\gamma < \frac{\pi}{2}.$$

Theorem 5. *Let m, n be integers, $m > 0$ and $n \geq -m+1$. Let α, β, γ, β_1 be real numbers so that $\alpha \in [0, m)$, $\gamma > 0$*

i) $1 = \beta_1 + \frac{2}{\pi}\operatorname{arctg}\frac{m+n}{\gamma}\beta_1$

ii) $\beta_1 < |\beta| < \frac{m+n}{2(m-\alpha)}$.

Let $F : U \to \mathbb{C}$ defined by $(*)$. If $f \in \Sigma^*_{m,n}(\alpha)$, then $\operatorname{Re}[z^m F(z)] > 0$, *consequently $F(z) \neq 0$ in $\dot{U}$.*

Proof. Since $f \in \Sigma^*_{m,n}(\alpha)$ we have $f(z) \neq 0$, for all $z \in U$ and, let $\varphi(z) = z^m f(z)$.

$$\frac{z\varphi'(z)}{\varphi(z)} = m + \frac{zf'(z)}{f(z)} \prec m - m\frac{1+\left(1-\frac{2\alpha}{m}\right)z}{1-z} = \frac{2(\alpha-m)z}{1-z} = h(z).$$

Applying Lemma D we obtain that

$$\varphi(z) = z^m f(z) \prec (1-z)^{\frac{2(\alpha-m)}{m+n}}. \tag{15}$$

This implies that $z^m f(z) \neq 0$ in U.

Let $g(z) = (z^m F(z))^\beta = 1 + g_n z^{m+n} + \cdots$. From (1) we obtain that

$$g(z)z^\gamma = \gamma\int_0^z f^\beta(t)t^{\gamma+m\beta-1}dt,$$

this implies that

$$g(z) + \frac{1}{\gamma}zg'(z) = (z^m f(z))^\beta \prec (1-z)^{\frac{2(\alpha-m)\beta}{m+n}}. \tag{16}$$

We observe that $|\arg(1-z)| < \frac{\pi}{2}$, when $z \in U$. From ii) and (16) it follows that

$$\operatorname{Re}\left(g(z) + \frac{1}{\gamma}zg'(z)\right) > 0,$$

consequently

$$g(z)+\frac{1}{\gamma}zg'(z)\prec\frac{1+z}{1-z}.$$

Applying Lemma C with $\alpha=1$, $n=m+n$ we obtain that

$$g(z)\prec\left(\frac{1-z}{1+z}\right)^{\beta_1},\tag{17}$$

where β_1 is given by i). We observe that

$$0<1-\beta_1=\frac{2}{\pi}\operatorname{arctg}\frac{n+m}{\gamma}\beta_1<1,$$

i.e. $\beta_1\in(0,1)$. From (17) it follows that

$$|\arg[z^mF(z)]^\beta|<\beta_1\frac{\pi}{2},$$

that implies

$$|\arg[z^mF(z)]|<\frac{\beta_1}{|\beta|}\frac{\pi}{2}.\tag{18}$$

Taking into account that $\beta_1<|\beta|$, from (18) we deduce that $\operatorname{Re}(z^mF(z))>0$ in $\dot{U}$, and this implies $F(z)\neq 0$ in $\dot{U}$.

Theorem 6. *Let m, n, α, β, γ, β_1 be real numbers satisfying all the conditions of Theorem 5 and suppose that $\beta<0$.*

If $f\in\Sigma^*_{m,n}(\alpha)$ and $F(z)$ is given by (1), then

$$F\in\Sigma^*_{m,n}(\delta),\tag{19}$$

where

$$\delta=-\frac{\gamma}{\beta}\frac{1}{{}_2F_1\left(\frac{2\beta(\alpha-m)}{m+n},1,\frac{\gamma}{m+n}+1;\frac{1}{2}\right)}+\frac{\gamma+\beta m}{\beta},$$

${}_2F_1$ given by $(**)$, and this result is sharp.

