Publ. Math. Debrecen
54 / 3-4 (1999), 313–320

Volume of tubes in noncompact symmetric spaces

By X. GUAL-ARNAU (Castellón) and A. M. NAVEIRA (Burjassot)

Abstract. From the duality between symmetric spaces of compact type and symmetric spaces of noncompact type, we deduce an expression for the "infinitesimal change of volume function" of geodesic balls and tubes around certain submanifolds of a noncompact symmetric space of arbitrary rank. We also give a formula for the volume of a geodesic ball in a noncompact symmetric space.

0. Introduction

The theory of tubes and geodesic balls in symmetric spaces of rank one has been fully developed without making an explicit reference mention to the theory of roots of the symmetric space [AGV], [G].

One important fact that arises for compact symmetric spaces of rank one is that the isotropy representation acts transitively on the unit sphere of the tangent space of any point. Since that property is not satisfied for symmetric spaces of rank greater than one, we have to use the theory of restricted roots and orbital geometry of the linear action of the isotropy group of a compact symmetric space, to obtain an expression for the area of geodesic balls and tubes around a submanifold of that compact symmetric space [N-G].

The aim of this work is to use the duality between compact and noncompact symmetric spaces to show that some of the formulae obtained in

Mathematics Subject Classification: 53C35, 53C21.
Key words and phrases: noncompact symmetric space, geodesic ball, restricted roots, tube, totally geodesic submanifold.
Work partially supported by the Spanish DGICYT grant No. PB94-0972 and the Mexican CONACyT grant No. 0329P-E.

[N-G] for geodesic balls and tubes in compact symmetric spaces of arbitrary rank are easily adaptable to the noncompact case.

Finally, if we restrict our results to symmetric spaces of rank one we can deduce the well-known formulae for geodesic balls and tubes in these spaces. For instance, the volume of a geodesic ball in an n-dimensional sphere of constant curvature $\lambda = 1$ is given by

$$V_m^1(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^r \sin^{n-1}(t)\,\mathrm{d}\,t, \tag{1}$$

and the volume of a geodesic ball in the hyperbolic space of constant curvature $\lambda = -1$ is

$$V_m^2(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^r \sinh^{n-1}(t)\,\mathrm{d}\,t. \tag{2}$$

So, the formulas for the case of sectional curvature $\lambda < 0$ can be found by changing all of the trigonometric functions, which appear for the case of sectional curvature $\lambda > 0$, to hyperbolic functions. Similar remarks apply to all rank one symmetric spaces and, as we will see in this work, also for symmetric spaces of arbitrary rank.

To illustrate this program we will use the symmetric spaces of type **AIII** as examples. Therefore, the complex projective space (compact case) and the complex hyperbolic space (noncompact case) are included in these examples.

1. Duality for symmetric spaces

Let $M_0 = G_0/K$ be a noncompact symmetric space and $M = G/K$ a compact symmetric space dual to M_0. We denote by g_0 the orthogonal involutive Lie algebra associated to M_0. Let

$$g_0 = k + m_0 \tag{3}$$

be the Cartan decomposition of g_0 (k is the Lie algebra of K). If g denotes the complexification of g_0, then the compact real form g_c of g, given by

$$g_c = k + m \quad \text{where } m = im_0, \quad (i = \sqrt{-1}), \tag{4}$$

is the orthogonal Lie algebra associated with M.

Let h_{m_0} denote a maximal abelian subspace of m_0 and let h_0 be any maximal abelian subalgebra of g_0 containing h_{m_0}. We denote by h the subspace of g generated by h_0 and $h_m = ih_{m_0}$ denotes a maximal abelian subspace of m.

We give M_0 the unique G_0-invariant Riemannian structure induced by the restriction of the Killing form B of g to $m_0 \times m_0$. Similarly, the space M will be given the unique G-invariant Riemannian structure induced by the restriction of $-B$ of g to $m \times m$.

Let Δ denote the set of all nonzero roots of g with respect to h, and let Δ_{m_0} denote the set of those roots which do not vanish identically on h_{m_0}. Each $\alpha \in \Delta$ is a $\mathbb{C}$-linear function on h which is real-valued on h_{m_0}; therefore, we get in this way an ordering of Δ_{m_0}. Let Δ^+ denote the subset of positive roots in Δ_{m_0}.

