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Some examples of almost complex manifolds with Norden metric

By V. OPROIU (Iaşi) and N. PAPAGHIUC (Iaşi)

A Norden metric on the almost complex manifold (M, J) is a pseudo-Riemannian metric g on M such that

$$g(JX, JY) = -g(X, Y)$$

for arbitrary vector fields X, Y on M. Then g has the signature (n, n) where $2n$ is the (real) dimension of M. In [2] the almost complex manifolds with Norden metric have been classified in eight classes. In this paper we present examples of almost complex manifolds with Norden metric belonging to some of the classes from [2]. Our examples are constructed on the cotangent bundle of a smooth manifold, endowed with a symmetric (nonlinear) connection and a nondegenerate symmetric M-tensor field of type (0,2). These examples cover five of the eight classes in the classification given in [2]. The remaining classes are those of the special complex, quasi-Kählerian and $\omega_1 \oplus \omega_3$ manifolds with Norden metric. Remark that in [1] the authors have obtained examples of Kählerian, special complex and semi-Kählerian manifolds with Norden metric. The examples given in this paper can be considered also on the tangent bundle endowed with an appropriate (nonlinear) connection. Throughout this paper the well known summation convention is used and the range for the indices $h, i, j, k, l, m, \dots$ is $\{1, 2, \dots, n\}$.

1. Almost complex manifolds with Norden metric

Let (M, J) be a $2n$-dimensional almost complex manifold, i.e. J is an endomorphism of the tangent bundle with $J^2 = -I$. A pseudo-Riemannian metric g on M is a Norden metric for J if

$$g(JX, JY) = -g(X, Y); \quad X, Y \in \mathfrak{X}(M). \tag{1}$$

Then g has necessarily the signature (n,n). If ∇ is the Levi Civita connection of g then the following tensor field F, of type (0,3) may be considered

$$F(X,Y,Z) = g((\nabla_X J)Y,Z); \quad X,Y,Z \in \mathfrak{X}(M).$$

The tensor field F has the following properties

$$F(X,Y,Z) = F(X,JY,JZ) = F(X,Z,Y). \tag{2}$$

The space of tensor fields F of type (0,3) with the properties (2) has been decomposed (see [2]) in to the direct sum of three terms. Using this decomposition, there are obtained eight classes of almost complex manifolds with Norden metric. To describe these classes, introduce the following 1-form φ, associated with F:

$$\varphi_p(X) = g^{ij}F(e_i,e_j,X); \quad p \in M,\ X \in T_pM,\ i,j = 1,\dots,2n, \tag{3}$$

where $(e_1,\dots,e_{2n})$ is a basis in T_pM and g_{ij}, g^{ij} are the components of the metric tensor field g and its inverse with respect to this basis.

Then M is called:

1. Kählerian manifold with Norden metric if

$$F(X,Y,Z) = 0 \tag{4}$$

or, equivalently, $\nabla J = 0$.

2. Conformally Kählerian manifold with Norden metric, or ω_1-manifold if

$$\begin{aligned} 2nF(X,Y,Z) =& g(X,Y)\varphi(Z) + g(X,Z)\varphi(Y) + g(X,JY)\varphi(JZ) + \\ &+ g(X,JZ)\varphi(JY). \end{aligned} \tag{5}$$

3. Special complex manifold with Norden metric, or ω_2-manifold if

$$\varphi = 0, \quad F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0. \tag{6}$$

4. Quasi-Kählerian manifold with Norden metric, or ω_3-manifold if

$$F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0. \tag{7}$$

5. Complex manifold with Norden metric, or $\omega_1 \oplus \omega_2$-manifold if

$$F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0. \tag{8}$$

6. Semi-Kählerian manifold with Norden metric, or $\omega_2 \oplus \omega_3$-manifold if

$$\varphi = 0. \tag{9}$$

7. $\omega_1 \oplus \omega_3$-manifold if

$$
\begin{aligned}
& n\left\{ F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) \right\} = g(X,Y)\varphi(Z) + \\
& + g(Z,X)\varphi(Y) + g(Y,Z)\varphi(X) + g(X,JY)\varphi(JZ) + \\
& + g(Y,JZ)\varphi(JX) + g(Z,JX)\varphi(JY).
\end{aligned}
\tag{10}
$$

8. Almost complex manifold with Norden metric if no special condition is fulfilled.

Remark that in [1] the classes of Kählerian, special complex and semi-Kählerian manifolds with Norden metric have been considered on the tangent bundle of an almost Hermitian manifold.

