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Perfect transsymmetric spaces

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Abstract. The transsymmetric spaces with an additional geometric property are considered and investigated. In particular, well known symmetric spaces of E. Cartan possess this property.

0. Introduction

The transsymmetric spaces have been introduced in [1] as a far going generalization of the symmetric spaces of E. Cartan. They are reductive spaces of a special kind and as such may be treated within the frame of homogeneous space theory. By a construction due to R. Baer–L. Sabinin [2], [3], they may be treated within the frame of smooth loop theory, as well. It has been observed that symmetric spaces possess some quite remarkable properties not valid in general transsymmetric spaces. Thus the problem to explore transsymmetric spaces with such a property (perfect ts-spaces) appeared to be solved.

In this article we present results on perfect ts-spaces.

1. Preliminaries

Definition 1.1. Let M be a smooth manifold and $(\sigma_x)_{x\in M}$ the family of local diffeomorphisms defined near $x \in M$ such that $ex = \sigma_x^{-1}x$ is defined.

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We say that σ_x is a *transsymmetry* (ts-symmetry) at a point $x \in M$ and $(M, (\sigma_x)_{x\in M})$ is *a transsymmetric structure (manifold)*, briefly *ts-structure (manifold)*.

Let us introduce

$$x \cdot y \overset{\text{def}}{=} \sigma_x y, \quad \rho_y x \overset{\text{def}}{=} x \cdot y. \tag{1}$$

Definition 1.2. A transsymmetric structure is correct if, for any x, ρ_{ex} is a local diffeomorphism near $x \in M$.

Definition 1.3. A correct ts-structure $(M, (\sigma_x)_{x\in M})$ with the property

$$\sigma_x \circ \sigma_y = \sigma_{\sigma_x y} \circ \sigma_{ex}, \tag{2}$$

or, in another notation,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (ex \cdot z) \quad \text{(Left F-property)} \tag{3}$$

is called *a transsymmetric space (ts-space)*.

Remark 1.4. One can treat a ts-space $(M, (\sigma_x)_{x\in M})$ as a smooth partial groupoid (magma) $\langle M, \cdot\rangle$. Moreover, since $a \cdot x = b$ is (locally) solvable ($x = \sigma_a^{-1} b$), a ts-space may be treated as a partial smooth left quasigroup $\langle M, \cdot, \backslash\rangle$ (where $a\backslash b \overset{\text{def}}{=} \sigma_a^{-1} b$). Henceforth we consider ts-spaces only.

Remark 1.5. If $\sigma_x x = x$ ($\forall x \in M$) then a ts-space is called a *generalized symmetric space* (O. Kowalski [13]). In this case it is evident that $ex = x$. If, moreover, $\sigma_x \circ \sigma_x = \mathrm{id}$ then a generalized symmetric space is called *symmetric* (E. Cartan).

1.6. Let $(M, (\sigma_x)_{x\in M}) = \langle M, \cdot\rangle$ be a ts-space. One can introduce

$$P(x, z, y) = P_x^z\, y = x \underset{z}{\times} y = \rho_{ez}^{-1} x \cdot \sigma_z^{-1} y. \tag{4}$$

Then $x \underset{z}{\times} z = x$, $z \underset{z}{\times} y = y$, and $a \underset{z}{\times} y = b$, $x \underset{z}{\times} a = b$ are uniquely solvable (locally) which means that $\langle M, \underset{z}{\times}, z\rangle$ is a local smooth loop with two-sided neutral element z.

In such a way we have obtained a so-called canonical loopuscular covering (structure) of a ts-space $(M, (\sigma_x)_{x\in M})$.

One can regard now a ts-space $\langle M, \cdot\rangle$ as a smooth partial algebra $\langle M, P\rangle$ with the ternary operation P.

1.7. The formula

$$\nabla_{X_a} Y = \left\{ \frac{d}{dt}[(P^a_{\gamma(t)})^{-1}_{*,a} Y_{\gamma(t)}] \right\}_{t=0}, \tag{5}$$

Y being a vector field, $\gamma(0) = a$, $\dot\gamma(0) = X_a$, introduces an affine connection into ts-spaces $(M, (\sigma_x)_{x\in M})$ (see [5], [6]), which is called *the canonical affine connection of a transsymmetric space* $(M, (\sigma_x)_{x\in M})$. It is known that the canonical connection is reductive ($\nabla T = 0$, $\nabla R = 0$). See [4].