Proof. From Theorem 1 we obtain that if $f\in\Sigma^*_{m,n}(\alpha)$, then $F(z)\neq 0$ hence

$$p(z)=-\frac{zF'(z)}{F(z)}=m+c_nz^{m+n}+\cdots$$

is an analytic function in U with $p(0)=m$. From (1) we easily obtain that

$$p(z)+\frac{zp'(z)}{-\beta p(z)+\gamma+m\beta}=-\frac{zf'(z)}{f(z)},\quad z\in U. \tag{20}$$

The condition $f\in\Sigma^*_{m,n}(\alpha)$ implies that

$$-\frac{zf'(z)}{f(z)}\prec m\frac{1+\left(1-\frac{2\alpha}{m}\right)z}{1-z}=h(z). \tag{21}$$

From (20) and (21) we obtain that

$$p(z)+\frac{zp'(z)}{-\beta p(z)+\gamma+m\beta}\prec h(z)=m\frac{1+\left(1-\frac{2\alpha}{m}\right)z}{1-z}.$$

It is easy to show that h is a convex function in U and for $\beta<0$, $\gamma>0$, $\mathrm{Re}(-\beta h(z)+\gamma)>0$. Hence we can apply Theorem A with $\beta'=-\beta$, $\gamma'=\gamma+\beta m$, $a=m$, $k=m+n$. Observe that $\mathrm{Re}(\beta' a+\gamma')=\gamma>0$ and we obtain that

$$p(z)\prec q(z)\prec h(z) \tag{22}$$

and this result is sharp, where

$$\begin{aligned} q(z)&=\left(\frac{-\beta}{m+n}\int_0^z t^{\frac{-\beta m}{m+n}}\left(\frac{1-tz}{1-z}\right)^{\frac{-2\beta(\alpha-m)}{m+n}} t^{\frac{\gamma+\beta m}{m+n}-1}dt\right)^{-1}+\frac{\beta m+\gamma}{\beta}\\ &=\left(\frac{-\beta}{m+n}\int_0^z t^{\frac{\gamma}{m+n}-1}\left(\frac{1-tz}{1-z}\right)^{\frac{-2\beta(\alpha-m)}{m+n}} dt\right)^{-1}+\frac{\beta m+\gamma}{\beta}. \end{aligned} \tag{23}$$

If we consider

$$Q(z)=\int_0^1 t^{\frac{\gamma}{m+n}-1}\left(\frac{1-tz}{1-z}\right)^{\frac{-2\beta(\alpha-m)}{m+n}} dt$$

and we denote with

$$b=\frac{2\beta(\alpha-m)}{m+n},\quad a=\frac{\gamma}{m+n},\quad c=a+1=\frac{\gamma}{m+n}+1,$$

we observe that $c > a > 0$ and

$$\begin{aligned} Q(z) &= (1-z)^b \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {}_2F_1(a,b,c,z) \\ &= (1-z)^b \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {}_2F_1\left(b, c-a, c\frac{z}{z-1}\right)(1-z)^{-b} \\ &= \int_0^1 \frac{1-z}{1-(1-t)z} d\mu(t) \end{aligned} \tag{24}$$

where

$$d\mu(t) = t^{b-1}(1-t)^{c-b-1}\frac{\Gamma(a)}{\Gamma(c-b)\Gamma(b)}dt.$$

Let $g(z,t) = \frac{1-z}{1-(1-t)z}$, we observe that $\operatorname{Re}\{g(t,z)\} > 0$, $|z| \leq r < 1$, $g(t,-r) \in \mathbb{R}$, $0 \leq r < 1$, $t \in [0,1]$ and

$$\operatorname{Re}\left\{\frac{1}{g(z,t)}\right\} = \operatorname{Re}\left\{\frac{1-(1-t)z}{1-z}\right\} \geq \frac{1+(1-t)r}{1+r} = \frac{1}{g(t,-r)}$$

for $|z| \leq r < 1$ and $t \in [0,1]$. Therefore, by using Lemma E we deduce that

$$\operatorname{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-r)}, \quad |z| \leq r < 1,$$

and by letting $r \to 1^-$, we obtain

$$\operatorname{Re}\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)}, \quad z \in U.$$

Taking into account that $\beta < 0$ we obtain that

$$\begin{aligned} \operatorname{Re} q(z) &= \operatorname{Re}\left(\frac{m+n}{-\beta Q(z)} + \frac{\beta m + \gamma}{\beta}\right) \geq \frac{m+n}{-\beta Q(-1)} + \frac{\gamma + \beta m}{\beta} \\ &= -\frac{\gamma}{\beta}\frac{1}{{}_2F_1\left(\frac{2\beta(\alpha-m)}{m+n}, 1, \frac{\gamma}{m+n}+1, \frac{1}{2}\right)} + \frac{\gamma+\beta m}{\beta} = \delta. \end{aligned} \tag{25}$$

From (22) and (24) it follows that

$$\operatorname{Re}\left(\frac{-zF'(z)}{F(z)}\right) > \min_{|z|<1} \operatorname{Re} q(z) = \delta,$$

consequently $F \in \Sigma^*_{m,n}(\delta)$ and this result is sharp.

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