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On Nemytskii operator for set-valued functions

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Abstract. This paper is devoted to Nemytskii operator mapping some subfamily of the space of class C^1 functions into a space of set-valued functions, and can be considered as generalization of [13].

J. MATKOWSKI has proved that every function h, which generates the Nemytskii operator mapping the Banach space of Lipschitzian functions into itself and satisfing the Lipschitz condition, must be Jensen function with respect to the second variable (see [3], [4], [5]). His result was generalized by A. SMAJDOR and W. SMAJDOR for set-valued functions ([10], [11]). We will restrict our attention to the Nemytskii operator defined on some subspace of the space considered by Matkowski.

We will start with some definitions and notations. Let $(X, \|\cdot\|)$ be a real normed space and let $C \subseteq X$ be nonempty. Let $f : \langle a, b\rangle \to X$, and $x_1, \dots, x_p \in \langle a, b\rangle$ be distinct points. The *divided difference* $[x_1, \dots, x_p; f]$ of f at points $x_1, \dots, x_p$ is defined by recurrence

$$[x_1, f] = f(x_1),$$

$$[x_1, \dots, x_p; f] = \frac{[x_2, \dots, x_p; f] - [x_1, \dots, x_{p-1}; f]}{x_p - x_1}, \quad p \geq 2$$

(cf. [2]).

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Let us consider

$$\mathrm{lip}^2(\langle a,b\rangle, C) := \left\{ f : \langle a,b\rangle \to C : \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} \|[t_1,t_2,t_3;f]\| < \infty \right\}.$$

It is clear that $\mathrm{lip}^2(\langle a,b\rangle, X)$ is a real normed space with the norm

$$\|f\| := \|f(a)\| + \|f(b)\| + \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} \|[t_1,t_2,t_3;f]\|.$$

It is a Banach space, if the space $(X, \|\cdot\|)$ is complete.

Indeed, let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{lip}^2(\langle a,b\rangle, X)$ and let $\varepsilon > 0$ be fixed. Then,

$$\|f_n(a) - f_m(a)\| \le \varepsilon,$$

$$\|f_n(b) - f_m(b)\| \le \varepsilon,$$

and

$$\frac{1}{|z-x|}\left\| \frac{f_n(z) - f_m(z) - f_n(y) + f_m(y)}{z-y} - \frac{f_n(y) - f_m(y) - f_n(x) + f_m(x)}{y-x} \right\| \le \varepsilon, \tag{1}$$

for all $x, y, z \in \langle a,b\rangle$, distinct and $n, m \ge n_0$, where $n_0 = n(\varepsilon)$ is some number.

Let us fix $x \in (a,b)$. By (1), for $n, m \ge n_0$,

$$\left\| \frac{f_n(x) - f_m(x) - f_n(b) + f_m(b)}{x-b} \right\|$$

$$\le \varepsilon(b-a) + \frac{\|f_n(b) - f_m(b) - f_n(a) + f_m(a)\|}{b-a} \le \varepsilon(b-a) + \frac{2\varepsilon}{b-a}.$$

Hence,

$$\|f_n(x) - f_m(x)\| \le \varepsilon\left((b-a)^2 + 2\right) + \|f_n(b) - f_m(b)\| \le \varepsilon\left((b-a)^2 + 3\right),$$

$$n, m \ge n_0.$$

So, $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, for each $x \in \langle a,b\rangle$. Define $f : \langle a,b\rangle \to X$ as follows $f(x) := \lim_{n\to\infty} f_n(x)$, $x \in \langle a,b\rangle$. By the continuity of the norm and (1),

$$\|[x,y,z;f_n - f]\| \le \varepsilon, \quad n \ge n_0,$$

thus $f \in \mathrm{lip}^2(\langle a,b\rangle, X)$, and $\|f_n - f\| \le 3\varepsilon$, $n \ge n_0$.

The space $\mathrm{lip}^2(\langle a,b\rangle, C) \subseteq \mathrm{lip}^2(\langle a,b\rangle, X)$ can be considered as a metric space with a metric generated by the norm from $\mathrm{lip}^2(\langle a,b\rangle, X)$.

First, we prove a few properties, which will be useful.
Let us consider the family

$$\mathrm{lip}^1(\langle a,b\rangle,\mathbb{R}) := \left\{ f : \langle a,b\rangle \to \mathbb{R} : \sup_{\substack{t_1,t_2\in\langle a,b\rangle \\ t_1\ne t_2}} |[t_1,t_2;f]| < \infty \right\}.$$

By [2] (Lemma 1, p. 390), $\mathrm{lip}^2(\langle a,b\rangle,\mathbb{R}) \subset \mathrm{lip}^1(\langle a,b\rangle,\mathbb{R})$, but not conversely (for example the function $f(x) = x^{\frac{3}{2}}$, $x \in \langle 0,1\rangle$ belongs to the class $\mathrm{lip}^1(\langle 0,1\rangle,\mathbb{R})$, but $f \notin \mathrm{lip}^2(\langle 0,1\rangle,\mathbb{R})$).

It is easy to check that $(\mathrm{lip}^1(\langle a,b\rangle,\mathbb{R}), \|\cdot\|)$ is a real Banach space with the norm

$$\|f\| := |f(a)| + \sup_{\substack{t_1,t_2\in\langle a,b\rangle \\ t_1\ne t_2}} |[t_1,t_2;f]|.$$

Lemma 1 (cf. the proof of Th. 1 in [2] p. 391)**.** *If $f \in \mathrm{lip}^2(\langle a,b\rangle,\mathbb{R})$, then f is of class C^1 in $\langle a,b\rangle$, and its derivative f' belongs to the class $\mathrm{lip}^1(\langle a,b\rangle,\mathbb{R})$ and*

$$\sup_{\substack{t_1,t_2\in\langle a,b\rangle \\ t_1\ne t_2}} |[t_1,t_2;f']| \le 2 \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\ne t_j,\ j\ne i}} |[t_1,t_2,t_3;f]|.$$

If a function $f : \langle a,b\rangle \to \mathbb{R}$ is differentiable in $\langle a,b\rangle$, and $f' \in \mathrm{lip}^1(\langle a,b\rangle,\mathbb{R})$, then $f \in \mathrm{lip}^2(\langle a,b\rangle,\mathbb{R})$, and

$$\sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\ne t_j,\ j\ne i}} |[t_1,t_2,t_3;f]| \le \sup_{\substack{t_1,t_2\in\langle a,b\rangle \\ t_1\ne t_2}} |[t_1,t_2;f']|.$$

PROOF. Let $f \in \operatorname{lip}^2(\langle a,b\rangle,\mathbb{R})$ and

$$M := \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}} |[t_1,t_2,t_3;f]|.$$