In this paper, we get examples for five of the eight classes listed above, i.e.: Kählerian, conformally Kählerian, compex, semi-Kählerian and almost complex manifolds with Norden metric. Our examples are obtained in the case of the cotangent bundle of a smooth manifold with a symmetric (nonlinear) connection.

2. The case 8: almost complex structures with Norden metric on cotangent bundles

Let M be an n-dimensional smooth manifold and denote by $\pi : T^*M \longrightarrow M$ its cotangent bundle with fibres the cotangent spaces to M. Then T^*M is a $2n$-dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Let (U,x) be a local chart on M with the domain U and the coordinate map $x = [x^i]$; $i = 1, \dots, n$. Then the local chart $(\pi^{-1}(U), (q,p))$ is induced on T^*M where the coordinate map $(q,p) = [q^i, p_i]$ is defined as follows. Firstly, $q^i = x^i \cdot \pi$, i.e. the first n local coordinates of a cotangent vector from $\pi^{-1}(U)$ are the local coordinates of its base point, thought of as functions on $\pi^{-1}(U)$. Then p_i; $i = 1, \dots, n$ are the vector space coordinates with respect to the natural local frame $(dx^1, \dots, dx^n)$ in T^*M defined by (U,x). The M-tensor fields and the linear M-connections may be considered on T^*M and the usual tensor fields and the linear connections on M may be thought of naturally as M-tensor fields and linear M-connections on T^*M.

Let $VT^*M = \operatorname{Ker} \pi_* \subset TT^*M$ be the vertical distribution over T^*M. Then VT^*M is involutive with fibre dimension n and the local vector fields $\frac{\partial}{\partial p_i} = \partial^i$; $i = 1, \dots, n$ define a local frame in VT^*M. A (nonlinear) connection on T^*M is defined by a complementary distribution HT^*M

(horizontal distribution) to VT^*M in TT^*M. A local frame in HT^*M is defined by the vector fields $\frac{\delta}{\delta q^i}$; $i = 1, \dots, n$ where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - N_{ij} \frac{\partial}{\partial p_j} \ . \tag{11}$$

The functions N_{ij}; $i, j = 1, \dots, n$ are the connection coefficients of the considered (nonlinear) connection in the induced local chart $(\pi^{-1}(U), (q, p))$. We shall assume $N_{ij} = N_{ji}$, i.e. the considered nonlinear connection is symmetric (there always exists a symmetric nonlinear connection on T^*M). Then

$$\left[\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right] = \Phi^i_{jk} \frac{\partial}{\partial p_k}, \quad \left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right] = -R_{kij} \frac{\partial}{\partial p_k} \tag{12}$$

where

$$\Phi^i_{jk} = -\frac{\partial N_{jk}}{\partial p_i}, \quad R_{kij} = \frac{\delta N_{kj}}{\delta q^i} - \frac{\delta N_{ki}}{\delta q^j}. \tag{13}$$

Remark that the components Φ^i_{jk}; $i, j, k = 1, \dots, n$ define a linear M-connection on T^*M and the components R_{kij}; $i, j, k = 1, \dots, n$ define an M-tensor field of type (0,3) on T^*M.

The pseudo-Riemannian metric G with the signature (n, n) is defined on T^*M by the following local coordinate expression:

$$G = 2(dp_i + N_{ij} dq^j) dq^i \tag{14}$$

(see [3]) and its Levi Civitia connection $\widetilde{\nabla}$ is given by

$$\begin{aligned} &\widetilde{\nabla}^i \frac{\partial}{\partial p_j} = 0, \quad \widetilde{\nabla}^i \frac{\delta}{\delta q^j} = 0, \quad \widetilde{\nabla}_i \frac{\partial}{\partial p_j} = -\Phi^j_{ik} \frac{\partial}{\partial p_k}, \\ &\widetilde{\nabla}_i \frac{\delta}{\delta q^j} = \Phi^k_{ij} \frac{\delta}{\delta q^k} + R_{ijk} \frac{\partial}{\partial p_k} \end{aligned} \tag{15}$$

where we have denoted:

$$\widetilde{\nabla}^i = \nabla_{\frac{\partial}{\partial p_i}}, \qquad \widetilde{\nabla}_i = \widetilde{\nabla}_{\frac{\delta}{\delta q^i}}.$$

