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Binary recurring sequences and powers, II

By PAULO RIBENBOIM (Kingston)

Dedicated to my friend Marco Fontanu on his 50th birthday

Abstract. We investigate when certain sums or differences of terms of Lucas sequences are powers (or multiples of powers). In the case of squares, we obtain explicit bounds, explicit solution in the case of odd parameters. For Fibonacci and Lucas numbers we obtain results involving squares, but also cubes.

1. Introduction

In this paper we investigate when certain sums or differences of terms of binary second-order recurrences may be equal to a power, or to a multiple of a power.

The starting impulse for this paper is the determination by Steiner [24], and more simply by Williams [25], of the Fibonacci numbers of the form $x^2 + 1$. Earlier, Cohn [1], as well as Wyler [26], had determined the squares among the Fibonacci numbers. The determination of squares among Lucas numbers and in Lucas sequences (with odd relatively prime parameters and non-zero discriminant) was obtained by various authors. Similar results concerning cubes were also obtained for specific sequences (like the Fibonacci, Lucas and Pell numbers).

We introduce a general method which allows to identify numbers in Lucas sequences of the form $x^2 \pm 1$ (or similar form), provided the squares and double squares are known. But in fact we obtain much more as we shall now indicate in a more systematic way.

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Section 2 consists of preliminaries where all the required facts are gathered for the convenience of the reader, including explicit references.

Section 3 is devoted to basic theorems which are proved using the theorem of PETHŐ [13] concerning multiples of powers in recurring sequences. Explicitly, we consider the relation

$$W_{s+2k} \pm Q^k W_s = ax^t,$$

where $W = U(P,Q)$ or $V(P,Q)$ (non-degenerate, $x \neq 0$, $t \geq 2$, $s \geq 1$, $k \geq 1$). a has prime factors in a given finite set of prime numbers. The two theorems of the section establish that if s (or k) is given, the other quantities remain effectively bounded (see the text for the precise statements).

In Section 4 we restrict our attention to the case where $t = 2$, H has at most one odd prime. We give explicit bounds which are directly linked to the occurrence of squares or double squares in the sequences U, V and also in another associated Lucas sequence and in a second sequence of Lehmer numbers. Thus if the required squares are known, the bounds obtained are far lower than what is normally provided by the theorems on diophantine approximation.

In the situation where P, Q are odd and relatively prime (and the discriminant is non-zero), the complete determination of squares and double squares is known. As a consequence, the complete and explicit solution of the original problem is possible. Namely, all pairs (s,k) such that

$$W_{s+2k} \pm Q^k W_s = \square$$

are completely determined (for arbitrary choice of the odd parameters). These results may be immediately applied to Fibonacci numbers and to Lucas numbers.

In Section 6 we show how the same method can be applied to determine Fibonacci numbers of the form $x^3 \pm 1$ (and similar results also for Lucas numbers). For this purpose many results are required to be established about Fibonacci and Lucas numbers which are multiples of cubes. The precise statements of the results obtained are found in the text.

2. Preliminaries

Let P, Q be non-zero integers, α, β the roots of the polynomial $X^2 - PX + Q$. The first and second *Lucas sequences with parameters* (P, Q) are defined as follows:

$$U_0 = 0, \qquad U_1 = 1, \qquad U_n = PU_{n-1} - QU_{n-2}, \tag{2.1}$$

$$V_0 = 2, \qquad V_1 = P, \qquad V_n = PV_{n-1} - QV_{n-2} \tag{2.2}$$

for all $n \geq 2$. We denote these sequences by $U = U(P, Q)$ and $V = V(P, Q)$; if required we use also the notations $U_n(P, Q) = U_n$, $V_n(P, Q) = V_n$.

If $P = 1$, $Q = -1$, then the numbers $U_n(1, -1)$, $V_n(1, -1)$ are the *Fibonacci numbers*, respectively the *Lucas numbers*. If $(P, Q) = (2, -1)$ the numbers $U_n(2, -1)$, $V_n(2, -1)$ are the *Pell numbers* (of first and second kind).

$D = P^2 - 4Q$ is the *discriminant*. We have $D \equiv 0$ or $1 \pmod 4$. For Fibonacci and Lucas numbers, $D = 5$, for Pell numbers $D = 8$.

It is convenient to extend the Lucas sequences also for negative indices:

$$U_{-n} = -\frac{U_n}{Q^n}, \qquad V_{-n} = \frac{V_n}{Q^n} \tag{2.3}$$

for all $n \geq 1$. With this definition, the relations (2.1), (2.2) hold for all integers n.

We note also:

$$U_n(-P, Q) = (-1)^n U_n(P, Q), \tag{2.4}$$

$$V_n(-P, Q) = (-1)^n V_n(P, Q). \tag{2.5}$$

Binet's formulas express the numbers U_n, V_n in terms of α, β:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.6}$$

$$V_n = \alpha^n + \beta^n. \tag{2.7}$$

Among the numerous identities and divisibility properties satisfied by the terms of Lucas sequences we list below some which will be used in this paper (here m, n are any integers):

$$V_n^2 - DU_n^2 = 4Q^n \tag{2.8}$$

(2.9) $$U_{m+n} = U_m V_n - Q^n U_{m-n}$$

(2.10) $$V_{m+n} = V_m V_n - Q^n V_{m-n}$$

(2.11) $$V_{m+n} = D U_m U_n + Q^n V_{m-n}$$

(2.12) $$U_{2n} = U_n V_n$$

(2.13) $$V_{2n} = V_n^2 - 2Q^n$$

(2.14) $$U_{3n} = U_n\left(V_n^2 - Q^n\right) = U_n\left(DU_n^2 + 3Q^n\right)$$

(2.15) $$V_{3n} = V_n\left(V_n^2 - 3Q^n\right).$$

More generally, the following essentially known two results are proven in [16]:

(2.16). *Let $k \geq 3$ be odd. Then there exist uniquely defined polynomials f_k^+, $f_k^- \in \mathbb{Z}[X]$ such that $\deg(f_k^\pm)=(k-1)/2$, $f_k^\pm(0)=(\pm 1)^{(k-1)/2}k$ and*

$$U_{km} = \begin{cases} U_m f_k^+\left(U_m^2\right) & \text{when } m \text{ is even,} \\ U_m f_k^-\left(U_m^2\right) & \text{when } m \text{ is odd.} \end{cases}$$

(2.17). *Let $k \geq 3$ be odd. Then there exist uniquely defined polynomials g_k^+, $g_k^- \in \mathbb{Z}[X]$ such that $\deg(g_k^\pm)=(k-1)/2$, $g_k^\pm(0)=\pm(-1)^{(k-1)/2}k$ and*

$$V_{km} = \begin{cases} V_m g_k^+\left(V_m^2\right) & \text{when } m \text{ is even,} \\ V_m g_k^-\left(V_m^2\right) & \text{when } m \text{ is odd.} \end{cases}$$

(2.18). *If $U_m \neq 1$ then U_m divides U_n if and only if $m|n$.*

(2.19). *If $V_m \neq 1$, $m \neq 0$, then V_m divides V_n if and only if $m|n$ and n/m is odd.*

For the next properties, we assume also that $\gcd(P,Q)=1$. If $m,n\neq 0$ let $d = \gcd(m,n)$.