Let $x_1, x_2, y_1, y_2 \in \langle a,b\rangle$ be fixed distinct points. Then

$$\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{f(y_1)-f(y_2)}{y_1-y_2}\right| \le \left|\frac{f(x_1)-f(x_2)}{x_1-x_2}-\frac{f(x_2)-f(y_1)}{x_2-y_1}\right| + \left|\frac{f(x_2)-f(y_1)}{x_2-y_1}-\frac{f(y_1)-f(y_2)}{y_1-y_2}\right| \le M(|x_1-y_1|+|x_2-y_2|).$$

Thus, for a fixed point $x \in \langle a,b\rangle$, there exists the limit

$$\lim_{\substack{x_1,x_2\to x\\ x_1\neq x_2}} \frac{f(x_1)-f(x_2)}{x_1-x_2} = f'(x).$$

So, the function f is differentiable at every point $x \in \langle a,b\rangle$. Moreover, letting $x_2 \to x_1$, and $y_2 \to y_1$, we obtain that

$$|f'(x_1)-f'(y_1)| \le 2M|x_1-y_1| \quad \text{for all } x_1,y_1\in\langle a,b\rangle,$$

which proves, that f is of class C^1, its derivative $f' \in \operatorname{lip}^1(\langle a,b\rangle,\mathbb{R})$ and

$$\sup_{\substack{x_1,y_1\in\langle a,b\rangle\\ x_1\neq y_1}} |[x_1,y_1;f']| \le 2M.$$

Conversely, let $f:\langle a,b\rangle\to\mathbb{R}$ be differentiable in $\langle a,b\rangle$, its derivative f' belong to $\operatorname{lip}^1(\langle a,b\rangle,\mathbb{R})$, and let $x,y,z\in\langle a,b\rangle$ be fixed distinct points. By Theorem 1 in [2] (p. 372), a function $(x,y,z)\mapsto[x,y,z;f]$ is symmetric, so we can assume, that $x<y<z$. Then, by the Lagrange Theorem, there exist $\xi_1\in(x,y)$ and $\xi_2\in(y,z)$ such that

$$\frac{1}{|z-x|}\left|\frac{f(z)-f(y)}{z-y}-\frac{f(y)-f(x)}{y-x}\right| = \frac{|f'(\xi_1)-f'(\xi_2)|}{|z-x|} \le \frac{|f'(\xi_1)-f'(\xi_2)|}{|\xi_1-\xi_2|} \le \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;f']|.$$

Thus

$$\sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}} |[t_1,t_2,t_3;f]| \le \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;f']|,$$

which ends the proof. □

Remark 1. If $f,g\in \operatorname{lip}^1(\langle a,b\rangle,\mathbb{R})$, then

$$\max\{f,g\},\min\{f,g\}\in \operatorname{lip}^1(\langle a,b\rangle,\mathbb{R}).$$

Indeed, let $h:\langle a,b\rangle\to\mathbb{R}$ be given as $h(x):=\max\{f(x),g(x)\}$, $x\in\langle a,b\rangle$. Since $\max\{u,v\}=\frac{u+v+|u-v|}{2}$, we have

$$\begin{aligned}\left|\frac{h(t_2)-h(t_1)}{t_2-t_1}\right| &\le \frac{1}{2}\left\{\left|\frac{f(t_2)-f(t_1)}{t_2-t_1}\right| + \left|\frac{g(t_2)-g(t_1)}{t_2-t_1}\right|\right.\\ &\qquad\left. + \left|\frac{|f(t_2)-g(t_2)|-|f(t_1)-g(t_1)|}{t_2-t_1}\right|\right\}\\ &\le \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;f]| + \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;g]|\end{aligned}$$

for all distinct $t_1,t_2\in\langle a,b\rangle$. Hence

$$\sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;\max\{f,g\}]| \le \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;f]| + \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;g]|.$$

For $\min\{f,g\}$, the proof and the inequality are the same. □

Remark 2. Let $u,v\in\mathbb{R}$, and $x,z\in\langle a,b\rangle$ $x<z$ be fixed. The function $g:\langle a,b\rangle\to\mathbb{R}$, given as follows

$$g(t):=\begin{cases} u, & t\in\langle a,x\rangle,\\ \dfrac{2(v-u)}{(z-x)^2}(t-x)^2+u, & t\in\left(x,\frac{x+z}{2}\right\rangle,\\ \dfrac{-2(v-u)}{(z-x)^2}(t-z)^2+v, & t\in\left(\frac{x+z}{2},z\right\rangle,\\ v, & t\in(z,b\rangle,\end{cases}$$

belongs to the class $\mathrm{lip}^2(\langle a,b\rangle,\mathbb{R})$ and $\sup\limits_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}} |[t_1,t_2,t_3;g]| \leq 8\frac{|u-v|}{(z-x)^2}$.

Proof. If $u=v$, the remark is obvious. Let $u<v$. Then it is easy to check that g is differentiable. Its derivative is equal to

$$g'(t) = \begin{cases} 0, & t\in\langle a,x\rangle,\\ \dfrac{4(v-u)}{(z-x)^2}(t-x), & t\in\left(x,\frac{x+z}{2}\right\rangle,\\ \dfrac{-4(v-u)}{(z-x)^2}(t-z), & t\in\left(\frac{x+z}{2},z\right\rangle,\\ 0, & t\in(z,b\rangle,\end{cases}$$

and g is of class C^1. Let us define functions $h_i:\langle a,b\rangle\to\mathbb{R}$, $i=1,2$ by formulas

$$h_1(t) := \frac{4(v-u)}{(z-x)^2}(t-x),\quad h_2(t) := \frac{-4(v-u)}{(z-x)^2}(t-z).$$

Then $g' = \min\{\max\{0,h_1\},\max\{0,h_2\}\}$, and

$$\sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;g']| \leq \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;h_1]| + \sup_{\substack{t_1,t_2\in\langle a,b\rangle\\ t_1\neq t_2}} |[t_1,t_2;h_2]| = 8\frac{|v-u|}{(z-x)^2}.$$

Thus, by Lemma 1,

$$\sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}} |[t_1,t_2,t_3;g]| \leq 8\frac{|u-v|}{(z-x)^2}.$$

If $v<u$, then define $g_0:\langle a,b\rangle\to\mathbb{R}$ as $g_0=-g$. Then

$$g_0(t) := \begin{cases} -u, & t\in\langle a,x\rangle,\\ \dfrac{2(u-v)}{(z-x)^2}(t-x)^2-u, & t\in\left(x,\frac{x+z}{2}\right\rangle,\\ \dfrac{-2(u-v)}{(z-x)^2}(t-z)^2-v, & t\in\left(\frac{x+z}{2},z\right\rangle,\\ -v, & t\in(z,b\rangle.\end{cases}$$

Since $-u < -v$, $g_0 \in \operatorname{lip}^2(\langle a, b\rangle, \mathbb{R})$ and

$$\sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} |[t_1, t_2, t_3; g_0]| \leq 8\,\frac{|-u+v|}{(z-x)^2}. \qquad \square$$

A set $C \subseteq X$ is said to be *a convex cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \in \langle 0, \infty)$.