From $\widetilde{\nabla}$ we get its Schouten connection $\overline{\nabla}$ given by

$$\begin{aligned} &\overline{\nabla}^i \frac{\partial}{\partial p_j} = 0, \quad \overline{\nabla}^i \frac{\delta}{\delta q^j} = 0, \quad \overline{\nabla}_i \frac{\partial}{\partial p_j} = -\Phi^j_{ik} \frac{\partial}{\partial p_k}, \\ &\overline{\nabla}_i \frac{\delta}{\delta q^j} = \Phi^k_{ij} \frac{\delta}{\delta q^k}. \end{aligned} \tag{16}$$

Next, we may consider the covariant derivatives with respect to $\overline{\nabla}$ of the M-tensor fields on T^*M. E.g. if the components $g_{jk};\ j,k=1,\dots,n$ define an M-tensor field of type (0,2) on T^*M then

$$\overline{\nabla}^i g_{jk} = \partial^i g_{jk}, \qquad \overline{\nabla}_i g_{jk} = \frac{\delta g_{jk}}{\delta q^i} - \Phi^h_{ij} g_{hk} - \Phi^h_{ik} g_{jh}.$$

The components $\overline{\nabla}_i g_{jk};\ i,j,k=1,\dots,n$ define an M-tensor field of type (0,3) on T^*M.

Assume that the components $g_{jk};\ j,k=1,\dots,n$ define a *nondegenerate* M-tensor field of type (0,2) on T^*M. Denote by $g^{jk};\ j,k=1,\dots,n$ the components of its inverse matrix, i.e.

$$g_{ih} g^{kh} = g_{hi} g^{hk} = \delta^k_i. \tag{17}$$

Then the components $g^{jk};\ j,k=1,\dots,n$ define an M-tensor field of type (2,0) on T^*M.

Definition. The almost complex structure J on T^*M determined by the nondegenerate M-tensor field g_{ij} is given by:

$$J\left(\frac{\delta}{\delta q^i}\right) = g_{ji} \frac{\partial}{\partial p_j}, \quad J\left(\frac{\partial}{\partial p_i}\right) = -g^{ij} \frac{\delta}{\delta q^j}. \tag{18}$$

J defines in fact an almost complex structure on T^*M, as it can be checked by a straightforward computation.

Proposition 1. *The pseudo-Riemannian metric G given by* (14) *is a Norden metric for the almost complex structure J given by* (18) *if and only if M-tensor field g_{ij} is symmetric, i.e. $g_{ij} = g_{ji}$.*

PROOF. By straightforward computation.

Thus, the existence of a symmetric (nonlinear) connection and of a symmetric nondegenerate M-tensor field of type (0,2) on the cotangent bundle T^*M of a smooth manifold M assures that T^*M may be organized as an almost complex manifold with Norden metric, i.e. a manifold from the class 8.

The following expressions of F and φ considered in the first section of this paper are obtained in the case of (T^*M, J, G) by a straightforward computation:

$$
(19)\qquad
\begin{aligned}
&F\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j},\frac{\partial}{\partial p_k}\right)=-\partial^i g^{jk},\\
&F\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j},\frac{\delta}{\delta q^k}\right)=F\left(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^k},\frac{\partial}{\partial p_j}\right)=0,\\
&F\left(\frac{\partial}{\partial p_i},\frac{\delta}{\delta q^j},\frac{\delta}{\delta q^k}\right)=\partial^i g_{jk},\quad F\left(\frac{\delta}{\delta q^i},\frac{\partial}{\partial p_j},\frac{\partial}{\partial p_k}\right)=-\overline{\nabla}_i g^{jk},\\
&F\left(\frac{\delta}{\delta q^i},\frac{\partial}{\partial p_j},\frac{\delta}{\delta q^k}\right)=F\left(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^k},\frac{\partial}{\partial p_j}\right)=R_{ikh}g^{jh},\\
&F\left(\frac{\delta}{\delta q^i},\frac{\delta}{\delta q^j},\frac{\delta}{\delta q^k}\right)=\overline{\nabla}_i g_{jk}\ ;
\end{aligned}
$$

$$
(20)\qquad \varphi\left(\frac{\partial}{\partial p_k}\right)=-\overline{\nabla}_i g^{ik},\quad \varphi\left(\frac{\delta}{\delta q^k}\right)=\partial^i g_{ik}-g^{ij}R_{ijk}.
$$