1.8. Any affine connection ∇ uniquely defines the system of geodesic odules centered at every point $z \in M$ (see [5], [6]) with local smooth operations $x \underset{z}{*} y = L^z_x\, y = L(x, z, y)$ and $t_z\, y = \omega_t(z, y)$, where $L^z_x y$ means the parallel displacement of the geodesic arc $\breve{zy}$ along the geodesic arc $\breve{zx}$.

Thus (M, ∇) may be considered as a smooth algebraic system $\langle M, L, (\omega_t)_{t\in\mathbb{R}}\rangle$, a geoodular manifold with characteristic geoodular identities. See [5], [6].

The affine connection may be restored from its geoodular manifold in a unique way (see [5], [6]).

For a reductive affine connection ∇

$$t_z x \underset{z}{*} (u_z x \underset{z}{*} y) = (t+u)_z x \underset{z}{*} y \quad \text{(Left monoalternativity)},$$

$$L^a_b(x \underset{z}{*} y) = L^a_b x \underset{L^a_b z}{*} L^a_b y \quad \text{(Left reductivity)}$$

are valid. See [5], [6].

2. Perfect ts-spaces

Definition 2.1. A ts-space $(M, (\sigma_x)_{x\in M})$ is called *perfect* if $\sigma_x \circ \sigma_y^{-1}$ induces a parallel displacement along the geodesic arc $\{t_y \sigma_x e y\}_{t\in[0,1]}$ joining y and $\sigma_x e y$.

Proposition 2.2. *Let $(M,(\sigma_x)_{x\in M})$ be a perfect ts-space. Then*

$$\sigma_{\rho_{ez}^{-1}y}\circ\sigma_z^{-1}=L_y^z, \tag{6}$$

where $L_y^z w = y \underset{z}{*} w$ is the composition in geodesic loop at a point $z\in M$, $\varphi_z=\sigma_z\circ e\in \mathrm{End}\langle Q,\underset{z}{*},z\rangle$.

Proof. According to [7], [8], $P_y^z = L^z_{\rho_{ez}^{-1}y}\circ L^z_{(-1)_z\varphi_z\rho_{ez}^{-1}y}$. On the other hand, due to (4), $P_x^z=\sigma_{\rho_{ez}^{-1}x}\circ\sigma_z^{-1}$. The Definition 1.6 shows that $P_y^z=L_y^z$ and $\sigma_{\rho_{ez}^{-1}y}\circ\sigma_z^{-1}=L_y^z=L^z_{\rho_{ez}^{-1}y}\circ L_{(-1)_z\varphi_z\rho_{ez}^{-1}y}$. □

Corollary 2.3. *If $(M,(\sigma_x)_{x\in M})$ is a perfect ts-space, then its canonical loopuscular structure coincides with its tangent geoodular structure and, consequently, is leftmonoalternative.*

Conversely, if a canonical loopuscular structure is leftmonoalternative (after adding the canonical unary operations), see [9], *then $(M,(\sigma_x)_{x\in M})$ is a perfect ts-space.*

2.4. Since $\langle M,P,(\omega_t)_{t\in\mathbb{R}}\rangle$ is reductive, the whole structure is determined by a geodesic odule $\langle M,\underset{\varepsilon}{*},(t_\varepsilon)_{t\in\mathbb{R}}\rangle$ at an arbitrarily fixed point $\varepsilon\in M$.

Using (3) one can obtain the characteristic identities of such an odule (see [7], [8]) in the form (for the sake of simplicity we set $x*y$, tx instead of $x\underset{\varepsilon}{*}y$, $t_\varepsilon y$, etc.):

$$x*[y*(\varphi x\backslash z)]=[x*(y*[\varphi x]^{-1})]*z,\quad (\varphi=\varphi_\varepsilon=\sigma_\varepsilon\circ e), \tag{7}$$

$$\varphi(x*y)=\varphi x*\varphi y. \tag{8}$$

(7) is the so called *M-identity*. See [10], [11].