Lemma 2. *Let $(X, \|\cdot\|)$ be a real normed space, $C \subseteq X$ be a convex cone and let $x, z \in \langle a, b\rangle$ be fixed, $x < z$. If $u, v \in C$ then the function $\varphi : \langle a, b\rangle \to X$, given by formula*

$$\varphi(t) = \begin{cases} u, & t \in \langle a, x\rangle, \\ \dfrac{2(v-u)}{(z-x)^2}(t-x)^2 + u, & t \in \left(x, \frac{x+z}{2}\right\rangle, \\ \dfrac{-2(v-u)}{(z-x)^2}(t-z)^2 + v, & t \in \left(\frac{x+z}{2}, z\right\rangle, \\ v, & t \in (z, b\rangle, \end{cases}$$

belongs to the class $\operatorname{lip}^2(\langle a, b\rangle, C)$ *and* $\sup\limits_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} \|[t_1, t_2, t_3; \varphi]\| \leq 8\frac{\|u-v\|}{(z-x)^2}$.

Proof. If $t \in \langle x, \frac{x+z}{2}\rangle$ then $0 \leq \left(\frac{t-x}{z-x}\right)^2 \leq \frac{1}{4}$. Thus

$$2\left(\frac{t-x}{z-x}\right)^2 (v-u) + u = 2\left(\frac{t-x}{z-x}\right)^2 v + \left(1 - 2\left(\frac{t-x}{z-x}\right)^2\right) u \in C.$$

Similarly, for $t \in \left(\frac{x+z}{2}, z\right\rangle$, $-2\left(\frac{t-z}{z-x}\right)^2 (v-u)+v \in C$. Hence $\varphi : \langle a, b\rangle \to C$.

Let us define a function $g : \langle a, b\rangle \to \mathbb{R}$ by formula

$$g(t) = \begin{cases} 0, & t \in \langle a, x\rangle, \\ \dfrac{2}{(z-x)^2}(t-x)^2, & t \in \left(x, \frac{x+z}{2}\right\rangle, \\ \dfrac{-2}{(z-x)^2}(t-z)^2 + 1, & t \in \left(\frac{x+z}{2}, z\right\rangle, \\ 1, & t \in (z, b\rangle. \end{cases}$$

It is easy to check that $\varphi(t) = (1-g(t))u + g(t)v$, $t \in \langle a,b\rangle$, and $[t_1,t_2,t_3;\varphi] = [t_1,t_2,t_3;g](v-u)$, for all $t_1,t_2,t_3 \in \langle a,b\rangle$, distinct. By Remark 2, $g \in \operatorname{lip}^2(\langle a,b\rangle,[0,+\infty))$ and $\sup\limits_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}} |[t_1,t_2,t_3;g]| \le \frac{8}{(z-x)^2}$.

Therefore

$$\|[t_1,t_2,t_3;\varphi]\| \le 8\frac{\|u-v\|}{(z-x)^2}. \qquad \square$$

Let us consider a real normed space $(Y,\|\cdot\|)$ and a convex cone $S\subset Y$.

Lemma 3. *If a function* $f:\langle a,b\rangle \to S$ *belongs to* $\operatorname{lip}^2(\langle a,b\rangle,S)$ *then* f *is continuous.*

Proof. Since $f\in\operatorname{lip}^2(\langle a,b\rangle,S)$, we have

$$\sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}} \|[t_1,t_2,t_3;f\| =: M < \infty.$$

Hence

$$\left\|\frac{f(t_3)-f(t_2)}{t_3-t_2} - \frac{f(t_2)-f(t_1)}{t_2-t_1}\right\| \le M|t_3-t_1|$$

for distinct $t_1,t_2,t_3\in\langle a,b\rangle$.

Let us fix a point $x\in\langle a,b\rangle$. Choose $y\in\langle a,b\rangle$ such that $x\neq y$.

Then for every $z\in\langle a,b\rangle$, $z\neq x$, $z\neq y$, we have

$$\left\|f(z)-f(y)-\frac{z-y}{y-x}\left(f(y)-f(x)\right)\right\| \le M|z-x||z-y|.$$

Thus

$$\begin{aligned}
\|f(z)-f(x)\| &= \|f(z)-f(y)+f(y)-f(x)\| \\
&= \|f(z)-f(y)+\frac{y-z+(z-x)}{y-x}\left(f(y)-f(x)\right)\| \\
&\le \|f(z)-f(y)-\frac{z-y}{y-x}\left(f(y)-f(x)\right)\| + \left|\frac{z-x}{y-x}\right|\cdot\|f(y)-f(x)\| \\
&\le M|z-x||z-y| + \left|\frac{z-x}{y-x}\right|\|f(y)-f(x)\|
\end{aligned}$$

for every $z \in \langle a, b\rangle$ such that $z \neq x$ and $z \neq y$. It implies

$$\lim_{z \to x} \|f(z) - f(x)\| = 0,$$

which proves the continuity of the function f. □

Let X, Z be real vector spaces and C be a convex subset of X. Let $f : C \to Z$ be an arbitrary function and $h \in X$. The *difference operator* Δ_h with the span h is given by the formula

$$\Delta_h f(x) := f(x + h) - f(x).$$

Thus $\Delta_h f$ is a function mapping the set $\{x \in C : x + h \in C\}$ into the space Z.

The iterates Δ_h^n of Δ_h are given by the recurrence

$$\Delta_h^0 f := f, \quad \Delta_h^{n+1} f := \Delta_h(\Delta_h^n f), \quad n = 0, 1, 2, \ldots$$

The expression $\Delta_h^n f$ is a function defined for all $x \in C$ such that $x + nh \in C$. It is easy to see that $x + kh \in C$ for $k = 1, \ldots, n-1$ whenever $x, x + nh \in C$.

A function $f : C \to Z$ is called *a polynomial function* of order at most n if

$$\Delta_h^{n+1} f(x) = 0$$

for every $x \in C$ and $h \in X$ such that $x + (n + 1)h \in C$.

Now, we can formulate the main result.

Theorem 1. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be real normed spaces, and let $C \subseteq X$, $S \subseteq Y$ be convex cones and let S be closed. Assume that a function $h : \langle a, b\rangle \times C \to S$ generates the Nemytskii operator $N : \mathrm{lip}^2(\langle a, b\rangle, C) \to \mathrm{lip}^2(\langle a, b\rangle, S)$, i.e.,*

$$N\varphi(x) := h(x, \varphi(x)), \quad x \in \langle a, b\rangle,\ \varphi \in \mathrm{lip}^2(\langle a, b\rangle, C) \tag{3}$$

and that N satisfies the Lipschitz condition

$$\|N\varphi_1 - N\varphi_2\| \leq L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \mathrm{lip}^2(\langle a, b\rangle, C). \tag{4}$$

Then there exist functions $a_o : \langle a, b\rangle \times C \to S$ and $b_o : \langle a, b\rangle \to S$ such that

(i) *$a_o(x, \cdot)$ is linear for every $x \in \langle a, b\rangle$,*

(ii) $a_o(x,\cdot)$ *is uniformily bounded on* C, *i.e., there exists a constant* $M>0$ *such that* $\|a_o(x,y)\|\le M\|y\|$, $y\in C$ *for every* $x\in\langle a,b\rangle$,

(iii) $h(x,y)=a_o(x,y)+b_o(x)$, $x\in\langle a,b\rangle$, $y\in C$,

(iv) b_o, $a_o(\cdot,y)$ *belong to* $\operatorname{lip}^2(\langle a,b\rangle,S)$ *for all* $y\in C$.