Finally, we shall present some auxiliary results for later use. Let ∇ be a torsion-free linear connection on M and denote by Γ^k_{ij} its connection coefficients. Then ∇ defines a symmetric (nonlinear) connection on T^*M with the coefficients $-\Gamma^k_{ij}p_k$ and we may express the connection coefficients of any symmetric nonlinear connection on T^*M as follows

$$
(21)\qquad N_{ij}=-\Gamma^k_{ij}p_k+c_{ij}
$$

where the components $c_{ij};\ i,j=1,\dots,n$ define a symmetric M-tensor field of type (0,2) on T^*M. Next, we may express the coefficients Φ^k_{ij} and the derivative defined by $\overline{\nabla}$ as follows:

$$
(22)\qquad \Phi^k_{ij}=\Gamma^k_{ij}-\partial^k c_{ij},
$$

$$
(23)\qquad \overline{\nabla}_i g_{jk}=\nabla_i g_{jk}-c_{ih}\partial^h g_{jk}+g_{jh}\partial^h c_{ik}+g_{hk}\partial^h c_{ij}
$$

where

$$
(23')\qquad \nabla_i g_{jk}=\left(\frac{\partial g_{jk}}{\partial q^i}+p_h\Gamma^h_{i\ell}\partial^\ell g_{jk}\right)-\Gamma^h_{ij}g_{hk}-\Gamma^h_{ik}g_{jh}.
$$

Finally, we get the following relation between the torsion of the nonlinear connection defined by N_{ij} and the curvature tensor field R^h_{kij} of ∇:

$$
(24)\qquad R_{kij}=-p_hR^h_{kij}+\nabla_i c_{jk}-\nabla_j c_{ik}-c_{ih}\partial^h c_{jk}+c_{jh}\partial^h c_{ik}.
$$

3. The case 1: a Kählerian structure with Norden metric on T^*M

Using (19), the condition $F = 0$ which must be fulfilled is reduced in the case of (T^*M, J, G) to:

$$\text{(25)} \qquad \text{(i) } \partial^i g_{jk} = 0, \quad \text{(ii) } \overline{\nabla}_i g_{jk} = 0, \quad \text{(iii) } R_{ijk} = 0.$$

From (25)(i) it follows that the g_{ij} are independent of p_k. Thus the M-tensor field g_{ij} is obtained from a tensor field on the base manifold, defining a pseudo-Riemannian metric. Then the condition (25)(ii) and the condition for N_{ij} to be symmetric imply that $\overline{\nabla}$ is the Levi Civita connection $\dot{\nabla}$ of g_{ij}, thought as a linear M-connection on T^*M. Then

$$\text{(26)} \qquad \text{(i) } \Phi\,{}^{k}_{ij} = \left\{ {k \atop ij} \right\}, \quad \text{(ii) } N_{ij} = -p_k \left\{ {k \atop ij} \right\} + c_{ij}$$

where $\left\{ {k \atop ij} \right\}$ are the Christoffel symbols and the components c_{ij}; $i, j = 1, \dots, n$ do not depend on p_k. From the conditions (25)(iii) and (24) we get

$$R_{kij} = -p_h \dot{R}^h_{kij} + \dot{\nabla}_i c_{jk} - \dot{\nabla}_j c_{ik} = 0.$$

It follows that $\dot{R}^h_{kij} = 0$, $\dot{\nabla}_i c_{jk} - \dot{\nabla}_j c_{ik} = 0$, i.e. the connection $\dot{\nabla}$ is flat and the tensor field with the components c_{jk} is a Codazzi tensor field. Hence

Theorem 2. *The almost complex manifold with Norden metric (T^*M, J, G) is Kählerian with Norden metric if and only if J and G are defined by a flat pseudo-Riemannian structure on M and a nonlinear connection given by (26)(ii) where the components c_{ij}; $i, j = 1, \dots, n$ define a Codazzi tensor field on M.*

The n-dimensional torus T^n has a flat Riemannian metric and every Codazzi tensor field on T^n defines a Kähler structure with Norden metric on its cotangent bundle T^*T^n.