(2.20) $$\gcd\left(U_m, U_n\right) = U_d$$

(2.21) $$\gcd\left(V_m, V_n\right) = \begin{cases} V_d & \text{if } m/d, n/d \text{ are odd,} \\ 1 \text{ or } 2 & \text{otherwise,} \end{cases}$$

$$(2.22)\qquad \gcd(U_m, V_n) = \begin{cases} V_d & \text{if } m/d \text{ is even,} \\ 1 \text{ or } 2 & \text{otherwise,} \end{cases}$$

$$(2.23)\qquad \gcd(U_n, Q) = 1, \qquad \gcd(V_n, Q) = 1,$$

$$(2.24)\qquad \gcd(D, U_n) = \begin{cases} \gcd(D, n) & \text{if } n \text{ is odd,} \\ \gcd\left(D, \frac{n}{2}\right) \text{ or } 2\gcd\left(D, \frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

(2.25). *If P, Q are odd, then U_n is even if and only if $3|n$, V_n is even if and only if $3|n$.*

In our study, we shall assume without loss of generality that $P > 0$. To exclude degenerate cases, we shall assume that $D = P^2 - 4Q > 0$. Accordingly, let $\mathcal{S}$ be the set of all pairs of non-zero integers (P, Q), with $\gcd(P, Q) = 1$, $P > 0$, $D > 0$. Let $\mathcal{S}^- = \{(P, Q) \in \mathcal{S} \mid P \text{ and } Q \text{ are odd}\}$. The following theorems were proved in [11]; see also [3] for the special case where $|Q| = 1$.

Theorem U1. *Let $(P, Q) \in \mathcal{S}^-$. If $n \geq 1$ and $U_n = \square$ then $n \in \{1, 2, 3, 6, 12\}$. Moreover*

1) $U_2 = \square$ *if and only if* $P = \square$.
2) $U_3 = \square$ *if and only if* $P^2 - Q = \square$.
3) $U_6 = \square$ *if and only if* $P = 3\square$, $P^2 - Q = 2\square$, $P^2 - 3Q = 6\square$; *this implies that* $Q \equiv 1 \pmod{24}$.
4) $U_{12} = \square$ *if and only if* $P = \square$, $P^2 - Q = 2\square$, $P^2 - 2Q = 3\square$, $P^2 - 3Q = \square$ *and* $(P^2 - 2Q)^2 - 3Q^2 = 6\square$; *this implies that* $Q \equiv -1 \pmod{120}$.

Theorem V1. *Let $(P, Q) \in \mathcal{S}^-$. If $V_n = \square$ then $n \in \{1, 3, 5\}$. Moreover*

1) $V_1 = \square$ *if and only if* $P = \square$.
2) $V_3 = \square$ *if and only if* $P = \square$, $P^2 - 3Q = \square$, *or* $P = 3\square$, $P^2 - 3Q = 3\square$; *this implies that* $Q \equiv 3 \pmod 4$.
3) $V_5 = \square$ *if and only if* $P = 5\square$ *and* $P^4 - 5P^2Q + 5Q^2 = 5\square$; *this implies that* $P \equiv Q \equiv 5 \pmod 8$.

Theorem U2. *Let* $(P,Q) \in \mathcal{S}^-$. *If* $n \geq 1$ *and* $U_n = 2\Box$ *then* $n \in \{3,6\}$.

1) $U_3 = 2\Box$ *if and only if* $P^2 - Q = 2\Box$.
2) $U_6 = 2\Box$ *if and only if* $P = \Box$, $P^2 - Q = 2\Box$, $P^2 - 3Q = \Box$; *this implies that* $Q \equiv -1 \pmod 8$.

Theorem V2. *Let* $(P,Q) \in \mathcal{S}^-$. *If* $n \geq 1$ *and* $V_n = 2\Box$ *then* $n \in \{3,6\}$.

1) $V_3 = 2\Box$ *if and only if* $P = 3\Box$, *and* $P^2 - 3Q = 6\Box$; *this implies that* $P \equiv 3 \pmod{24}$ *and* $Q \equiv 1,3 \pmod 8$.
2) $V_6 = 2\Box$ *if and only if* $P^2 - 2Q = 3\Box$ *and* $(P^2-2Q)^2 - 3Q^2 = 6\Box$; *this implies that* $Q \equiv 3 \pmod 4$.

The special cases of Fibonacci and Lucas numbers were obtained earlier (see [1], [2], [26]).

(2.26). *The only square Fibonacci numbers are* $U_1 = U_2 = 1$, $U_{12} = 144$. *The only square Lucas number is* $V_3 = 4$. *The only double square Fibonacci numbers are* $U_3 = 2$, $U_6 = 8$. *The only double square Lucas numbers are* $V_0 = 2$, $V_6 = 18$.

We turn our attention to square classes. Let T be a set of positive integers. The numbers $a, b \in T$ are said to be *square-equivalent* if there exist integers $h, k \neq 0$ such that $ah^2 = bk^2$. It is equivalent to say that $ab = \Box$.

The relation of square-equivalence is an equivalence relation. The equivalence classes are called the *square-classes* of T. The square-classes of Fibonacci numbers and of Lucas numbers were determined by Cohn [3]; an independent proof was given in [14].

(2.27). *The square-classes with more than one term (non-trivial square-classes) of the sequences of Fibonacci numbers and of Lucas numbers are, respectively:* $\{U_1 = 1, U_2 = 1, U_{12} = 144\}$, $\{U_3 = 2, U_6 = 8\}$ *an* $\{V_1 = 1, V_3 = 4\}$, $\{V_0 = 2, V_6 = 18\}$.

In [17], Ribenboim and McDaniel proved:

(2.28). *If* $(P,Q) \in \mathcal{S}$ *then for each* U_n *(respectively* V_n*) there exists an effectively computable bound* B, *respectively a bound* C *(depending on* P, Q, n*) such that if* U_m *is in the square class of* U_n *(respectively* V_m *is in the square class of* V_n*) then* $m < B$ *(respectively* $m < C$*).*

The following theorems were also proved by McDaniel and Ribenboim in [11] (see Cohn [3] for the case when $|Q| = 1$).