Due to the leftmonoalternativity we get further

$$x*[(y*[\varphi x]^{-1})*z]=[x*(y*[\varphi x]^{-1})]*z,\quad \text{(Left half Bol identity)}. \tag{9}$$

Note that due to invertibility of ρ_{ez} we get (differentiating $\gamma(t)*e(\gamma(t))=\gamma(t)$ by t at $t=0$, $\gamma(0)=\varepsilon$) that $(\varphi_\varepsilon)_{*,\varepsilon}-\mathrm{id}_\varepsilon=(\rho_{e\varepsilon})_{*,\varepsilon}$ is invertible.

We can present (9) as

$$L_x\circ L_y\circ L_{(\varphi x)^{-1}}=L_{x*(y*(\varphi x)^{-1})},\quad (L_xy\overset{\mathrm{def}}{=}x*y). \tag{10}$$

Due to leftmonoalternativity we get further $L^{-1}_{(\varphi x)^{-1}} \circ L^{-1}_y \circ L^{-1}_x = L^{-1}_q$, or (putting $y^{-1} = z$, $x^{-1} = Ix = p$, $w = q^{-1}$)

$$L_{\varphi Ip} \circ L_z \circ L_p = L_w. \tag{11}$$

Applying both parts of (11) to ε, we get $w = (\varphi Ip) * (z * p)$ and

$$L_{\varphi Ip} \circ L_z \circ L_p = L_{(\varphi Ip)*(z*p)}. \tag{12}$$

Combining now (12) and (10) we obtain

$$\begin{aligned} L_{x*(y*(\varphi x)^{-1})} &= (L_x \circ L_{\varphi Ix}) \circ (L^{-1}_{\varphi Ix} \circ L_y \circ L^{-1}_x) \circ L_x \circ L_{(\varphi x)^{-1}} \\ &= (L_x \circ L_{(\varphi x)^{-1}}) \circ (L_{\varphi x} \circ L_y \circ L_{Ix}) \circ (L_x \circ L_{(\varphi x)^{-1}}) \\ &= L_{x*(\varphi x)^{-1}} \circ L_{\varphi x*(y*x^{-1})} \circ L_{x*(\varphi x)^{-1}}. \end{aligned}$$

Thus

$$L_{\lambda x} \circ L_w \circ L_{\lambda x} = L_{\lambda x*(w*\lambda x)}, \quad (\forall w), \tag{13}$$

where $\lambda x = x * (\varphi x)^{-1}$.

But $\lambda : x \mapsto \lambda x = x * (\varphi x)^{-1}$ is locally invertible near $\varepsilon \in M$, since $(\lambda)_{*,\varepsilon} = \mathrm{id}_\varepsilon - \varphi_{*,\varepsilon}$ (which is invertible).

Consequently, in a suitable neighbourhood of ε,

$$L_y \circ L_w \circ L_y = L_{y*(w*y)}, \quad (\forall y, w) \quad \text{(Left Bol Property)}. \tag{14}$$

We have proved the following:

Proposition 2.5. *Any geodesic loop of a perfect ts-space is a Bol loop.*

Proposition 2.6. *For any geodesic loop of a perfect ts-space, $(\varphi x)^{-1} * x^{-1} \in N_l = \{w; w*(a*b) = (w*a)*b,\ \forall a, b\}$ – the left nucleus of $\langle M, *, \varepsilon\rangle$.*

Proof.

$$\begin{aligned} L_{(\varphi x)^{-1}} \circ L_y \circ L_x = L_{(\varphi x)^{-1}*(y*x)} &\Longrightarrow L_{(\varphi x)^{-1}*(y*x)} \\ = (L_{(\varphi x)^{-1}} \circ L_{x^{-1}}) \circ (L_x \circ L_y \circ L_x) &= (L_{(\varphi x)^{-1}} \circ L_{x^{-1}}) \circ L_{x*(y*x)}. \end{aligned}$$