The sectional curvature of the symmetric spaces M_0 and M is given, by means of the roots in Δ^+, as follows:

Let H_0 and X_0 be two unit vectors in h_{m_0} and m_0, respectively; the sectional curvature of M_0 is

$$\begin{aligned} K_{H_0X_0} &= \langle R(H_0, X_0)H_0, X_0 \rangle \\ &= \langle -[H_0, [H_0, X_0]], X_0 \rangle = -\alpha^2(H_0), \quad \alpha \in \Delta^+. \end{aligned} \tag{5}$$

Therefore, since $\alpha(H_0) \in \mathbb{R}$, $K_{H_0X_0} < 0$.

Now, let $H = iH_0$ and $X = iX_0$ be two unit vectors in h_m and m, respectively; the sectional curvature of M is

$$K_{HX} = \langle R(H, X)H, X \rangle = -\alpha^2(H) = -\alpha^2(iH_0) = \alpha^2(H_0) > 0. \tag{6}$$

On the other hand, if we consider the linear actions

$$\mathrm{Ad} : K \times m_0 \longrightarrow m_0 \quad \text{and} \quad \mathrm{Ad} : K \times m \longrightarrow m,$$

it can be shown that the two actions have the same principal orbit type. Moreover, if we denote by K/K_{H_0} this principal orbit type, (K_{H_0} is the isotropy subgroup of K), the volume of the principal orbits $K(H_0) = \bigcup_{k \in K} \mathrm{Ad}(k)H_0$, ($H_0 \in m_0$), and $K(H) = \bigcup_{k \in K} \mathrm{Ad}(k)H$, ($H \in m$) is ([H-H], [H-L]):

$$v^2(K(H_0)) = c^2 \prod_{j=q+1}^{n} \alpha_j^2(H_0) = c^2 \prod_{j=q+1}^{n} \alpha_j^2(H) = v^2(K(H)); \tag{7}$$

where c denotes the constant given by the volume of K/K_{H_0}.

2. The volume function of a geodesic ball in a noncompact symmetric space

We begin this section by giving an expression for the infinitesimal change of the volume function from the roots in Δ^+. The infinitesimal change of the volume function appears in the volume formula of a tube and it is defined as follows: given a submanifold P of M_0, the infinitesimal change of the volume function $\vartheta_{H_0}(t)$ of P in the direction of a unit vector H_0, normal to P, is given by $\omega(\xi(t)) = \vartheta_{H_0}(t)\,\mathrm{d}\,H_0 \wedge \mathrm{d}\,P \wedge \mathrm{d}\,t$, where $\xi(t)$ is a geodesic of M satisfying $\xi(0) = p \in P$ and $\xi'(0) = H_0$, ω is the Riemannian volume element of M_0, $\mathrm{d}\,P$ that of P, and $\mathrm{d}\,H_0$ that of the unit sphere in the normal space $T_pP^{\perp}$. This definition makes sense when P is a point $p \in M_0$ and H_0 is a unit vector in m_0; in this case a geodesic ball (tube around a point) is considered and we obtain the following expression for the infinitesimal change of the volume function:

Proposition 2.1. *Let M_0 be a noncompact symmetric space and $\xi(t)$ a geodesic in M_0 such that $\xi(0) = p$ and $\xi'(0) = H_0$ (H_0 a unit vector in a maximal abelian subspace h_{m_0} of m_0). Then, the infinitesimal change of the volume function $\vartheta_{H_0}(t)$ is given by*

$$\vartheta_{H_0}(t) = t^{-(n-q)} \prod_{j=q+1}^{n} \frac{\sinh(t\alpha_j(H_0))}{\alpha_j(H_0)}. \tag{8}$$

where q is the rank of M_0, (i.e. $\dim(h_{m_0})$), and α_j varies in Δ^+, where without loss of generality we assume that $0 = \alpha_1(H_0) = \cdots = \alpha_q(H_0) < \alpha_{q+1}(H_0) \leq \cdots \leq \alpha_n(H_0)$.