Theorem *SCU***.** *Let* $(P,Q) \in \mathcal{S}^-$.

a) *If* $1 \le m < n$ *and* $U_m U_n = \square$ *then* $(m,n) \in \{(1,2),(1,3),(1,6),(1,12),(2,3),(2,12),(3,6),(5,10)\}$ *or* $n = 3m$ *where* $m > 1$, m *is odd,* $3 \nmid m$ *and in this case* $Q \equiv 1 \pmod 4$, $(-Q/P) = +1$, *and* $P \le |Q+1|$.

b1) $U_1 U_n = \square$ *if and only if* $U_n = \square$ *and this implies (see theorem* U1*) that* $n \in \{2,3,6,12\}$.

b2) $U_2 U_3 = \square$ *if and only if* $P = \square$ *and* $P^2 - Q = \square$*; this implies that* $Q \equiv 1 \pmod 4$.

b3) $U_3 U_6 = \square$ *if and only if* $P = \square$ *and* $P^2 - 3Q = \square$, *or else if* $P = 3\square$ *and* $P^2 - 3Q = 3\square$*; this implies that* $Q \equiv 3 \pmod 4$.

b4) $U_5 U_{10} = \square$ *if and only if* $P = 5\square$ *and* $P^4 - 5P^2Q + 5Q^2 = 5\square$*; this implies that* $P \equiv Q \equiv 5 \pmod 8$.

b5) $U_2 U_{12} = \square$ *if and only if* $P = \square$, $P^2 - 3Q = \square$, $P^2 - Q = 2\square$, $P^2 - 2Q = 3\square$, *and* $(P^2-2Q)^2 - 3Q^2 = 6\square$*; this implies that* $Q \equiv 4 \pmod 4$.

Theorem *SCV***.** *Let* $(P,Q) \in \mathcal{S}^-$.

a) *If* $1 \le m < n$ *and* $V_m V_n = \square$, *then* $(m,n) = (1,3)$ *or* $n = 3m$, *with* $m > 1$, m *odd,* $3 \nmid m$*; this implies that* $Q \equiv 3 \pmod 4$, $3 \nmid P$, $(-3Q/P) = +1$, *and* $P \le \left|\frac{7}{6}Q + \frac{6}{7}\right|$.

b) $V_1 V_3 = \square$ *if and only if* $P^2 - 3Q = \square$*; this implies that* $Q \equiv 3 \pmod 8$ *and* $3 \nmid P$.

Concerning cubes we quote the following results (see London and Finkelstein [8], Lagarias and Weissel [6], Pethő [12]):

(2.29).

a) $U_1 = U_2 = 1$, $U_6 = 8$ *are the only Fibonacci numbers which are cubes.*

b) $V_1 = 1$ *is the only Lucas number which is a cube.*

We shall also require the Lehmer numbers (see [7]). Let $R > 0$, Q be integers with $D = R - 4Q \neq 0$. Let α, β be the roots of $X^2 - \sqrt{R}X + Q$, so $\alpha + \beta = \sqrt{R}$, $\alpha\beta = Q$.

The first and second sequences of *Lehmer numbers*, with parameters $\sqrt{R}$, Q are defined as follows (see [7]):

$$K_n\left(\sqrt{R},Q\right)=\begin{cases}\dfrac{\alpha^n-\beta^n}{\alpha-\beta}, & \text{if } n \text{ is odd},\\ \dfrac{\alpha^n-\beta^n}{\alpha^2-\beta^2}, & \text{if } n \text{ is even}.\end{cases}$$

$$L_n\left(\sqrt{R},Q\right)=\begin{cases}\dfrac{\alpha^n+\beta^n}{\alpha+\beta}, & \text{if } n \text{ is odd},\\ \alpha^n+\beta^n, & \text{if } n \text{ is even}.\end{cases}$$

If $R=P^2$ then

$$K_{2n+1}\left(\sqrt{P^2},Q\right)=U_{2n+1}(P,Q),$$

$$K_{2n}\left(\sqrt{P^2},Q\right)=U_n\left(V_2(P,Q),Q^2\right)=\frac{U_{2n}(P,Q)}{U_2(P,Q)},$$

$$L_{2n+1}\left(\sqrt{P^2},Q\right)=K_{2n+1}\left(\sqrt{D},-Q\right),$$

$$L_{2n}\left(\sqrt{P^2},Q\right)=V_n(P,Q).$$

The square terms in certain sequences of Lehmer numbers were determined by McDaniel [10].

Theorem K1. *Let* $Q\equiv 3 \pmod 4$, *or let* $Q\equiv 5 \pmod 8$ *and* $R\equiv 5 \pmod 8$.

a) *If* $n\geq 1$ *and* $K_n=\square$ *then* $n\in\{1,2,3,4,6,12\}$.

b1) $K_3=\square$ *if and only if* $R-Q=\square$.

b2) $K_4=\square$ *if and only if* $R-2Q=\square$.

b3) $K_6=\square$ *if and only if* $R-Q=2\square$ *and* $R-3Q=2\square$.

b4) $K_{12}=\square$ *if and only if* $R-Q=\square$, $R-3Q=2\square$, $R-2Q=3\square$, *and* $R^2-4RQ+Q^2=6\square$.

Theorem L1. *Let* $Q\equiv 1 \pmod 4$ *and* $R\equiv 1,5,$ *or* $7 \pmod 8$, *or else let* $Q\equiv 3 \pmod 4$ *and* $R\equiv 1 \pmod 8$. *Then:* $L_n=\square$ *if and only if* $n=1$ *or else* $n=3$ *and* $R-3Q=\square$.

To conclude the preliminaries we quote the following result of Pethő [13] which will be used in Section 3. The special case when a is fixed was proved earlier by Shorey and Stewart [23]. We shall use the following notation. If H is a finite set of prime numbers, then $H^\times$ denote the set of all numbers whose prime factors are in H.

Theorem P. *Let $(P,Q) \in \mathcal{S}$, let $W = U(P,Q)$ or $V(P,Q)$ and let H be a finite set of primes. Then there exists and effectively computable number (depending on P, Q, H) such that if $a \in H^\times$, $k \geq 2$, $|x| \geq 1$, $n \geq 1$ and $W_n = ax^k$ then*

i) *if $|x| = 1$ then $n, |a| < C$;*

ii) *if $|x| > 1$ then $n, |a|, |x|, k < C$.*

3. Basic results

Let $\mathcal{S}$ be the set of non-degenerate pairs (P,Q) with $\gcd(P,Q) = 1$. Let $W = U(P,Q)$ or $V(P,Q)$, let H be a finite set of prime numbers. We shall discuss the relation

$$W_{s+2k} \pm Q^k W_s = ax^t \tag{3.1}$$

where $s \geq 1$, $k \geq 1$, $a \in H^\times$, x is a non-zero integer, $t \geq 2$. We shall prove the following theorems:

Theorem s. *Given $s \geq 1$, there exists $C > 0$, effectively computable, depending on P, Q, H, s such that if* (3.1) *holds then:*

i) *if $|x| = 1$ then $k, |a| < C$,*

ii) *if $|x| > 1$ then $k, |a|, |x|, t < C$.*

The next theorem is analogous, for given k (instead of given s):

Theorem k. *Given $k \geq 1$, there exists $C > 0$, effectively computable, depending on P, Q, H, k such that if* (3.1) *holds then:*

i) *if $|x| = 1$ then $s, |a| < C$,*

ii) *if $|x| > 1$ then $s, |a|, |x|, t < C$.*

Special noteworthy cases occur for $Q = \pm 1$, $t = 2$ or 3 leading to the relations:

$$U_{s+2k} \pm U_s = x^t, \qquad V_{s+2k} \pm V_s = x^t.$$

In particular, the results apply to Fibonacci numbers, Lucas numbers, and Pell numbers.