Thus

$$(L_{(\varphi x)^{-1}} \circ L_{x^{-1}}) \circ L_z = L_w, \quad (\forall z). \tag{15}$$

Further, at $z = \varepsilon$, $L_{(\varphi x)^{-1}} \circ L_{x^{-1}} = L_{(\varphi x)^{-1} * x^{-1}}$.
Consequently,

$$L_{((\varphi x)^{-1} * x^{-1})} \circ L_z = L_{((\varphi x)^{-1} * x^{-1}) * z}, \quad (\forall z), \tag{16}$$

which proves our assertion. □

Remark 2.7. It is known that for a left Bol loop the left nucleus is a normal subloop. See [10], [11].

Proposition 2.8. *If for a loop*

$$L_x \circ L_y \circ L_x = L_{x*(y*x)}, \quad \textit{(Left Bol property)},$$

and

$$L_{((\varphi x)^{-1} * x^{-1})} \circ L_z = L_{((\varphi x)^{-1} * x^{-1}) * z},$$

then

$$L_{(\varphi x)^{-1}} \circ L_y \circ L_x = L_{(\varphi x)^{-1} * (y * x)}.$$

PROOF.

$$\begin{aligned} L_{(\varphi x)^{-1}} \circ L_y \circ L_x &= (L_{(\varphi x)^{-1}} \circ L_{x^{-1}}) \circ (L_x \circ L_y \circ L_x) \\ &= (L_{(\varphi x)^{-1}} \circ L_{x^{-1}}) \circ L_{x*(y*x)}. \end{aligned} \tag{17}$$

Taking $y = x^{-1}$, due to leftmonoalternativity we get $L_{((\varphi x)^{-1} * x^{-1})} \circ L_x = L_{(\varphi x)^{-1}}$, or

$$L_{(\varphi x)^{-1}} \circ L_{x^{-1}} = L_{((\varphi x)^{-1} * x^{-1})}. \tag{18}$$

Now (17) and (18) imply

$$L_{(\varphi x)^{-1}} \circ L_y \circ L_x = L_{((\varphi x)^{-1} * x^{-1})} \circ L_z = L_w,$$

which proves our assertion. □

As a result, due to the above considerations, we have reduced the problem to investigate a perfect *ts*-space to the exploration of a smooth local loop $\langle M, *, \varepsilon \rangle$ with certain properties. We summarize it in the following proposition.

Proposition 2.9. *Any geodesic loop $\langle M, *, \varepsilon\rangle$ of a perfect ts-space $(M, (\sigma_x)_{x\in M})$ satisfies the identities*

(1) $L_x \circ L_y \circ L_x = L_{x*(y*x)}$ *(Left Bol property), (which implies, as known, leftmonoalternativity, see* [5], [6]*),*

(2) $L_{\varphi x * x} \circ L_y = L_{(\varphi x * x)*y}$, *where* $\sigma_\varepsilon \circ e = \varphi \in \operatorname{End}\langle M, *, \varepsilon\rangle$, $\varphi_{*,\varepsilon} - \mathrm{id}_{*,\varepsilon}$ *is invertible (meaning that* $\varphi x * x \in N_l$ *(left nucleus) which is a normal subloop; moreover it is a group),*

(3) $l(a,b) = L_{a*b}^{-1} \circ L_a \circ L_b \in \operatorname{Aut}\langle M, *, \varepsilon\rangle$, *(Left A-property). (This last property is due to the reductivity of* $(M, (\sigma_x)_{x\in M})$, *see* [1].*)*

2.10. Since any smooth leftmonoalternative loop with the left A-property (4) can be realized as a reductant Q in an appropriate reductive space G/H (locally at least, see [5], [6]), it is evident that $N_l \subset Q$ will be a normal subgroup in G. Moreover, since in this case, as well known (see [8], [9]), $(x*y)*(x^{-1}*y^{-1}) \in N_l$, we get a symmetric space $(G/N_l)/(H/N_l)$.