Proof. By analogy with the compact case (see [N-G]), we omit the explicit proof of the proposition. The main difference between the two cases is that for noncompact symmetric spaces the sectional curvature is negative and the Riccati differential equations for the principal curvature functions are:

$$\begin{aligned} &k_i'(t) = k_i(t)^2, && i = 2, \ldots, q; \\ &k_j'(t) = k_j(t)^2 - \alpha_j(H_0)^2, && j = q+1, \ldots, n. \end{aligned} \tag{9}$$

Now, starting with the expression (8) for the infinitesimal change of volume function, our purpose is to simplify the integral over the unit sphere which appears in the next identity for the volume function of a geodesic ball:

$$V_p^{M_0}(r) = \int_0^r A_p^{M_0}(t)\,\mathrm{d}\,t = \int_0^r \int_{S^{n-1}(1)} t^{n-1}\vartheta_{H_0}(t)\,\mathrm{d}\,H_0\,\mathrm{d}\,t. \tag{10}$$

In order to achieve this, we consider the next two properties, which are satisfied for compact and noncompact symmetric spaces.

Proposition 2.2. *The infinitesimal change of volume function $\vartheta_{H_0}(t)$ does not depend on the vector chosen in the orbit $\bigcup_{k\in K} Ad(k)H_0$.*

Proposition 2.3. *Let $S^{n-1}(1)$ and $S^{q-1}(1)$ be the unit spheres in m_0 and h_{m_0}, respectively. Given a Weyl chamber D_0 in h_{m_0} we consider $C_0 = \bar{D}_0 \cap S^{q-1}(1)$, where $\bar{D}_0$ is the closure of D_0; then,*

$$S^{n-1}(1) = \bigcup_{k\in K} Ad(k)\,C_0. \tag{11}$$

(See [N-G] for the proofs of both propositions.)

Theorem 2.4. *The volume of a geodesic ball in the noncompact symmetric space M_0 is given by*

$$\begin{aligned} V_p^{M_0}(r) &= \int_0^r A_p^{M_0}(t)\,\mathrm{d}\,t \quad \textit{where} \\ A_p^{M_0}(t) &= ct^{q-1}\int_{C_0}\prod_{j=q+1}^{n}\sinh(t\alpha_j(H_0))\,\mathrm{d}\,H_0 \end{aligned} \tag{12}$$

where c denotes the constant given by the volume of the principal orbit K/K_{H_0}.

Proof. From Propositions 2.2 and 2.3, we can express tha area function of the geodesic ball as

$$A_p^{M_0}(t) = t^{n-1}\int_{C_0} v(H_0)\vartheta_{H_0}(t)\,\mathrm{d}\,H_0, \tag{13}$$

where $v(H_0)$ is the volume of the orbit $\bigcup_{k\in K} Ad(k)H_0$.

Now, replacing $\vartheta_{H_0}(t)$ and $v(H_0)$ in (13), and having in mind (8) and (7), respectively, we obtain the desired result. □

3. Example: Symmetric spaces of type AIII

Now, we shall consider the noncompact symmetric space $M_0 = SU(p,q)/S(U_p \times U_q)$, and we specialize the expression (12) for this space. Here,

$$m = \left\{ \begin{pmatrix} 0 & iZ \\ -i\bar{X}^t & 0 \end{pmatrix} : Z \text{ complex } p \times q \text{ matrix} \right\}; \tag{14}$$

and

$$h_{m_0} = \sum_{j=1}^{q} i\mathbb{R}(E_{j,p+j} - E_{p+j,j}) \tag{15}$$

where E_{jk} denotes the matrix $(\delta_{aj}\delta_{bk})$.

Let H_0 be a unit vector in h_{m_0} which we identify with $(x_1, \ldots, x_q)$, such that $x_1^2 + \cdots + x_q^2 = 1$. Then, from (5), the eigenvalues of R_{H_0} are: 0 with multiplicity q, $-x_j^2$ with multiplicity $2(p-q)$, $-4x_j^2$ with multiplicity 1 and $-(x_j - x_k)^2$, $-(x_j + x_k)^2$ with multiplicity 2; for $1 \leq j < k \leq q$.

A Weyl chamber in h_{m_0} is given by $D_0 = \{(x_1, \ldots, x_q) : x_1 > \cdots > x_q > 0\}$.

From (12) we have that

$$\begin{aligned} A_p^{M_0}(t) = ct^{q-1} \int_{C_0} \prod_{j=1}^{2(p-q)} \sinh(tx_j) \prod_{j=1}^{q} \sinh(2tx_j) \\ \times \prod_{\substack{j=1 \\ j<k}}^{q} \sinh^2(t(x_k - x_j)) \sinh^2(t(x_k + x_j)) \, \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_q. \end{aligned} \tag{16}$$

Remark 1. For $q = 1$, M_0 is the complex hyperbolic space and the area function of the geodesic ball is, [G],

$$A_p^{M_0}(t) = c \sinh^{2(p-1)}(t) \sinh(2t) = 2c \sinh^{2p-1}(t) \cosh(t). \tag{17}$$

Remark 2. The area function $A_p^M(t)$ for the compact symmetric space dual to M_0 is given by changing in (16) all the hyperbolic functions to trigonometric functions.