Proof of Theorem s. *Case* (+)*:* $W_{s+k}V_k = W_{s+2k} + Q^k W_s = ax^t$ (where a, x, t depend on k). Let $d_k = \gcd(s+k, k)$. Then $d_k|s$, and $e_k = \gcd(W_{s+k}, V_k)$ is equal to 1, 2 or V_{d_k}. Let $H_1 = H \cup \{2\} \cup \{\text{prime factors of } V_d \text{ for all } d|s\}$. So H_1 is a finite set of primes which depends on H, s, but not on k.

We have $e_k^2|W_{s+k}V_k = ax^t$. Let $x = yz$, with $\gcd(y,z) = 1$, $y \in H_1^{\times}$. Then $b = ay^t/e^2$ is an integer belonging to $H_1^{\times}$; note that y, z, and b depend on k. Then

$$\frac{W_{s+k}}{e_k} = b_1 z_1^t \qquad \frac{V_k}{e_k} = b_2 z_2^t$$

with $b_1 b_2 = b$, $z_1 z_2 = z$. Then $V_k = (e_k b_2)\, z_2^t$ with $e_k b_1 \in H_1^{\times}$. By Pethő's theorem [13] k is effectively bounded and from (3.1) we deduce that if $|x| = 1$ then $|a| < C$, while if $|x| > 1$ then also $|x|, t < C$.

Case (−)*:* We first consider the case $W = U$:

$$U_k V_{s+k} = U_{s+2k} - Q^k U_s = ax^t.$$

The proof continues like in Case (+). If $W = V$, then

$$DU_{s+k}U_k = V_{s+2k} - Q^k V_s = ax^t.$$

Let $d_k = \gcd(s+k, k)$ so $d_k|s$ and $\gcd(U_{s+k}, U_k) = U_{d_k}$. Hence $e_k = \gcd(DU_{s+k},\, U_k)$ divides DU_{d_k}. Let $H_1 = H \cup \{\text{prime factors of } DU_s\}$, so H_1 is finite. Then, as in the proof of Case (+), $U_k = (e_k, b_2)\, z_2^t$ where $e_k b \in H_1^{\times}$ and by Pethő's theorem [3], $k < C$ and the proof is concluded as before. □

Proof of Theorem k. *Case* (+)*:* $W_{s+k}V_k = W_{s+2k} + Q^k W_s = ax^t$. Let $d_s = \gcd(s+k, k)$. Then $e_s = \gcd(W_{s+k}, V_k) = 1, 2$ or V_{d_s}. Let $H_1 = H \cup \{2\} \cup$prime factors of V_d, for all $d|k\}$. As in the proof of Theorem (s), $W_{s+k} = (e_s b_1)\, z^t$ with $e_s b_1 \in H_1^{\times}$. The proof is concluded using Pethő's theorem.

Case (−)*:* We have

$$U_k V_{s+k} = U_{s+2k} - Q^k U_s = ax^t$$

and

$$DU_{s+k}U_k = V_{s+2k} - Q^k V_s = ax^t.$$

The proof continues as in Case (−) of Theorem (s). □

4. Bounds in the case of squares

In this section we consider the special case of squares and investigate the relations

$$W_{s+2k} \pm Q^k W_s = \square$$

where $W = U(P,Q)$ or $W = V(P,Q)$, $(P,Q) \in \mathcal{S}$.

The purpose is to give effective explicit upper bounds. For this purpose we introduce the following notations. If $l \geq 1$ let

$$T_l = \{h\square, 2h\square \mid h|l\}.$$

In particular, $T_1 = T_2 \subseteq T_l$ for $l \geq 1$.

For each $r \geq 1$ and $l \geq 1$ let

$$R_l^r = \{U_d h\square, 2U_d h\square \mid \; d|r, h|l\},$$

$$S_l^r = \{V_d h\square, 2V_d h\square \mid \; d|r, h|l\}.$$

In particular, $T_l = R_l^1 \subseteq R_l^r$ for all $r \geq 1$, $l \geq 1$. For $r \geq 1$, $l \geq 1$ let

$$B_l^{(r)} = \max\{n \mid U_n \text{ or } V_n \text{ belongs to } T_l \cup S_l^r\}$$

and

$$C_l^{(r)} = \max\{n \mid U_n \text{ or } V_n \text{ belongs to } S_l^r \cup R_l^r\}.$$

Then $B_l^{(2)}$, $C_l^{(r)}$ are finite and effectively computable. We note also that $T_1 \subseteq T_l \subseteq R_l^r$ (since $U_1 = 1$) so $B_1^{(r)} \leq C_1^{(r)}$.

$$S_1^1 = \{V_1\square,\ 2V_1\square\},$$

$$S_1^2 = \{V_1\square,\ 2V_1\square,\ V_2\square,\ 2V_2\square\}.$$

(4.1). *For every $s \geq 1$, if $k \geq 1$ is such that*

$$U_{s+2k} \pm Q^k U_s \in \bigcup_{p \text{ prime}} T_p$$

then $k \leq B_1^{(s)}$.

Proof. *Case* $(+)$*:* We have

$$U_{s+k}V_k = U_{s+2k} + Q^k U_s \in \bigcup_p T_p$$

so there exists a prime $p \geq 2$ such that $U_{s+k}V_k \in T_p$. Let $e = \gcd(U_{s+k}, V_k)$, so $e = 1, 2$ or V_d, where $d = \gcd(s+k, k) = \gcd(s,k)$ with $(s+k)/d$ even, k/d odd, hence s/d odd. If $e = 1$ then

$$\begin{cases} U_{s+k} = \square \\ V_k = \square \end{cases} \quad \text{or} \begin{cases} = 2\square \\ = \square \end{cases} \quad \text{or} \begin{cases} = \square \\ = 2\square \end{cases}$$

$$\text{or} \begin{cases} = p\square \\ = \square \end{cases} \quad \text{or} \begin{cases} = \square \\ = p\square \end{cases} \quad \text{or} \begin{cases} = 2p\square \\ = \square \end{cases}$$

$$\text{or} \begin{cases} = p\square \\ = 2\square \end{cases} \quad \text{or} \begin{cases} = 2\square \\ = p\square \end{cases} \quad \text{or} \begin{cases} = \square \\ = 2p\square \end{cases}$$

Thus either $U_{s+k} \in T_1$ or $V_k \in T_1$. Hence $k \leq B_1^{(s)}$. Let $e = 2$. Proceeding similarly from $\frac{U_{s+k}}{2}\frac{V_k}{2} \in T_p$ we deduce that either U_{s+k} or V_k belong to T_1, hence $k \leq B_1^{(s)}$.