Passing to the corresponding Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$ for G, H, N_l, respectively, we have

$$\begin{gathered}\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}, \quad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}, \quad \mathfrak{n} \subset \mathfrak{m}, \quad \exp \mathfrak{m} = Q, \\ \text{where } \mathfrak{n} \text{ is an ideal of } \mathfrak{g}, \quad (\mathfrak{g}/\mathfrak{n})/(\mathfrak{h}/\mathfrak{n}) \text{ is a symmetric pair.}\end{gathered} \tag{19}$$

As a result, we have got the following:

Proposition 2.11. *If* $(M, (\sigma_x)_{x\in M})$ *is a perfect transsymmetric space, then it is reductive,* $M = G/H$ *(at least locally). Moreover if* $\mathfrak{g}$, $\mathfrak{h}$ *are the Lie algebras for* G *and* H *and* $\mathfrak{m}$ *is a reductive complement to* $\mathfrak{h}$ *in* $\mathfrak{g}$ $(\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m})$, *then* $\mathfrak{m}$ *contains an ideal* $\mathfrak{n}$ *of* $\mathfrak{g}$ *such that* $(\mathfrak{g}/\mathfrak{n})/(\mathfrak{h}/\mathfrak{n})$ *is a symmetric pair.*

Proposition 2.12. *With notations and conditions as above, if*

$$\tilde{\varphi} = \begin{cases} \varphi_{*,\varepsilon} & \text{on } \mathfrak{m}, \\ \mathrm{Id} & \text{on } \mathfrak{h}, \end{cases}$$

then $\tilde{\varphi} \in \operatorname{End} \mathfrak{g}$, $\tilde{\varphi}\mathfrak{m} = \mathfrak{m}$, $\tilde{\varphi}\zeta = \zeta \iff \zeta \in \mathfrak{h}$, $\tilde{\varphi}\mathfrak{n} \subset \mathfrak{n}$. *Moreover* $[\tilde{\varphi}\zeta, \zeta] \subset \mathfrak{n}$ *for any* $\zeta \in \mathfrak{m}$.

Proof. Due to the above construction, all assertions except the last one are evident.

Let us consider $((\varphi x)^{-1} * x^{-1}) \in N_l$ (the left nucleus of $\langle M, *, \varepsilon\rangle$), $\forall x \in M$. We may denote x^{-1} by y due to the arbitrariness of x. Thus $(\varphi y * y) \in N_l$, $\forall y \in Q$. Further, $[\varphi(\gamma(t)) * \gamma(t)] \in N_l$ for any smooth arc $(\gamma(0) = \varepsilon)$. Since $N_l = \exp \mathfrak{n}$, we get, after differentiation, $(\tilde{\varphi} + \mathrm{Id})\mathfrak{m} \subset \mathfrak{n} \subset \mathfrak{m}$. □

Finally, we get the following infinitesimal description of a perfect transsymmetric space.

Proposition 2.13. *Any perfect transsymmetric space M is reductive $M = G/H$ (at least locally).*

If $(\mathfrak{g}, \mathfrak{h})$ are the Lie algebras for G and H, respectively, and $\mathfrak{m}$ is a reductive complement to $\mathfrak{h}$ in $\mathfrak{g}$ ($\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$) then there exists an endomorphism $\tilde{\varphi}$ of $\mathfrak{g}$ such that $\tilde{\varphi}\mathfrak{m} = \mathfrak{m}$, $\tilde{\varphi}\zeta = \zeta \Longleftrightarrow \zeta \in \mathfrak{h}$ and an ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{m}$, $(\tilde{\varphi} + \mathrm{Id})\mathfrak{m} \subset \mathfrak{n}$, which means in particular, that $(\mathfrak{g}/\mathfrak{n})/(\mathfrak{h}/\mathfrak{n})$ is a symmetric pair with the defining symmetry $\tilde{\varphi}/\mathfrak{n} = s$.

The above said means that a perfect transsymmetric space is "glued" in a nontrivial way by means of some normal subgroup and a symmetric space.

In particular, if G is simple, then the corresponding perfect transsymmetric space is either a group space, or a symmetric space.

Remark 2.14. The theory of perfect spaces appeared for the first time in [12] for s-spaces (a particular subcase of Transsymmetric spaces).

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