4. The case 2: a conformally Kählerian structure with Norden metric on T^*M

The condition (5) which must be fulfilled is reduced in the case of (T^*M, J, G) to the following relations:

$$\text{(27)} \qquad \begin{aligned} &\text{(i) } (n+1)\partial^i g_{jk} = \delta^i_j \partial^h g_{hk} + \delta^i_k \partial^h g_{hj}, \\ &\text{(ii) } \qquad \overline{\nabla}_i g_{jk} = 0, \\ &\text{(iii) } (n+1) R_{ijk} = g_{ik} \partial^h g_{hj} - g_{ij} \partial^h g_{hk}. \end{aligned}$$

Consider a fixed pseudo-Riemannian metric on M, defined by the symmetric tensor field t_{ij} and denote by $\dot{\nabla}$ the Levi Civita connection of (M, t_{ij}). Take the Christoffel symbols $\left\{ {k \atop ij} \right\}$ of $\dot{\nabla}$ for Γ^k_{ij} in (21). We also take

$$g_{ij} = t_{ij} + \eta_i p_j + \eta_j p_i + a p_i p_j, \quad c_{ij} = \theta_i p_j + \theta_j p_i + b p_i p_j \tag{28}$$

where η_i, θ_j define 1-forms on M, thought of as M-1-forms on T^*M and a, b are constants.

It follows by a straightforward computation that the condition (27)(i) is identically fulfilled by the g_{ij} given in (28). Then, by using (23), (23') we get from (27)(ii) that

$$\begin{aligned} b = 0, \quad 0 = \dot{\nabla}_i t_{jk} &= -t_{ij}\theta_k - t_{ik}\theta_j - 2t_{jk}\theta_i, \\ \dot{\nabla}_i \eta_j &= -\theta_i \eta_j - \theta_j \eta_i. \end{aligned} \tag{29}$$

From the second condition (29) we get $\theta_i = 0$.

Finally, from the condition (27)(iii) we get, by using (24), the following relations:

$$\begin{aligned} &\text{(i)} \quad \dot{R}^h_{ijk} = -(a t_{ik} - \eta_i \eta_k)\delta^h_j + (a t_{ij} - \eta_i \eta_j)\delta^h_k, \\ &\text{(ii)} \quad t_{jk}\eta_i - t_{ik}\eta_j = 0 \end{aligned} \tag{30}$$

where $\dot{R}^h_{ijk}$ are the components of the curvature tensor field of $\dot{\nabla}$. Since the tensor field defined by t_{ij} is nondegenerate and we assume $n > 1$, it follows from (30)(ii) that $\eta_k = 0$. So we get the following expression

$$g_{ij} = t_{ij} + a p_i p_j, \quad c_{ij} = 0 \tag{31}$$

i.e. the nonlinear connection on T^*M is defined by the Levi Civita connection of (M, t_{ij}):

$$N_{ij} = -\left\{ {k \atop ij} \right\} p_k. \tag{32}$$

Then, from (30)(i) we get

$$\dot{R}^h_{ijk} = a(t_{ij}\delta^h_k - t_{ik}\delta^h_j), \tag{33}$$

i.e. (M, t_{ij}) must have the constant curvature $-a$. Thus we may state:

Theorem 3. *Let (M, t_{ij}) be a pseudo-Riemannian manifold. Assume that the almost complex structure with Norden metric on T^*M is obtained from the Levi Civita connection $\dot{\nabla}$ of (M, t_{ij}) by (21) and from the M-tensor fields g_{ij}, c_{ij} given in (31). Then (T^*M, J, G) is conformally Kähler with Norden metric if and only if the pseudo-Riemannian manifold (M, t_{ij}) has constant sectional curvature $-a$.*

5. The case 5: a complex structure with Norden metric on T^*M

The condition (8) which must be fulfilled is reduced in the case of (T^*M, J, G) to the following relations

$$\text{(34)} \qquad \text{(i) } \overline{\nabla}_i g_{jk} = \overline{\nabla}_j g_{ik}, \quad \text{(ii) } R_{kij} = \partial^h g_{ki} g_{hj} - \partial^h g_{kj} g_{hi}.$$