If $e = V_d$ then $\frac{U_{s+k}}{V_d}\frac{V_k}{V_d} \in T_p$ with $\gcd\left(\frac{U_{s+k}}{V_d}, V_k/V_d\right) = 1$ As before, $\frac{U_{s+k}}{V_d} \in T_1$ or $\frac{V_k}{V_d} \in T_1$, hence $U_{s+k} \in \mathcal{S}_1^s$ or $V_k \in \mathcal{S}_1^s$ and therefore $k \leq B_1^{(s)}$.

Case $(-)$*:* The proof is very similar. We have $U_kV_{s+k} = U_{s+2k} - Q^kU_s \in T_p$ for some prime p. Again, $e = \gcd(U_k, V_{s+k}) = 1$, or 2, or V_d where $d = \gcd(k, s+k) = \gcd(s,k)$, k/d even, $(s+k)/d$ odd, so s/d odd. The proof continues as in Case $(+)$, by interchanging $s+k$ and k. $\square$

(4.2). *For every $s \geq 1$, if $k \geq 1$ is such that*

$$V_{s+2k} \pm Q^kV_s \in \bigcup_{p \text{ prime}} T_p$$

then $k \leq C_w^{(s)}$, where $D = P^2 - 4Q = wz^2$, with $w \geq 1$, w square-free.

Proof. *Case* $(+)$*:* We have

$$V_{s+k}V_k = V_{s+2k} + Q^kV_s \in T_p$$

for some prime $p \geq 2$. Let $e = \gcd(V_{s+k}, V_k)$, so $e = 1, 2$ or V_d where $d = \gcd(s+k, k) = \gcd(s,k)$, with $\frac{s+k}{d}$, $\frac{k}{d}$ both odd. If $e = 1$ we see as in the proof of (4.1) that either $V_{s+k} \in T_1$ or $V_k \in T_1$ hence $k \leq C_w^{(s)}$. If $e = 2$ the same argument shows that $V_{s+k} \in T_1$ or $V_k \in T_1$ so $k \leq C_w^{(s)}$. If

$e = V_d$ we proceed similarly getting again $V_{s+k} \in V_d T_1$ or $V_k \in V_d T_1$, that is V_{s+k} or $V_k \in S_1^s$ hence $k \le C_w^{(s)}$.

Case $(-)$*:* We have

$$DU_{s+k}U_k = V_{s+2k} - Q^k V_s \in T_p$$

for some prime $p \ge 2$. Since $D = wz^2$ with w square free, then

$$U_{s+k}U_k \in \{w\square, 2w\square, wp\square, 2wp\square\}\,.$$

Let $d = \gcd(s+k, k) = \gcd(s, k)$ so $U_d = \gcd(U_{s+k}, U_k)$. Then

$$\frac{U_{s+k}}{U_d}\frac{U_k}{U_d} \in \{w\square, 2w\square, wp\square, 2wp\square\}$$

hence

$$\begin{aligned}
&\begin{cases} U_{s+k} = U_d a\square \\ U_k = U_d b\square \end{cases}
&&\text{or} \begin{cases} = U_d \times 2a\square \\ = U_d b\square \end{cases}
&&\text{or} \begin{cases} = U_d a\square \\ = U_d \times 2b\square \end{cases}\\
\text{or} &\begin{cases} = U_d ap\square \\ = U_d b\square \end{cases}
&&\text{or} \begin{cases} = U_d a\square \\ = U_d bp\square \end{cases}
&&\text{or} \begin{cases} = U_d \times 2ap\square \\ = U_d b\square \end{cases}\\
\text{or} &\begin{cases} = U_d ap\square \\ = U_d \times 2b\square \end{cases}
&&\text{or} \begin{cases} = U_d \times 2a\square \\ = U_d bp\square \end{cases}
&&\text{or} \begin{cases} = U_d a\square \\ = U_d \times 2bp\square \end{cases}
\end{aligned}$$

where $ab = w$, $\gcd(a, b) = 1$. In all cases U_{s+k} or U_k belongs to R_w^s, hence $k \le C_w^{(s)}$. $\square$

We indicate now corresponding results where k and l are given.

(4.3). *Let $k \ge 1$, $l \ge 1$ be given. If $s \ge 1$ is such that*

$$U_{l+2k} \pm Q^k U_s \in T_l,$$

then $s \le B_l^{(k)}$.

Proof. *Case* $(+)$*:* We have $U_{s+2k} + Q^k U_s = U_{s+k}V_k \in T_l$. Let $e = \gcd(U_{s+k}, V_k)$ so $e = 1, 2$, or V_d where $d = \gcd(s+k, k)$, $(s+k)/d$ is even and k/d is odd.

Proceeding as in the proof of (4.1), if $e = 1$ or 2 then

$$\begin{cases} U_{s+k} = \square \\ V_k = \square \end{cases} \quad \text{or} \quad \begin{cases} = 2\square \\ = \square \end{cases} \quad \text{or} \quad \begin{cases} = \square \\ = 2\square \end{cases}$$

$$\text{or} \quad \begin{cases} = a\square \\ = b\square \end{cases} \quad \text{or} \quad \begin{cases} = 2a\square \\ = b\square \end{cases} \quad \text{or} \quad \begin{cases} = a\square \\ = 2b\square \end{cases}$$

where $ab = l$, $\gcd(a, b) = 1$. Then $U_{s+k} \in T_l$ so $s \leq B_l^{(k)}$. If $e = V_d$ then $U_{s+k} \in S_l^k$ hence $s \leq B_l^{(k)}$.

Case $(-)$*:* We have $U_{s+2k} - Q^k U_s = U_k V_{s+k} \in T_l$. Proceeding as before we obtain $V_{s+k} \in T_l \cup S_l^k$ hence $s \leq B_l^{(k)}$. $\square$

(4.4). *Let $k \geq 1$, $l \geq 1$ be given. If $s \geq 1$ is such that*

$$V_{s+2k} \pm Q^k V_s \in T_l$$

then $s \leq \max\left\{B_l^{(k)}, C_{lw}^{(k)}\right\}$.

Proof. *Case* $(+)$*:* We have $V_{s+2k} + Q^k V_s = V_{s+k} V_k \in T_l$. Let $e = \gcd(V_{s+k}, V_k)$, so $e = 1, 2$, or V_d where $d = \gcd(s+k, k)$ and $(s+k)/d$, k/d are odd. If $e = 1$ or 2, by the same argument we see that $V_{s+k} \in T_l$, hence $s \leq B_l^{(k)}$. If $e = V_d$, from $\frac{V_{s+k}}{V_d} \frac{V_k}{V_d} \in T_l$ with coprime factors, then $V_{s+k} \in S_l^k$, so $s \leq B_l^{(k)}$.