Moreover, if $(X,\|\cdot\|)$ *is a real Banach space (or the set of the second category) and* $\operatorname{int}C\neq\emptyset$ *then the function* $a_o(x,\cdot):C\to S$ *is continuous for each* $x\in\langle a,b\rangle$.

Proof. The function $\varphi_0(x)=y_0$, $x\in\langle a,b\rangle$ belongs to $\operatorname{lip}^2(\langle a,b\rangle,C)$ for a fixed $y_0\in C$. Consequently

$$N\varphi_0=h(\cdot,y_0)\in\operatorname{lip}^2(\langle a,b\rangle,S)\quad\text{for all } y_0\in C.$$

Let us fix $x,z\in\langle a,b\rangle$, $x<z$. By (4),

$$\begin{aligned}
L\|\varphi_1-\varphi_2\| &\ge \|N\varphi_1-N\varphi_2\| \\
&\ge \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle\\ t_i\neq t_j,\ j\neq i}}\|[t_1,t_2,t_3;N\varphi_1-N\varphi_2]\|\ge\left\|\left[x,\frac{x+z}{2},z;N\varphi_1-N\varphi_2\right]\right\| \\
&=\frac{1}{|z-x|}\left\|\frac{N\varphi_1(z)-N\varphi_2(z)-N\varphi_1\left(\frac{x+z}{2}\right)+N\varphi_2\left(\frac{x+z}{2}\right)}{z-\frac{x+z}{2}}\right. \\
&\quad\left.-\frac{N\varphi_1\left(\frac{x+z}{2}\right)-N\varphi_2\left(\frac{x+z}{2}\right)-N\varphi_1(x)+N\varphi_2(x)}{\frac{x+z}{2}-x}\right\| \\
&=\frac{2}{(z-x)^2}\left\|N\varphi_1(z)-N\varphi_2(z)-2N\varphi_1\left(\frac{x+z}{2}\right)\right. \\
&\quad\left.+2N\varphi_2\left(\frac{x+z}{2}\right)+N\varphi_1(x)-N\varphi_2(x)\right\| \\
&=\frac{2}{(z-x)^2}\left\|h(z,\varphi_1(z))-h(z,\varphi_2(z))-2h\left(\frac{x+z}{2},\varphi_1\left(\frac{x+z}{2}\right)\right)\right. \\
&\quad\left.+2h\left(\frac{x+z}{2},\varphi_2\left(\frac{x+z}{2}\right)\right)+h(x,\varphi_1(x))-h(x,\varphi_2(x))\right\|
\end{aligned}$$

for every $\varphi_1, \varphi_2 \in \operatorname{lip}^2(\langle a, b\rangle, C)$. Hence

$$\begin{aligned} &\Big\| h(z,\varphi_1(z)) + 2h\left(\frac{x+z}{2}, \varphi_2\left(\frac{x+z}{2}\right)\right) + h(x,\varphi_1(x)); \\ &\qquad - h(z,\varphi_2(z)) - 2h\left(\frac{x+z}{2}, \varphi_1\left(\frac{x+z}{2}\right)\right) - h(x,\varphi_2(x)) \Big\| \\ &\quad \le \frac{L}{2}(z-x)^2 \|\varphi_1 - \varphi_2\| \end{aligned} \tag{5}$$

for every $\varphi_1, \varphi_2 \in \operatorname{lip}^2(\langle a, b\rangle, C)$.

Let us fix $y_1, y_3, \bar{y}_1, \bar{y}_3 \in C$ and define finctions $\varphi_1, \varphi_2 : \langle a, b\rangle \to X$ as follows

$$\varphi_1(t) = \begin{cases} y_1, & t \in \langle a, x\rangle, \\ \dfrac{2(y_3-y_1)}{(z-x)^2}(t-x)^2 + y_1, & t \in \left(x, \frac{x+z}{2}\right\rangle, \\ \dfrac{-2(y_3-y_1)}{(z-x)^2}(t-z)^2 + y_3, & t \in \left(\frac{x+z}{2}, z\right\rangle, \\ y_3, & t \in (z, b\rangle, \end{cases}$$

$$\varphi_2(t) = \begin{cases} \bar{y}_1, & t \in \langle a, x\rangle, \\ \dfrac{2(\bar{y}_3-\bar{y}_1)}{(z-x)^2}(t-x)^2 + \bar{y}_1, & t \in \left(x, \frac{x+z}{2}\right\rangle, \\ \dfrac{-2(\bar{y}_3-\bar{y}_1)}{(z-x)^2}(t-z)^2 + \bar{y}_3, & t \in \left(\frac{x+z}{2}, z\right\rangle, \\ \bar{y}_3, & t \in (z, b\rangle. \end{cases}$$

By Lemma 2, $\varphi_1, \varphi_2 \in \operatorname{lip}^2(\langle a, b\rangle, C)$, $\varphi_1(x) = y_1$, $\varphi_1\left(\frac{x+z}{2}\right) = \frac{y_1+y_3}{2}$, $\varphi_1(z) = y_3$, $\varphi_2(x) = \bar{y}_1$, $\varphi_2\left(\frac{x+z}{2}\right) = \frac{\bar{y}_1+\bar{y}_3}{2}$, $\varphi_2(z) = \bar{y}_3$ and

$$\sup_{\substack{t_1,t_2,t_3 \in \langle a,b\rangle \\ t_i \neq t_j,\ j \neq i}} \|[t_1, t_2, t_3; \varphi_1 - \varphi_2]\| \le 8\frac{\|y_3 - \bar{y}_3 - y_1 + \bar{y}_1\|}{(z-x)^2}.$$

Thus, by (5), we have

$$\begin{aligned}\Bigg\| h(z,y_3) - h(z,\bar{y}_3) + 2h\left(\frac{x+z}{2}, \frac{\bar{y}_1+\bar{y}_3}{2}\right) \\ - 2h\left(\frac{x+z}{2}, \frac{y_1+y_3}{2}\right) + h(x,y_1) - h(x,\bar{y}_1)\Bigg\| \\ \leq \frac{L}{2}(z-x)^2\left(\|y_1-\bar{y}_1\| + \|y_3-\bar{y}_3\| + 8\frac{\|y_3-\bar{y}_3-y_1+\bar{y}_1\|}{(z-x)^2}\right) \\ = 4L\|y_3-\bar{y}_3-y_1+\bar{y}_1\| + \frac{L}{2}(z-x)^2\left(\|y_3-\bar{y}_3\| + \|y_1-\bar{y}_1\|\right)\end{aligned}$$

for all $x, z \in \langle a, b\rangle$, $x < z$, $y_1, y_3, \bar{y}_1, \bar{y}_3 \in C$.