Fix an arbitrary torsion-free linear connection ∇ on M and express the connection coefficients of the nonlinear connection on T^*M by (21). Assume that the M-tensor fields g_{ij}, c_{ij} are given by (28). Then from the condition (34)(i) we get

$$\text{(35)} \quad \text{(i) } \nabla_i t_{jk} - \nabla_j t_{ik} = t_{ik}\theta_j - t_{jk}\theta_i, \quad \text{(ii) } \nabla_i \eta_k = b t_{ik} - \eta_k \theta_i - \eta_i \theta_k.$$

Next, from the condition (34)(ii) we get by using (24) the following relations:

$$\text{(36)} \quad \begin{aligned} &\text{(i)} \quad R^h_{kij} = \delta^h_k(\nabla_i\theta_j - \nabla_j\theta_i) - \delta^h_i(\nabla_j\theta_k + \theta_j\theta_k + a t_{jk}) + \\ &\qquad\qquad + \delta^h_j(\nabla_i\theta_k + \theta_i\theta_k + a t_{ik}), \\ &\text{(ii)} \quad \eta_i t_{jk} - \eta_j t_{ik} = 0. \end{aligned}$$

Since $n > 1$, we get from (36)(ii) that $\eta_i = 0$, and (35)(ii) implies $b = 0$. Thus we find the expression (31) of g_{ij} and the following expression for c_{ij}

$$\text{(28')} \qquad c_{ij} = \theta_i p_j + \theta_j p_i \ .$$

Moreover, the expression (36)(i) of R^h_{kij} gives us the condition that be ∇ projectively flat. Hence we may state:

Theorem 4. *Let (M, t_{ij}) be a pseudo-Riemannian manifold and assume that there exist a torsion free linear connection ∇ and a 1-form θ_i on M such that the condition (35)(i) is fulfilled. Then (T^*M, J, G) defined by ∇ and g_{ij} given by (31) and c_{ij} given by (28') is a complex manifold with Norden metric if and only if the linear connection ∇ is projectively flat.*

Remark. If we take $\theta_i = 0$, the conditions which must be fulfilled become

$$\text{(37)} \qquad \nabla_i t_{jk} - \nabla_j t_{ik} = 0, \quad R^h_{kij} = a(t_{ki}\delta^h_j - t_{kj}\delta^h_i)$$

i.e. ∇ is always projectively flat and t_{ij} is a kind of Codazzi tensor field with respect to ∇.

We may get another example of a complex manifold with Norden metric by considering a Kähler manifold as the base manifold of T^*M.

Denote by F^i_j the components of the complex structure on M, by t_{ij} the components of the Kähler metric on M and by $\dot\nabla$ the corresponding Levi Civita connection. Then:

$$t_{ij}F^i_kF^j_h = t_{kh}, \quad \dot\nabla_i t_{jk} = 0, \quad \dot\nabla_i F^j_k = 0. \tag{38}$$

Take in (21) $\nabla = \dot\nabla$ and $c_{ij} = 0$, i.e.

$$N_{ij} = -p_k \left\{ \begin{matrix} k \\ ij \end{matrix} \right\},$$

and consider the M-tensor field g_{ij} on T^*M given by

$$g_{ij} = t_{ij} + a(p_ip_j - F^k_iF^h_jp_kp_h) \tag{39}$$

where a is a constant. Then it follows easily from (23),(38) that $\overline{\nabla}_k g_{ij} = 0$, thus the condition (34)(i) is automatically fulfilled. Next from (24) and (34)(ii) we get

$$R^h_{kij} = a\left\{\delta^h_j t_{ki} - \delta^h_i t_{kj} + F^h_i t_{j\ell}F^\ell_k - F^h_j t_{i\ell}F^\ell_k - 2t_{i\ell}F^\ell_jF^h_k\right\}, \tag{40}$$

i.e. (M, F^i_j, t_{ij}) must have constant holomorphic sectional curvature $-4a$. Hence we may state:

Theorem 5. *Let (M, F^i_j, t_{ij}) be a Kähler manifold of constant holomorphic sectional curvature $-4a$. Then (T^*M, J, G) with the nonlinear connection defined by the Levi Civita connection of (M, t_{ij}) and the M-tensor field g_{ij} given by (39) is a complex manifold with Norden metric.*

Remark. Under the assumptions from Theorem 5, $(T^*M, J, G,)$ cannot be a conformally Kähler manifold with Norden metric.