To end this section we plot the volume of a geodesic ball, up to the same constant factor, in the compact and noncompact symmetric spaces of type **AIII** from 0 to the injectivity radius of the compact symmetric space, which is $\pi/2$, [N-G].

$SU(4)/S(U(2)\times U(2))$ $\qquad$ $SU(2,2)/S(U(2)\times U(2))$

4. Tubes in noncompact symmetric spaces

In the preceding section we have computed the volume of a geodesic ball in a noncompact symmetric space M_0 as an adaptation of the volume function of geodesic balls in the compact symmetric space M dual to M_0. In this section we consider tubes around totally geodesic submanifolds compatible with M_0.

Let P_0 be a totally geodesic submanifold in M_0; then ([He] and [W]), $P_0 = U_0/L$ is a noncompact symmetric space. Let μ_0 denote the Lie algebra of U_0 and $\mu_0 = \rho_0 + l$ its canonical decomposition.

In order to get the compatibility condition of P_0 with M_0 we assume that for all vectors H_0 in $\rho_0^{\perp}$ the curvature operator satisfies the condition of preserving the curvature $R_{H_0}(\rho_0) \subset \rho_0$ and $R_{H_0}(\rho_0^{\perp}) \subset \rho_0^{\perp}$, (the first inclusion is immediate because ρ_0 is a Lie triple system).

Let P denote the compact symmetric space dual to P_0, which is, in its turn, a totally geodesic submanifold compatible with M.

In [N-G] we have computed the infinitesimal change of volume function of P; therefore, as we have seen in Proposition 2.1 for geodesic balls, the infinitesimal change of the volume function of P_0 is given by changing all of the trigonometric functions, which appear in the infinitesimal change of the volume function of P, to hyperbolic functions.

However, although we have an expression for the infinitesimal change of the volume function of P_0, it is not possible to obtain a finite value for volume of a tube around P_0 because P_0 is a noncompact submanifold of M_0.

Acknowledgement. We wish to express our gratitude to Dr. A. Sanchez-Valenzuela for his fruitful comments about symmetric spaces and for making possible the visit of the first author to the CIMAT (Guanajuato, Mexico).

References

[AGV] E. Abbena, A. Gray and L. Vanhecke, Steiner's Formula for the volumes of parallel hypersurfaces in a Riemannian manifold, *Ann. Scuola Norm. Pisa Cl. Sci.* **8** (1981), 473–493.

[G-N] X. Gual and A.M. Naveira, The volume of geodesic balls and tubes in $S^{\mathrm{n}}(1) \times S^{\mathrm{m}}(1)$, Proceedings of the 23rd conference on geometry and topology, *Babes-Bolyai University*, *Cluj-Napoca, Romania*, 1993, 71–79.

[Gr] A. Gray, Tubes, *Addison-Wesley*, *Reading*, 1990.

[He] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, *Academic Press*, 1978.

[H-H] Wu-teh Hsiang and Wu-Yi Hsiang, On the construction of constant mean curvature imbeddings of exotic and/or knotted spheres into $S^{\mathrm{n}}(1)$, *Invent. Math.* **82** (1985), 423–445.

[H-L] Wu-Yi Hsiang and H. Blaine Lawson, Jr., Minimal submanifolds of low cohomogeneity, *J. Diff. Geom.* **5** (1971), 1–38.

[N-G] A. M. Naveira and X. Gual, The volume of geodesic balls and tubes about totally geodesic submanifolds in compact symmetric spaces, *Diff. Geom. Appl.* **7** no. 2 (1997), 101–113.

[W] J.A. Wolf, Spaces of constant curvature, *Publish or Perish*, *Boston*, 1974.

X. GUAL ARNAU
DEPARTAMENT DE MATEMÁTIQUES
UNIVERSITAT JAUME I
12071 CASTELLÓN
SPAIN

E-mail: gual@ mat.uji.es

A.M. NAVEIRA.
DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSITAT DE VALENCIA
46100 BURJASSOT (VALENCIA)
SPAIN

E-mail: naveira@ uv.es

(Received July 10, 1997; accepted March 17, 1998)