By Lemma 3, the function $h(\cdot, y_0)$ is continuous in $\langle a, b\rangle$ for every $y_0 \in C$. Letting $z \to x$, we get the inequality:

$$\begin{aligned}\Bigg\| h(x,y_3) - h(x,\bar{y}_3) + 2h\left(x, \frac{\bar{y}_1+\bar{y}_3}{2}\right) - 2h\left(x, \frac{y_1+y_3}{2}\right) \\ + h(x,y_1) - h(x,\bar{y}_1)\Bigg\| \leq 4L\|y_3-\bar{y}_3-y_1+\bar{y}_1\|\end{aligned} \tag{6}$$

for all $x \in \langle a, b\rangle$, $y_1, y_3, \bar{y}_1, \bar{y}_3 \in C$.

Now, let us fix $y \in C$, $h \in X$ such that $y + 3h \in C$. Putting in (6), $y_3 := y + 3h$, $\bar{y}_3 := y + 2h$, $y_1 := y + h$, $\bar{y}_1 := y$ we get

$$\|h(x, y + 3h) + 3h(x, y + h) - 3h(x, y + 2h) - h(x, y)\| = 0$$

for $x \in \langle a, b\rangle$, $y \in C$, $h \in X$ such that $y + 3h \in C$.

This means that for every $x \in \langle a, b\rangle$ the function $h(x, \cdot) : C \to S$ is a polynomial function of order at most 2. According to Corollary 2 in [1], there exist functions $\psi_0 : \langle a, b\rangle \to Y$ and $\psi_k : \langle a, b\rangle \times C^k \to Y$, $k = 1, 2$ such that $\psi_1(x, \cdot)$ is additive and $\psi_2(x, \cdot, \cdot)$ is biadditive, $x \in \langle a, b\rangle$ and

$$h(x, y) = \psi_0(x) + \psi_1(x, y) + \psi_2(x, y, y)$$

for all $x \in \langle a, b\rangle, y \in C$.

Let us fix $y \in C$ and put in (6) $y_3 := y$, $y_1 = \bar{y}_1 = \bar{y}_3 := 0$. Then

$$\left\| h(x,y) - h(x,0) + 2h(x,0) - 2h\left(x, \frac{y}{2}\right) + h(x,0) - h(x,0) \right\| = \left\| h(x,y) - 2h\left(x, \frac{y}{2}\right) + h(x,0) \right\| \le 4L\|y\|$$

for all $x \in \langle a,b\rangle, y \in C$.

Since $\psi_2(x,\cdot\,,\,\cdot)$ is biadditive and $\|\psi_2(x,y,y)\| \le 8L\|y\|$, $x \in \langle a,b\rangle$, $y \in C$, we conclude that $\psi_2(x,y,y) = 0$, $x \in \langle a,b\rangle$, $y \in C$. Hence

$$h(x,\cdot) = \psi_0(x) + \psi_1(x,\cdot), \quad x \in \langle a,b\rangle, \tag{7}$$

is polynomial of order at most one, for each $x \in \langle a,b\rangle$. There exist unique constants $b_o(x) \in Y$ and the additive functions $a_o(x,\cdot) : C \to Y$ such that

$$h(x,y) = a_o(x,y) + b_o(x) \quad \text{for } x \in \langle a,b\rangle, y \in C. \tag{8}$$

It is easy to check, that $b_o(x) \in S$ and $a_o(x,\cdot) : C \to S$.

To prove the linearity of the function $a_o(x,\cdot)$, $x \in \langle a,b\rangle$, let us fix $x,z \in \langle a,b\rangle$, $x \ne z$, $y_1,y_3 \in C$ such that $y_1 - y_3 \in C$. Consider a function $\bar{\varphi}_1 : \langle a,b\rangle \to C$ given as follows

$$\bar{\varphi}_1(t) := \frac{y_1 - y_3}{(z-x)^2}(t-x)^2 + y_3, \quad t \in \langle a,b\rangle.$$

Then $\bar{\varphi}_1(x) = y_3$, $\bar{\varphi}_1(z) = y_1$, $\bar{\varphi}_1(\frac{x+z}{2}) = \frac{y_1+3y_3}{4}$, and

$$\begin{aligned}
&\|[t_1,t_2,t_3;\bar{\varphi}_1]\| \\
&\quad = \frac{\|y_3 - y_1\|}{(z-x)^2}\frac{1}{|t_3 - t_1|}\left| \frac{(t_3-x)^2 - (t_2-x)^2}{t_3 - t_2} - \frac{(t_2-x)^2 - (t_1-x)^2}{t_2 - t_1} \right| \\
&\quad = \frac{\|y_3 - y_1\|}{(z-x)^2}, \quad \text{for distinct } t_1,t_2,t_3 \in \langle a,b\rangle.
\end{aligned}$$

Hence $\bar{\varphi}_1 \in \mathrm{lip}^2(\langle a,b\rangle, C)$. Let $\bar{\varphi}_2(t) = 0$, $t \in \langle a,b\rangle$. By (5), we have

$$\begin{aligned}
&\Bigg\| h(z,y_1) - h(x,0) + 2h\left(\frac{x+z}{2}, 0\right) \\
&\qquad + h(x,y_3) - 2h\left(\frac{x+z}{2}, \frac{y_1+3y_3}{4}\right) - h(x,0)\Bigg\| \le \frac{L}{2}(z-x)^2\|\bar{\varphi}_1\| \\
&\quad \le \frac{L}{2}\|y_3 - y_1\|\left(1 + (a-x)^2 + (b-x)^2\right) + L\|y_3\|(z-x)^2
\end{aligned}$$

for all $x, z \in \langle a, b\rangle$, $x \neq z$, $y_1, y_3 \in C$, such that $y_1 - y_3 \in C$. Letting $z \to x$, by the continuity of $h(\cdot, y), y \in C$ we obtain

$$\left\|h(x,y_1) - h(x,0) + 2h(x,0) - 2h\left(x, \frac{y_1+3y_3}{4}\right) + h(x,y_3) - h(x,0)\right\|$$
$$\leq \frac{L}{2}\left(1 + (a-x)^2 + (b-x)^2\right)\|y_3 - y_1\| \leq \frac{L}{2}\left(1 + (a-b)^2\right)\|y_3 - y_1\|$$

for $x \in \langle a, b\rangle$, $y_1, y_3 \in C$, $y_1 - y_3 \in C$. Thus, by (8)

$$\|a_o(x,y_1) - a_o(x,y_3)\| \leq L\left(1 + (a-b)^2\right)\|y_3 - y_1\| \tag{9}$$

for all $x \in \langle a, b\rangle$, $y_1, y_3 \in C$ such that $y_1 - y_3 \in C$. In particular

$$\|a_o(x,y)\| \leq L\left(1 + (a-b)^2\right)\|y\|$$

for $x \in \langle a, b\rangle$, $y \in C$, which proves (ii).