Case $(-)$*:* We have $V_{s+2k} - Q^k V_s = D U_{s+k} U_k \in T_l = \{h\square, 2h\square \mid h|l\}$. Since $D = wz^2$, with w square-free then

$$U_{s+k} U_k \in \{wh\square, 2wh\square \mid h|l\}.$$

Let $d = \gcd(s+k, k)$, so $U_d = \gcd(U_{s+k}, U_k)$ hence

$$\begin{cases} U_{s+k} = U_d ag\square \\ U_k = U_d bh\square \end{cases}$$

where $ab = w$, gh divides $2l$ with $\gcd(a, b) = 1$, $\gcd(g, h) = 1$. Thus $U_{s+k} \in R_{lw}^k$ so $s \leq C_{lw}^{(k)}$. $\square$

When $s = 1$ we were led in some of the cases of the above proofs to determine when $\frac{U_{1h}}{V_1} = \frac{U_{1h}}{U_2}$ (for $h \geq 1$) or $\frac{V_k}{V_1}$ (for k odd) is in $\{\square, 2\square\}$, We note that $\frac{U_{2h}}{U_2} = U_h\left(V_2(P, Q), Q^2\right)$ and $\frac{V_k}{V_1} = L_k\left(\sqrt{P^2}, Q\right)$ (kth term of the second Lehmer sequence). The problem becomes the determination of squares and double squares in the above sequences.

5. Applications to sequences with odd parameters and squares

Let $(P,Q)\in\mathcal{S}$ and assume that P, Q are odd. We shall apply the results in [11], [18] to determine the pairs (s,k) such that

$$U_{s+2k}\pm Q^kV_s=\square \tag{5.1}$$

or

$$V_{s+2k}\pm Q^kV_s=\square \tag{5.2}$$

hold.

(5.3). *Let $s\geq 1$, $k\geq 1$. If $U_{s+2k}-Q^kU_s=\square$ then*

$$(s,k)\in\{(1,2),\ (2,1),\ (4,1),\ (2,3),\ (3,3)\}.$$

Proof. We have

$$\square=U_{s+2k}-Q^kU_s=U_kV_{s+k}.$$

Let $e=\gcd(U_k,V_{s+k})$, so $e=1,2$, or V_d where $d=\gcd(s+k,k)$, k/d is even.

a) If $e=1$ then $U_k=\square$ and $V_{s+k}=\square$. Then $k=1,2,3,6$, or 12 and $s+k=1,3,5$. Since $k<s+k$ then

$$(s,k)\in\{(2,1),\ (4,1),\ (1,2),\ (2,3)\}.$$

Note that $(3,2)$ is not possible, since $U_2=\square$ means that $P=\square$, while $V_5=\square$ implies that $P=5\square$.

b) If $e=2$ then $U_k=2\square$, $V_{s+k}=2\square$, hence $k=3$ or 6 and $s+k=3$ or 6. Since $k<s+k$ then $(s,k)=(3,3)$.

c) If $e=V_d$ with k/d even, $(s+k)/d$ odd, so s/d odd. Then $U_kV_d=\square, V_{s+k}V_d=\square$. Since $d\leq k<s+k$ then $s+k=3d$, with d odd, $3\nmid d$, hence $k=2d$. So $(U_dV_d)\,V_d=\square$ hence $U_d=\square$. Hence $d=1$ and $(s,k)=(1,2)$ $\square$

(5.4). *Let* $s \geq 1$, $k \geq 1$. *If* $U_{s+2k} + Q^k U_s = \square$ *then*

$$(s,k) \in \{(1,1),\ (2,1),\ (11,1),\ (9,3),\ (2,2),\ (3,3),\ (10,2),\ (6,6)\}.$$

Proof. We have

$$\square = U_{s+2k} + Q^k U_s = U_{s+k} V_k.$$

Let $e = \gcd(U_{s+k}, V_k)$, so $e = 1, 2$, or V_d where $d = \gcd(s+k, k)$ and $(s+k)/d$ is even.

a) If $e = 1$ then $U_{s+k} = \square, V_k = \square$, hence $s+k = 1, 2, 3, 6$, or 12 and $k = 1, 3, 5$. Since $k < s+k$ then

$$(s,k) \in \{(1,1),\ (2,1),\ (5,1),\ (11,1),\ (3,3),\ (9,3),\ (1,5),\ (7,5)\}.$$

But $(3,3)$ is excluded, since $U_6 = \square$ implies that $Q \equiv 1 \pmod{24}$ and $V_3 = \square$ implies that $Q \equiv 3 \pmod 4$. Also $(1,5)$ is impossible, since $U_6 = \square$ implies that $P = 3\square$ while $V_5 = \square$ implies that $P = 5\square$. Similarly $(7,5)$ is not possible, because $U_{12} = \square$ implies that $P = \square$. Also $U_6 V_1 = U_6 U_2$ is not a square.

The pair (5,1) is also excluded because $U_6 = \square$ implies that $P = 3\square$, while $V_1 = \square$ means that $P = \square$, a contradiction.

Thus

$$(s,k) \in \{(1,1),\ (2,1),\ (11,1),\ (9,3)\}.$$

b) If $e = 2$ then $U_{s+k} = 2\square, V_k = 2\square$. Hence $s+k = 3$ or 6 and $k = 3$ or 6. Since $k < s+k$ then $(s,k) = (3,3)$, but this is excluded, since $U_6 = 2\square$ implies that $P = \square$, while $V_3 = 2\square$ implies that $P = 3\square$.

c) If $e = V_d$ then $(s+k)/d$ is even, so $k/d, s/d$ are odd and $U_{s+k} V_d = \square$, $V_k V_d = \square$.

Case c 1)*:* $d < k$. Then $k = 3d$, and d is odd.

Case c 1.1) Let $d = 1$ so $k = 3$ and $s + 3 = 2^f g$ with g odd, $f \geq 1$. We have

$$U_g V_g V_{2g} \cdots V_{2^{f-1}g} = P\square.$$

But $\gcd\left(U_g, V_g V_{2g} \cdots V_{2^{f-1}g}\right)$ is 1 or a proper power of 2. From

$$U_g V_g V_{2g} \cdots V_{2^{f-1}g} = U_{2^f g} = P\square$$

then $U_g = P\square$ or $2P\square$. Since $U_2 = P$ then $U_2U_g = \square$ or $2\square$. In the first case $g = 3$; in the second case U_g is even so $3|g$. In both cases $3|d$, which is absurd.

Case c 1.2) Let $d > 1$ then d is odd. $3 \nmid d$, $3 \nmid P$. Let $s+k = 2^f g$ with $f \geq 1$, g odd. Since d is odd, then $d|g$. Also if $3|g$ then $3|s+k$, $k = 3d$ so $3|d$, which is a contradiction. So $3 \nmid g$. We have $V_dU_{s+k} = \square$ so

$$V_dU_gV_gV_{2g}\cdots V_{2^{f-1}g} = \square,$$

with $\gcd\big(U_g, V_{2^ig}\big) = 1$ (for $0 \leq i$), $\gcd(V_{2^ig}, V_{2^jg}) = 1$ (for $i < j$), $\gcd(V_d, U_g) = 1$ and $\gcd\big(V_d, V_{2^ig}\big) = 1$ (for $i \leq i$). Therefore

$$\frac{V_g}{V_d}U_gV_{2g}\cdots V_{2^{f-1}g} = \square$$

hence $V_dV_g = \square$, therefore $g = 3d$, which is a contradiction.