6. The case 6: a semi-Kählerian structure with Norden metric on T^*M.

The condition $\varphi = 0$ which must be fulfilled is expressed in the case of (T^*M, J, G) by:

$$\overline{\nabla}_i g^{ik} = 0, \quad g^{ij}R_{ijk} = \partial^i g_{ik}. \tag{41}$$

To get an example for this case we shall take $g_{ij} = t_{ij}$ independent of p_k and the nonlinear connection on T^*M defined by the Levi Civita connection $\dot\nabla$ of (M, t_{ij}) i.e.

$$N_{ij} = -p_k \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}. \tag{42}$$

With this choice, the conditions (41) become

$$\dot{\nabla}_i t^{ik} = 0, \qquad t^{ij}\dot{R}^h_{ijk} = 0. \tag{41'}$$

Remark that the first condition (41') is trivially fulfilled. The second condition (41') is equivalent to the property for the pseudo-Riemannian manifold (M, t_{ij}) to be Ricci flat. Thus we may state:

Theorem 6. *Assume that the pseudo-Riemannian manifold (M, t_{ij}) is Ricci flat. Then (T^*M, J, G) defined by the nonlinear connection N_{ij} given by (42) and $g_{ij} = t_{ij}$ is a semi-Kählerian manifold with Norden metric.*

7. Some final remarks concerning the remaining cases

In this section we shall discuss the possibility of finding examples for special complex structures with Norden metric (case 3), quasi-Kählerian structures with Norden metric (case 4) and $\omega_1 \oplus \omega_3$ - structures (case 7) on the cotangent bundle T^*M.

In the case 3 (the special complex manifolds with Norden metric) the conditions which must be fulfilled by (T^*M, J, G) are (34) and (41). Then we get after a straightforward computation

$$g_{ih}\partial^h g_{jk} g^{jk} = 2\partial^h g_{hi}. \tag{43}$$

Thus we must find a nondegenerate symmetric M-tensor field g_{ij} on T^*M satisfying (43). Such an M-tensor field could be $g_{ij} = t_{ij}$ -independent of p_k. Then the conditions which must be fulfilled are

$$\text{(i) } \overline{\nabla}_i g_{jk} = \overline{\nabla}_j g_{ik}, \quad \text{(ii) } \overline{\nabla}_i g^{ik} = 0, \quad \text{(iii) } g^{ki}R_{kij} = 0. \tag{44}$$

Denoting by $\dot{\nabla}$ the Levi Civita connection of (M, g_{ij}), the conditions (44) may be written in the following equivalent form:

$$\begin{gathered} g_{jh}\partial^h c_{ik} = g_{ih}\partial^h c_{jk}, \quad g^{ki}\partial^h c_{hk} + \partial^i c_{hk} g^{hk} = 0, \\ g^{ki}(-\dot{R}^h_{kij}p_h + \dot{\nabla}_i c_{jk} - \dot{\nabla}_j c_{ik} - c_{ih}\partial^h c_{jk} + c_{jh}\partial^h c_{ik}) = 0 \end{gathered} \tag{45}$$

where $\dot{\nabla}_i c_{jk}$ is defined similarly to (23') by using $\dot{\nabla}$ instead of ∇. Now, the point is to find an M-tensor field c_{ij} on T^*M in order that the conditions (45) be fulfilled.

In the case 4 (the quasi-Kählerian manifolds with Norden metric) the conditions which must be fulfilled by (T^*M, J, G) are:

$$\text{(i) } \overline{\nabla}_k g_{ij} = 0, \quad \text{(ii) } R_{kji} + R_{jki} + g_{ih}\partial^h g_{jk} = 0. \tag{46}$$

Due to the identity

$$R_{ijk}+R_{jki}+R_{kij}=0 \tag{47}$$

the relation (46)(ii) is equivalent to the following two relations

$$\begin{aligned} &\text{(i)} \qquad g_{ih}\partial^h g_{jk}+g_{jh}\partial^h g_{ki}+g_{kh}\partial^h g_{ij}=0,\\ &\text{(ii)} \qquad 3R_{ijk}=g_{jh}\partial^h g_{ik}-g_{kh}\partial^h g_{ij}. \end{aligned} \tag{48}$$

So we must find a nondegenerate symmetric M-tensor field g_{ij} and a symmetric nonlinear connection on T^*M in order that conditions (46)(i) and (48)(i),(ii) be fulfilled.