To prove (i), let us fix $\lambda > 0$. Then $\lambda = \lim_{n\to\infty} \lambda_n$, $\lambda_n \in \mathbb{Q} \cap \langle 0, \infty)$, $\lambda_n \geq \lambda$, $n \in \mathbb{N}$. Therefore $\lambda_n y - \lambda y = (\lambda_n - \lambda)y \in C$ for $y \in C$, $n \in \mathbb{N}$, and by (9)

$$\|\lambda_n a_o(x,y) - a_o(x,\lambda y)\|$$
$$= \|a_o(x,\lambda_n y) - a_o(x,\lambda y)\| \leq L\left(1 + (a-b)^2\right)|\lambda_n - \lambda|\|y\|$$

for $n \in \mathbb{N}$, $y \in C$, $x \in \langle a, b\rangle$. Letting $n \to \infty$, we get $a_o(x, \lambda y) = \lambda a_o(x, y)$ for all $y \in C$, $x \in \langle a, b\rangle, \lambda > 0$.

Since $\tilde{\varphi} = 0$ belongs to $\mathrm{lip}^2(\langle a, b\rangle, C)$,

$$b_o(\cdot) = a_o(\cdot, \tilde{\varphi}) + b_o(\cdot) = h(\cdot, \tilde{\varphi}) = N\tilde{\varphi}(\cdot) \in \mathrm{lip}^2(\langle a, b\rangle, S).$$

Similary, let $\tilde{\varphi}_1 = 0$, $\tilde{\varphi}_2 = y$, where $y \in C$ is fixed. Then, by the Lipschitz condition

$$\|[t_1, t_2, t_3; a_o(\cdot, y)]\| = \|[t_1, t_2, t_3; a_o(\cdot, \tilde{\varphi}_2) - a_o(\cdot, \tilde{\varphi}_1)]$$
$$= \|[t_1, t_2, t_3; N\tilde{\varphi}_2 - N\tilde{\varphi}_1]\| \leq L\|\tilde{\varphi}_1 - \tilde{\varphi}_2\| = 2L\|y\|$$

for all $t_1, t_2, t_3 \in \langle a, b\rangle$, which imply $a_o(\cdot, y) \in \mathrm{lip}^2(\langle a, b\rangle, S)$ and ends the first part of the proof.

Now, let us assume that $(X, \|\cdot\|)$ is a real Banach space, and $\operatorname{int} C \neq \emptyset$. There exists an open ball $B \subset C$. Fix $x \in \langle a, b\rangle$. Then the function $f : C \to Y$ given by

$$f(y) := a_o(x, y), \quad y \in C$$

is linear. By Theorem 2 ([1]), there exists the unique linear extension $\bar{f} : X \to Y$ of the function f. The function $\bar{f}$ is linear and bounded on B, which is a set of the second category. Therefore, by Theorem 5 in [6], the function $\bar{f}$ is continuous, so the function f does, which ends the proof. □

Now we formulate a multivalued version of Theorem 1.

Let $(Y, \|\cdot\|)$ be a real normed space. We denote as $n(Y)$ the family of all nonempty subsets of Y, as $cc(Y)$ the family of all convex and compact members of $n(Y)$, and as θ the singleton $\{0\}$. The class $cc(Y)$ may be considered as a metric space with the Hausdorff distance δ given as follows

$$\delta(A, B) := \inf\left\{t > 0 : A \subseteq B + tS,\ B \subseteq A + tS\right\},$$

where S is closed unit ball in Y. Of course, $\delta(A, \theta) = \sup\{\|a\| : a \in A\}$ for $A \in cc(Y)$. We define

$$\|A\| := \delta(A, \theta).$$

Let $F : \langle a, b\rangle \to cc(Y)$ be a set-valued function (abbreviated to s.v. function in the sequel). Hans Rådström defined the equivalence relation $\sim$ on $cc(Y)$ (see [8]) by stating $(A, B) \sim (C, D)$ if $A + D = B + C$. The equivalence class containing (A, B) is denoted by $[A, B]_\sim$. The quotient space $\mathcal{Y} = (cc(Y))^2/_\sim$, with addition defined by

$$[A, B]_\sim + [D, E]_\sim := [A + D, B + E]_\sim,$$

and scalar multiplication

$$\lambda[A, B]_\sim := \begin{cases} [\lambda A, \lambda B]_\sim\,, & \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A]_\sim\,, & \lambda < 0, \end{cases}$$

is a real normed space with the norm defined by the Hausdorff metric

$$\|[A, B]_\sim\| := \delta(A, B), \quad A, B \in cc(Y).$$

Let define $\mathcal{S} := \{[A,\theta]_\sim, A \in cc(Y)\}$. It is obvious that $\mathcal{S}$ is a convex cone in the space $\mathcal{Y}$. By Lemma 6 in [12], the cone $\mathcal{S}$ is closed in the normed space $\mathcal{Y}$. Consider

$$\mathrm{Lip}^2(\langle a,b\rangle, cc(Y)) := \mathrm{lip}^2(\langle a,b\rangle, \mathcal{S}).$$

Then

$$\mathrm{Lip}^2(\langle a,b\rangle, cc(Y))$$
$$= \left\{ F : \langle a,b\rangle \to cc(Y) : \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} \|[t_1,t_2,t_3;[F,\theta]_\sim\| < \infty \right\},$$

where $[F,\theta]_\sim$ denotes the function $\langle a,b\rangle \ni x \mapsto [F(x),\theta]_\sim \in \mathcal{S}$ and $(\mathrm{Lip}^2(\langle a,b\rangle, cc(Y)), d)$ is a metric space with the metric given

$$d(F,G) := \delta(F(a),G(a)) + \delta(F(b),G(b)) + \sup_{\substack{t_1,t_2,t_3\in\langle a,b\rangle \\ t_i\neq t_j,\ j\neq i}} \|[t_1,t_2,t_3;[F,G]_\sim\|.$$

By the definition, $F \in \mathrm{Lip}^2(\langle a,b\rangle, cc(Y))$ iff the function $\langle a,b\rangle \ni x \mapsto [F(x),\theta]_\sim \in \mathcal{S}$ belongs to the class $\mathrm{lip}^2(\langle a,b\rangle, \mathcal{S})$.