Case c 2) $d = k$. Hence $U_{s+k}V_k = \square$ with $k|s$ and s/k odd, so $(s+k)/k$ even. Let $s+k = 2^f g$, with $f \geq 1$, g odd. Then $k = 2^l h$ with $0 \leq l < f$, h odd, $h|g$. We have

$$U_gV_gV_{2g}\cdots V_{2^{f-1}g}V_{2^lh} = \square$$

with $\gcd(U_g, V_gV_{2g}\cdots V_{2^lh}) = 1$ or a power of 2. So $U_g = \square$ or $2\square$, hence $g = 1$ or 3 or $g = 3$.

Case c 2.1) Let $g = 1$, so $h = 1$, $s+k = 2^f$, $k = 2^l$ and $U_{2^f}V_{2^l} = \square$. If $f = l+1$ then $U_{2^l} = \square$ so $2^l = 1$ or 2, $2^f = 2$ or 4. So $(s,k) = (1,1)$, $(2,2)$. Now, if $f > l+1$ then

$$V_{2^{f-1}}V_{2^{f-2}}\cdots V_{2^{l+1}}U_{2^l} = \square.$$

Since $U_{2^l}, V_{2^{l+1}}, \ldots, V_{2^{f-1}}$ are pairwise relatively prime, then $V_{2^{l+1}} = \square$ which is impossible.

Case c 2.2) Let $g = 3$ so $s+k = 2^f \times 3$, $k = 2^l h$ with $0 \leq l < f$, h odd, $h|3$ so $h = 1$ or 3. From

$$V_{2^{f-1}\times 3}\cdots V_{2^{l+1}\times 3}V_{2^l\times 3}V_{2^l\times h}U_{2^l\times 3} = \square$$

and $\gcd(U_{2^l\times 3}, V_{2^{f-1}\times 3}\cdots V_{2^l\times h}) = 1$ or a power of 2, then $U_{2^f\times 3} = \square$ or $2\square$. Hence $2^f \times 3 = 6$ or 12, respectively $2^f \times 3 = 6$, so $f = 1$ or 2, respectively $f = 1$. Since $(s+k)/k$ is even, then $(s,k) = (5,1)$, $(3,3)$, $(11,1)$, $(10,2)$, $(9,3)$, $(6,6)$; $(s,k) = (5,1)$ is not possible, since $U_6V_1 = U_6U_2$ is not a square.

We apply (5.3) and (5.4) to the sequence of Fibonacci numbers (see Robbins [19] [21] [22] for (c), (d), (e), (f)):

(5.5). *Let U be the sequence of Fibonacci numbers.*

a) *The only sums of Fibonacci numbers with indices of the same parity which are equal to a square are: U_4+U_2, U_9+U_3, U_6+U_2.*

b) *The only differences of Fibonacci number with indices of the same parity, which are equal to a square are: U_3-U_1, U_5-U_1, $U_{13}-U_{11}$, $U_{15}-U_9$.*

c) $U_n=\square+1$ *if and only if* $n=3,5$.

d) $U_n=\square-1$ *if and only if* $n=4,6$.

e) $U_n\neq\square+2$ *for all odd* n.

f) $U_n=\square-2$ *with n odd if and only if* $n=9$.

Proof. This is just a particular case of (5.3) and (5.4), excluding the pairs (s,k) which do not yield a square. $\square$

Before proving the next result, we observe that if P, Q are odd then $D\neq\square$. Indeed, let α, β be the roots of X^2-PX+Q, so $\alpha+\beta=P$, $\alpha\beta=Q$. If $D=\square$ then $\alpha,\beta\in\mathbb{Z}$ so α, β are odd, hence $P=\alpha+\beta$ would be even. Now we show:

(5.6). *Let $s\geq 1$, $k\geq 1$ be such that* $\gcd(s,k)=1$. *If* $V_{s+2k}-Q^kV_s=\square$ *then* $\gcd(D,s+k)\neq 1$.

Proof. We have $\square=V_{s+2k}-Q^kV_s=DU_{s+k}U_k$. We recall that

$$\gcd(D,U_n)=\begin{cases}\gcd(D,n) & \text{for all odd } n,\\ \gcd\left(D,\dfrac{n}{2}\right)\ \text{or}\ 2\gcd\left(D,\dfrac{n}{2}\right) & \text{for all even } n.\end{cases}$$

We note that if n is even, since D is odd then the second alternative above cannot occur.

Let $d_k=\gcd(D,U_k)$, $d_{s+k}=\gcd(D,U_{s+k})$. Since $\gcd(U_k,U_{s+k})=1$ then $\frac{U_k}{d_k}$, $\frac{U_{s+k}}{d_{s+k}}$, $\frac{D}{d_kd_{s+k}}$ are coprime integers whose product is a square. So $\frac{U_k}{d_k}=\square$, $\frac{U_{s+k}}{d_{s+k}}=\square$, $\frac{D}{d_kd_{s+k}}=\square$. But

$$d_{s+k}=\gcd(D,U_{s+k})=\begin{cases}\gcd(D,s+k) & \text{when } s+k \text{ is odd},\\ \gcd\left(D,\dfrac{s+k}{2}\right) & \text{when } s+k \text{ is even}.\end{cases}$$

So if $\gcd(D, s+k) = 1$ then $d_{s+k} = 1$ hence $U_{s+k} = \square$. By Section 2, Theorem (U1) $s+k = 1, 2, 3, 6$, or 12. Since $\gcd(s,k) = 1$ then

$$(s,k) \in \{(1,1),\ (2,1),\ (5,1),\ (11,1),\ (1,2),\ (1,5),\ (7,5),\ (5,7),\ (1,11)\}.$$

Now we note that $k = 1$ or 2. If $k = p \in \{5, 7, 11\}$ then $d_k = \gcd(D, U_k) = \gcd(D, k) = 1$ or p. If $p|U_p$ then p is the rank of appearance of p, thus $p \mid p - \left(\frac{D}{p}\right)$, which is absurd. Thus $d_k = 1$ and therefore $D = \square$, which is again impossible. So $k = 1$ or 2, therefore $d_k = 1$ hence $U_k = \square$. From $DU_{s+k}U_k = \square$, it follows that $D = \square$ which is absurd. $\square$

We obtain the following more precise result for the sequences of Fibonacci numbers and Lucas numbers ($P = 1$, $Q = -1$):

(5.7). *Let $s \geq 1$, $k \geq 1$ with $\gcd(s,k) = 1$. If $P = 1$, $Q = -1$ then $V_{s+2k} - (-1)^k V_s \neq \square$ except when $(s,k) = (4,1)$ or $(3,2)$.*

Proof. We have $D = 5$. If

$$\square = V_{s+2k} - (-1)^k V_s = 5U_{s+k}U_k,$$

by (5.6) $5|s+k$ hence $5 \nmid U_k$ and $\gcd(5U_{s+k}, U_k) = 1$, so $U_k = \square$ and $5U_{s+k} = U_5 U_{s+k} = \square$. The first equation implies $k \in \{1, 2, 12\}$ by Theorem $U1$. If $s+k = 5$, then we obtain the solutions. On the other hand, if $s+k > 5$ then by Theorem SCU $s+k = 10$, which is impossible, because in that case $P \equiv Q \equiv 5 \pmod 8$. $\square$