Finally, in the case 7 (the $\omega_1\oplus\omega_3$ - structures) the conditions which must be fulfilled by (T^*M,J,G) are

$$\begin{aligned} &\text{(i)} \qquad n\overline{\nabla}_k g_{ij}=(g_{ik}\overline{\nabla}_h g_{j\ell}+g_{jk}\overline{\nabla}_h g_{i\ell}-g_{ij}\overline{\nabla}_h g_{k\ell})g^{h\ell},\\ &\text{(ii)} \qquad n g_{ih}\partial^h g_{jk}+nR_{jki}+nR_{kji}=g_{ij}(\partial^h g_{hk}-g^{h\ell}R_{h\ell k})+\\ &\qquad\qquad +g_{ik}(\partial^h g_{hj}-g^{h\ell}R_{h\ell j})-g_{jk}(\partial^h g_{hi}-g^{h\ell}R_{h\ell i}. \end{aligned} \tag{49}$$

Using again (47) the condition (49)(ii) may be transformed into

$$\begin{aligned} (n+2)\{g_{ih}\partial^h g_{jk}+g_{jh}\partial^h g_{ki}+g_{kh}\partial^h g_{ij}\}&=g_{ij}(2\partial^h g_{hk}+\\ &+g_{kh}\partial^h g_{m\ell}g^{m\ell})+g_{jk}(2\partial^h g_{hi}+g_{ih}\partial^h g_{m\ell}g^{m\ell})+\\ &+g_{ki}(2\partial^h g_{hj}+g_{jh}\partial^h g_{m\ell}g^{m\ell}), \end{aligned} \tag{50}$$

$$\begin{aligned} 3R_{kij}=&g_{ih}\partial^h g_{jk}-g_{jh}\partial^h g_{ik}+\\ &+\frac{2}{n+2}\Big\{\,g_{jk}(2\partial^h g_{hi}+g_{ih}\partial^h g_{m\ell}g^{m\ell})-\\ &-g_{ik}(2\partial^h g_{hj}+g_{jh}\partial^h g_{m\ell}g^{m\ell})\,\Big\}. \end{aligned} \tag{51}$$

The following nondegenerate symmetric M-tensor fields on T^*M can be considered:

a) $g_{ij}=t_{ij}$ -independent of p_k,

b) $g_{ij}=\dfrac{1}{1+a\alpha}(t_{ij}+ap_ip_j);\ g^{ij}=(1+a\alpha)at^{ij}-t^{ik}t^{jh}p_kp_h$,

c) $g_{ij}=\dfrac{1}{\alpha\gamma-(1+\beta)^2}(t_{ij}+\eta_ip_j+\eta_jp_i);\ g^{ij}=\big\{\alpha\gamma-(1+\beta)^2\big\}t^{ij}+$
$\quad +t^{ik}t^{jh}\big\{-\gamma p_kp_h+(1+\beta)(\eta_kp_h+\eta_hp_k)-\alpha\eta_k\eta_h\big\}$,

d) $$g_{ij} = \frac{1}{\alpha(\gamma - a) - (1+\beta)^2}(t_{ij} + \eta_i p_j + \eta_j p_i + a p_i p_j);$$
$$g^{ij} = \{\ \alpha(\gamma - a) - (1+\beta)^2\ \}\ t^{ij} + t^{ik} t^{jh}\ \{\ (a-\gamma) p_k p_h + {} + (1+\beta)(\eta_k p_h + \eta_h p_k) - \alpha \eta_k \eta_h\ \},$$

where a is a constant and $\alpha = t^{kh} p_k p_h$, $\beta = t^{kh} \eta_k p_h$, $\gamma = t^{kh} \eta_k \eta_h$.

Remark. Some of the examples presented above may be adapted for the remaining cases. However, they cannot be considered as specific examples for the cases 3, 4, 7.

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V. OPROIU
FACULTY OF MATHEMATICS
UNIVERSITY OF IAŞI
ROMANIA

N. PAPAGHIUC
DEPARTMENT OF MATHEMATICS
POLYTECHNIC INSTITUTE OF IAŞI
ROMANIA

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