Consider an s.v. function

$$H : \langle a,b\rangle \times C \to cc(Y)$$

which generates the Nemytskii operator

$$\boldsymbol{N} : \mathrm{lip}^2(\langle a,b\rangle, C) \to \mathrm{Lip}^2(\langle a,b\rangle, cc(Y)),$$

$$\boldsymbol{N}\varphi(x) := H(x,\varphi(x)), \quad x\in\langle a,b\rangle, \quad \varphi \in \mathrm{lip}^2(\langle a,b\rangle, C).$$

Assume that $\boldsymbol{N}$ satisfies the Lipschitz condition

$$d(\boldsymbol{N}\varphi_1, \boldsymbol{N}\varphi_2) \le L\|\varphi_1 - \varphi_2\|, \quad \varphi_1,\varphi_2 \in \mathrm{lip}^2(\langle a,b\rangle, C).$$

Then the function

$$h(x,y) : \langle a,b\rangle \times C \to \mathcal{S}$$

given by

$$h(x,y) := [H(x,y),\theta]_\sim, \quad x\in\langle a,b\rangle, \quad y \in C$$

generates the Nemytskii operator

$$N : \mathrm{lip}^2(\langle a,b\rangle, C) \to \mathrm{lip}^2(\langle a,b\rangle, \mathcal{S}),$$
$$N\varphi(x) = h(x,\varphi(x)) = [H(x,\varphi(x)),\theta]_\sim = [\boldsymbol{N}\varphi(x),\theta]_\sim.$$

The operator N is Lipschitzian, so by Theorem 1, there exist functions $a_o : \langle a,b\rangle \times C \to \mathcal{S}$ and $b_o : \langle a,b\rangle \to \mathcal{S}$ such that

(i) $a_o(x,\cdot)$ is linear for all $x \in \langle a,b\rangle$,

(ii) $a_o(x,\cdot)$ is uniformily bounded on the cone C, $x \in \langle a,b\rangle$,

(iii) $h(x,y) = a_o(x,y) + b_o(x)$, $x \in \langle a,b\rangle$, $y \in C$,

(iv) $b_o, a_o(\cdot,y) \in \mathrm{lip}^2(\langle a,b\rangle, \mathcal{S})$ for all $y \in C$.

By the definition of $\mathcal{S}$, there exist s.v. functions

$$A : \langle a,b\rangle \times C \to cc(Y),$$
$$B : \langle a,b\rangle \to cc(Y)$$

such that

$$a_o(x,y) = [A(x,y),\theta]_\sim, \quad b_o(x) = [B(x),\theta]_\sim, \quad x \in \langle a,b\rangle,\ y \in C.$$

The linearity of the function $A(x,\cdot)$ follows from the linearity of $a_o(x,\cdot)$. By (ii) there exists a constant $M > 0$ such that $\|a_o(x,y)\| \le M\|y\|$, $x \in \langle a,b\rangle$. Thus, for all $x \in \langle a,b\rangle$,

$$\|A(x,y)\| = \sup\{\|z\|; z \in A(x,y)\} = \|[A(x,y),\theta]_\sim\| = \|a_o(x,y)\| \le M\|y\|.$$

By (iii), (iv) and the definition of $\mathcal{S}$, $H(x,y) = A(x,y) + B(x)$, $x \in \langle a,b\rangle$, $y \in C$ and $A(\cdot,y)$, $B \in \mathrm{Lip}^2(\langle a,b\rangle, cc(Y))$.

If the values of an s.v. function F are compact, then the continuity of F (cf. [7]) is equivalent to the continuity of the function F treated as an ordinary function with values in the metric space $c(Y)$-nonempty and compact subsets of the space Y with the Hausdorff metric. So, if we assume that $(X, \|\cdot\|)$ is a Banach space and $\mathrm{int}\, C \ne \emptyset$, then continuity of the function $a_o(x,\cdot)$ implies the continuity of the function $A(x,\cdot)$, for each $x \in \langle a,b\rangle$.

So, we have the following collorary:

Corollary 1. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be real normed spaces, and let $C \subseteq X$ be a convex cone. Let a function $H : \langle a, b\rangle \times C \to cc(Y)$ generates the Nemytskii operator $N : \mathrm{lip}^2(\langle a, b\rangle, C) \to \mathrm{Lip}^2(\langle a, b\rangle, cc(Y))$, i.e.,*

$$N\varphi(x) := H(x, \varphi(x)), \quad x \in \langle a, b\rangle,\ \varphi \in \mathrm{lip}^2(\langle a, b\rangle, C)$$

and let N satisfies the Lipschitz condition

$$d(N\varphi_1, N\varphi_2) \le L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \mathrm{lip}^2(\langle a, b\rangle, C).$$

Then there exist s.v. functions $A : \langle a, b\rangle \times C \to cc(Y)$ and $B : \langle a, b\rangle \to cc(Y)$ such that

- (i) *$A(x, \cdot)$ is linear for every $x \in \langle a, b\rangle$,*
- (ii) *$A(x, \cdot)$ is uniformily bounded on C, i.e., there exists a constant $M > 0$ such that $\|A(x, y)\| \le M\|y\|$, $y \in C$ for every $x \in \langle a, b\rangle$,*
- (iii) *$H(x, y) = A(x, y) + B(x)$, $x \in \langle a, b\rangle$, $y \in C$,*
- (iv) *B, $A(\cdot, y)$ belong to $\mathrm{Lip}^2(\langle a, b\rangle, cc(Y))$ for all $y \in C$.*

Moreover, if $(X, \|\cdot\|)$ is a real Banach space (or the set of the second category) and $\mathrm{int}\, C \ne \emptyset$ then the function $A(x, \cdot) : C \to cc(Y)$ is continuous for each $x \in \langle a, b\rangle$.

In the real case, we have

Theorem 2. *Let $C = \mathbb{R}$ or $C = [0, \infty)$ or $C = (-\infty, 0]$ and let $(Y, \|\cdot\|)$ be a real normed space. A function $H : \langle a, b\rangle \times C \to cc(Y)$ generates the Nemytskii operator $N : \mathrm{lip}^2(\langle a, b\rangle, C) \to \mathrm{Lip}^2(\langle a, b\rangle, cc(Y))$, satisfying the Lipschitz condition* (4), *if and only if there exist s.v. functions $A, B : \langle a, b\rangle \to cc(Y)$ such that*

- (i) *$A, B \in \mathrm{Lip}^2(\langle a, b\rangle, cc(Y))$,*
- (ii) *A is bounded, i.e., there exists a constant $M > 0$ such that $\|A(x)\| \le M$ for $x \in \langle a, b\rangle$,*
- (iii) *$H(x, y) = A(x)y + B(x)$; $x \in \langle a, b\rangle$, $y \in C$.*

Proof. Assume that H generates the Nemytskii operator, which is Lipschitzian. Let us fix $x \in \langle a, b\rangle$. By the last theorem, $A(x, \cdot) : C \to cc(Y)$ is a linear function. Hence, there exists a unique set $A(x) \in cc(Y)$ such that

$$A(x, y) = A(x)y \quad \text{for all } y \in C.$$

Boundedness of the function

$$\langle a, b\rangle \ni x \mapsto A(x) \in cc(Y)$$

follows from (ii) of Corollary 1. Similary, (i) and (iii) are clear from Corollary 1.

