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A note on equal values of polygonal numbers

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Abstract. Using the theory of simultanous Pell equations we consider the equal values of polygonal numbers.

1. Introduction

The equal values of certain combinatorial numbers, including binomial coefficients, Stirling numbers, have been investigated by several authors (see [4], [5], [8]). In a recent paper, Brindza, Pintér and Turjányi, [6] have dealt with the equal values of the polygonal and pyramidal numbers. The purpose of this note is to study the equation

$$\mathrm{Pol}_x^m = \mathrm{Pol}_y^n = \mathrm{Pol}_z^o, \tag{1}$$

where Pol_x^m denotes the xth polygonal number of order m, that is

$$\mathrm{Pol}_x^m = \frac{m-2}{2}x^2 - \frac{m-4}{2}x.$$

By using the following simple transformations

$$X = 2(m-2)x - (m-4)$$
$$Y = 2(m-2)(n-2)y - (m-2)(n-4)$$

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and

$$Z = 2(m-2)(o-2)z - (m-2)(o-4)$$

the equation (1) leads to the simultaneous equations

$$\begin{aligned} Y^2 - (m-2)(n-2)X^2 = &-(m-2)(n-2)(m-4)^2 \\ &+ (m-2)^2(n-4)^2 \end{aligned} \tag{2}$$

$$\begin{aligned} Z^2 - (m-2)(o-2)X^2 = &-(m-2)(o-2)(m-4)^2 \\ &+ (m-2)^2(o-4)^2. \end{aligned} \tag{3}$$

Applying a result of Baker [1] on hyperelliptic equations (cf. Brindza [3]) we obtain

Theorem 1. *If (m, n, o) is not a permutation of the triplet $(3, 6, k)$, $(k > 3)$, then all the solutions x, y, z to the equation (1) satisfy*

$$\max(x, y, z) < c,$$

where c is an effectively computable constant depending only on the parameters m, n, o.

Remark. The situation in the remaining case is more difficult. If $(m, n, o) = (3, 6, 5)$ then the system of equations

$$x^2 + x = 4y^2 - 2y = 3z^2 - z$$

has infinitely many solutions in positive integers x, y and z (see [7, pp. 19–20]). On the other hand, if $(m, n, o) = (3, 6, l^2 + 2)$ then the simultaneous equations (2) and (3) possess finitely many effectively determinable solutions.

2. Proof of Theorem 1

By the above-mentioned result of Baker it is enough to show that the polynomial

$$f(X) = (AX^2 + B)(CX^2 + D),$$

where

$$A = (m-2)(n-2), B = (m-2)^2(n-4)^2 - (m-2)(n-2)(m-4)^2$$

and

$$C = (m-2)(o-2), D = (m-2)^2(o-4)^2 - (m-2)(o-2)(m-4)^2,$$

has simple zeros, only. Supposing the contrary we immediately get

$$\left(\frac{B}{A} - \frac{D}{C}\right) BD = 0,$$

therefore,

$$\begin{aligned} &[on - 2(o+n)](n-o)(2n - m(n-2))(m-n)(m-2) \\ &\qquad \cdot (2o - m(o-2))(m-o)(m-2) = 0. \end{aligned} \tag{4}$$

Since $m > 2$ and m, n, o are pairwise distinct, (4) yields that (m, n, o) is a permutation of the triplet $(3, 6, k)$, $k > 3$.

3. Some numerical examples

The results of this section are based upon the next theorem of Rickert [11] (see also Bennett [2])

Theorem A. *All the integer solutions x, y, z of the simultaneous Pell-type equations*

$$x^2 - 2z^2 = u, \quad y^2 - 3z^2 = v$$

satisfy

$$\max\{|x|, |y|, |z|\} \le (10^7 \max\{|u|, |v|\})^{12}.$$

A straightforward consequence of Theorem A provides a sharp bound for several special cases of the equation (1).

Theorem 2. *If $(m-2)(n-2) = 2L^2$ and $(m-2)(o-2) = 3L^2$ for some positive integer L, then equation* (1) *implies*

$$\max\{x, y, z\} \le 10^{84}(\max\{m, n, o\})^{48}. \tag{5}$$

Baker-type results also make it possible to derive effective estimates for the solutions, however this does not lead to bounds depending polynomially on $\max\{m, n, o\}$.

We may use the inequality (5) to solve explicit equations quickly. To illustrate it take the example $(m,n,o)=(3,4,5)$. Then the unique solution of (1) is given by $(x,y,z)=(1,1,1)$. Indeed, our estimate shows that $\max\{x,y,z\}\le 10^{84}\cdot 5^{48}$ and we have the system of equations

$$(4y)^2-2(2x+1)^2=-2,(6z-1)^2-3(2x+1)^2=-2.$$

It is easy to see that the only solutions to the first equation, which can be rewritten as $u^2-2v^2=1$ with $u=2x+1$ and $v=2y$, are given by

$$2y=a_r=\frac{\alpha^r-\alpha^{-r}}{2\sqrt{2}},\alpha=3+2\sqrt{2}.$$

Using the program package MATHEMATICA we obtain $r\le 154$ and these values can be tested by computer. For a similar approach (without using computer) see Rickert [11] or Pintér [10].

References

[1] A. Baker, Bounds for the solutions of the hyperelliptic equation, *Proc. Camb. Phil. Soc.* **65** (1969), 439–444.

[2] M. A. Bennett, Simultaneous Approximation to pairs of Algebraic Numbers, Canadian Mathematical Society Conference Proceeding (K. Dilcher, ed.), vol. 15, *AMS*, 55–65.

[3] B. Brindza, On S-integral solutions of the equation $y^m=f(x)$, *Acta Math. Hungar.* **44** (1984), 133–139.

[4] B. Brindza and Á. Pintér, On the irreducibility of some polynomials in two variables, *Acta Arith.* (*to appear*).

[5] B. Brindza and Á Pintér, On equal values of power sums, *Acta Arith.* **77** (1996), 97–101.

[6] B. Brindza, Á. Pintér and S.Turjányi, On equal values of pyramidal and polygonal numbers, *Indag. Math.* (*to appear*).

[7] L. E. Dickson, History of the theory of numbers, Vol. II, Diophantine Analysis, *Chelsea Publishing Company*, *New York*, 1971.

[8] L. Hajdu and Á.Pintér, Combinatorial diophantine equations, (manuscript).

[9] Á. Pintér, On a diophantine problem concerning Stirling numbers, *Acta Math. Hungar.* **65** (1994), 361–364.

[10] Á. Pintér, A new solution of two old diophantine equations, (manuscript).

[11] J. H. Rickert, Simultaneous rational approximations and related diophantine equations, *Math. Proc. Camb. Phil. Soc.* **113** (1993), 461–472.

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