Conversely, let us fix $\varphi \in \operatorname{lip}^2(\langle a, b\rangle, C)$. Then the map

$$\langle a, b\rangle \ni x \mapsto N\varphi(x) = \varphi(x)A(x) + B(x) \in cc(Y),$$

belongs to $\operatorname{Lip}^2(\langle a, b\rangle, cc(Y))$.
Indeed, for fixed distinct points $t_1, t_2, t_3 \in \langle a, b\rangle$, we have

$$\begin{aligned} &\|[t_1, t_2, t_3; [N\varphi, \theta]_\sim]\| \\ &\leq \|[t_1, t_2, t_3; [B, \theta]_\sim]\| + \|[t_1, t_2, t_3; [\varphi A, \theta]_\sim]\|. \end{aligned}$$

Let us consider the case $C = (-\infty, 0]$. Then

$$\begin{aligned} &\|[t_1, t_2, t_3; [\varphi A, \theta]_\sim]\| \\ &\leq \frac{1}{|t_3 - t_1|}\left\|\frac{[A(t_2)\varphi(t_3), A(t_2)\varphi(t_2)]_\sim}{t_3 - t_2} - \frac{[A(t_2)\varphi(t_2), A(t_2)\varphi(t_1)]_\sim}{t_2 - t_1}\right\| \\ &\quad + \frac{1}{|t_3 - t_1|}\left\|\frac{[A(t_3)\varphi(t_3), A(t_2)\varphi(t_3)]_\sim}{t_3 - t_2} - \frac{[A(t_2)\varphi(t_3), A(t_1)\varphi(t_3)]_\sim}{t_2 - t_1}\right\| \\ &\quad + \frac{1}{|t_3 - t_1|}\left\|\frac{[A(t_2)\varphi(t_3), A(t_1)\varphi(t_3)]_\sim}{t_2 - t_1} - \frac{[A(t_2)\varphi(t_1), A(t_1)\varphi(t_1)]_\sim}{t_2 - t_1}\right\| \\ &= |[t_1, t_2, t_3; \varphi] \cdot \|[-A(t_2), \theta]_\sim\| \\ &\quad + |\varphi(t_3)| \cdot \frac{1}{|t_3 - t_1|}\left\|\frac{[-A(t_3), -A(t_2)]_\sim}{t_3 - t_2} - \frac{[-A(t_2), -A(t_1)]_\sim}{t_2 - t_1}\right\| \\ &\quad + \left|\frac{\varphi(t_3) - \varphi(t_1)}{t_3 - t_1}\right| \cdot \left\|\frac{[-A(t_2), -A(t_1)]_\sim}{t_2 - t_1}\right\| \end{aligned}$$

for $t_1, t_2, t_3 \in \langle a, b\rangle$, distinct. Since,

$$\begin{aligned} |\varphi(t_3)| &\leq \left|\frac{\varphi(t_3) - \varphi(b)}{t_3 - b}\right|(b - a) + |\varphi(b)| \\ &\leq \left|\frac{\varphi(t_3) - \varphi(b)}{t_3 - b} - \frac{\varphi(b) - \varphi(a)}{b - a}\right|(b - a) + \left|\frac{\varphi(b) - \varphi(a)}{b - a}\right|(b - a) + |\varphi(b)| \end{aligned}$$

$$\leq |[a, b, t_3; \varphi]|(b-a)^2 + |\varphi(a)| + 2|\varphi(b)| \leq \left((b-a)^2 + 2\right) \|\varphi\|,$$

$$\left\| \frac{[-A(t_2), -A(t_1)]_\sim}{t_2 - t_1} \right\|$$
$$\leq \left\| \frac{[-A(t_2), -A(t_1)]_\sim}{t_2 - t_1} - \frac{[-A(t_1), -A(a)]_\sim}{t_1 - a} \right\| + \left\| \frac{[-A(t_1), -A(a)]_\sim}{t_1 - a} \right\|$$
$$\leq \|[a, t_1, t_2; [-A, \theta]_\sim]\|(b-a) + \|[b, a, t_1; [-A, \theta]_\sim]\|(b-a)$$
$$+ \frac{\delta(A(a), A(b))}{b-a} \leq \left(2(b-a) + \frac{1}{b-a}\right) \|[-A, \theta]_\sim\|,$$

and similarly,

$$\|[-A(t_2), \theta]_\sim\| \leq \left((b-a)^2 + 2\right) \|[-A, \theta]_\sim\|,$$
$$\left| \frac{\varphi(t_3) - \varphi(t_1)}{t_3 - t_1} \right| \leq \left(2(b-a) + \frac{1}{b-a}\right) \|\varphi\|,$$

the following inequality holds:

$$\sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j,\ j \neq i}} \|[t_1, t_2, t_3; [\varphi A, \theta]_\sim]\|$$
$$\leq 2\left(2 + (b-a)^2\right) \|[-A, \theta]_\sim\| \, \|\varphi\| + \left(2(b-a) + \frac{1}{b-a}\right)^2 \|[-A, \theta]_\sim\| \, \|\varphi\|$$
$$= M_A \|\varphi\|$$

for all $\varphi \in \mathrm{lip}^2(\langle a, b \rangle, C)$, where M_A is a constant independent on φ. The last inequality shows that $N\varphi \in \mathrm{Lip}^2(\langle a, b \rangle, cc(Y))$ as well as N is Lipschitzian. The same proof works for $C = [0, \infty)$. For $C = \mathbb{R}$, the s.v. function A is an ordinary function $A : \langle a, b \rangle \to Y$. Then $[A(u), A(v)]_\sim = [A(u) - A(v), 0]_\sim$, which simplifies the above considerations. □

Since every function belonging to the class $\mathrm{lip}^2(\langle a, b \rangle, C)$ is bounded, the following consequence of Theorem 2 holds:

Corollary 2. *Assume that $C = \mathbb{R}$ or $C = [0, \infty)$ or $C = (-\infty, 0]$ and $(Y, \|\cdot\|)$ is a real normed space. A function $h : \langle a, b \rangle \times C \to Y$ generates* *the Nemytskii operator* $N : \mathrm{lip}^2(\langle a,b\rangle, C) \to \mathrm{lip}^2(\langle a,b\rangle, Y)$, *satisfying the Lipschitz condition*

$$\|N\varphi_1 - N\varphi_2\| \le L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in \mathrm{lip}^2(\langle a,b\rangle, C) \tag{10}$$

if and only if there exist functions $a_o, b_o : \langle a,b\rangle \to Y$, $a_o, b_o \in \mathrm{lip}^2(\langle a,b\rangle, Y)$ *such that*

$$h(x,y) = a_o(x)y + b_o(x) \quad \text{for } x \in \langle a,b\rangle,\ y \in C.$$

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