(5.8). *Let $s \geq 1$, $k \geq 1$. If $V_{s+2k} + Q^k V_s = \square$ then $(s,k) = (2,1)$ or $1 < k$, $s = 2k$. The first case happens if and only if $P^2 3Q = \square$; this implies that $Q \equiv 3 \pmod 8$, $3 \nmid P$. If the second case happens then k is odd, $Q \equiv 3 \pmod 4$, $3 \nmid P$, $\left(\frac{-3Q}{P}\right) = +1$ and also $P < \left|\frac{7}{6}Q + \frac{6}{7}\right|$.*

Proof. Let

$$\square = V_{s+2k} + Q^k V_s = V_{s+k}V_k.$$

By Section 2, Theorem ($S \subset V$), we must have $(k, s+k) = (1,3)$, so $(s,k) = (2,1)$, or $1 < k$, $s = 2k$. Finally, the conditions indicated in the statement must be satisfied as it was proved in [18]. $\square$

From the preceding results we deduce (see Robbins [19] for (c), (d)):

(5.9). *Let V be the sequence of Lucas numbers:*

a) $V_6 + V_4$ *is the only sum of Lucas numbers with indices of the same parity which is a square.*

b) $V_4 - V_2$, $V_7 - V_3$ *are the only differences of Lucas numbers with indices of the same parity which are equal to a square.*

c) $V_n \neq \square \pm 1$ *for all odd* n.

d) $V_n = \square + 3$ *with* n *even if and only if* $n = 4$; $V_n \neq \square - 3$ *for all even* n.

Proof. This is just a rephrasement of the preceding results for the sequence of Lucas numbers. $\square$

As a supplement we show:

(5.10). *Let* $a \geq 1$.

a) *If* $a + 2$ *is a prime then* $V_{2n} = \square + (-1)^n a$ *if and only if* $V_n = (a + 2 + (-1)^n)/2$.

b) *If* $a - 2$ *is a prime or equal to 1 then* $V_{2n} = \square - (-1)^n a$ *if and only if* $V_n = (a - 2 - (-1)^n)/2$.

Proof. a) Let $x \geq 0$ and $x^2 + (-1)^n a = V_{2n} = V_n^2 - 2(-1)^n$ so $V_n^2 - x^2 = (-1)^n(2 + a)$. Then

$$\begin{cases} V_n + x = a + 2 \\ V_n - x = 1 \end{cases} \quad (n \text{ even}) \quad \text{or} \quad \begin{cases} x + V_n = a + 2 \\ x - V_n = 1. \end{cases} \qquad \square$$

6. Applications to cubes and Fibonacci or Lucas numbers

In this section we determine some sums and differences of Fibonacci numbers or of Lucas numbers which are cubes. We recall that London and Finkelstein [8], as well as Pethő [12], Langarias and Weissel [6], showed that the only Fibonacci numbers which are cubes are $U_1 = U_2 = 1$, $U_6 = 8$. On the other hand, $V_1 = 1$ is the only Lucas number which is a cube.

Our first result is the following:

(6.1). *Let $S = \{2C, 4C, 3C, 9C, 6C, 12C, 18C, 36C\}$. Then $U_n \in S$ if and only if $n = 3, 4, 12$. In particular $U_n \notin \{4C, 9C, 6C, 12C, 36C\}$.*

Proof. Clearly for $n \leq 12$, $U_n \in S$ if and only if $n = 3, 4, 12$: $U_3 = 2$, $U_4 = 3$, $U_{12} = 18 \times 8$. Assume that n is the smallest index, $n > 12$, such that $U_n \in S$. Then either $2|U_n$ or $3|U_n$.

First case. $2|U_n$ then $3|n$. Let $n = 3k$. So $U_n = U_k\left(5U_k^2 + 3(-1)^k\right)$. Let $d = \gcd\left(U_k, 5U_k^2 + 3(-1)^k\right)$ so $d = 1$ or 3. If $d = 1$ then $U_k \in S \cup C$. If $U_k = C$ then $k = 1$ or 6 so $n = 3$ (excluded) or $n = 18$. However, $U_{18} \notin S$, as verified by computation. If $U_k \in S$ then $k \leq 12$ (by the minimality of n), hence $k = 3$, giving $n = 9$ (excluded) or $k = 4$, but then $d = 3$, which is excluded; or $k = 12$, $n = 36$, but $U_{36} \notin S$. If $d = 3$ then

$$\frac{U_k}{3}\,\frac{5U_k^2 + 3(-1)^k}{2} \in S \cup C.$$

Then $U_k \in S \cup C$ and we conclude as in the previous case that $k = 3, 4$, or 12, giving again $n = 9, 12, 36$ which are excluded.

Second case. $2 \nmid U_n$ then $3|U_n$ thus $4|n$. Let $n = 2k$, so $U_kV_k = U_n \in S$. Let $e = \gcd(U_k, V_k)$. So $e = 1$ or 2. By hypothesis, $2 \nmid U_n$, so $e = 1$ then $U_k \in S \cup C$. If $U_k = C$ then $k = 1$ so $n = 2$ (excluded), or $k = 6$, $n = 12$, which was already considered. If $U_k \in S$ then $k \leq 12$ so $k = 3$, hence $n = 6$, which is excluded; or $k = 4$ and $n = 8$, also excluded; or $k = 12$, $n = 24$, but $U_{24} \notin S$. $\square$

Similarly,

(6.2). *Let $S = \{2C, 4C, 3C, 9C, 6C, 12C, 18C, 36C\}$. Then $V_n \in S$ if and only if $n = 2, 3, 6$. In particular, $V_n \notin \{2C, 6C, 9C, 12C, 36C\}$.*

Proof. For $n \leq 6$, $V_n \in S$ if and only if $n = 2, 3, 6$. Let $n > 6$ be the smallest index such that $V_n \in S$. Either $2|V_n$ or $3|V_n$.

First case. $2|V_n$ so $3|n$. Let $n = 3k$, hence $V_k\left(V_k^2 - 3(-1)^k\right) \in S$. Let $d = \gcd\left(V_k, V_k^2 - 3(-1)^k\right)$, so $d = 1$ or 3. If $d = 1$ then $V_k \in S \cup C$. If $V_k = C$ then $k = 1$ and $n = 3$, excluded. If $V_k \in S$ then by the minimality of n, $k = 2, 3, 6$ hence $n = 6$ (excluded), or $n = 9, 18$ but $V_9, V_{18} \notin S$. If $d = 3$ then

$$\frac{V_k}{3}\,\frac{V_k^2 - 3(-1)^k}{3} \in S \cup C,$$

so $V_k \in S \cup C$. As before $k = 1, 2, 3, 6$ leading to no allowed value for $n = 3k$.

Second case. $2 \nmid V_n$, hence $3|V_n$, thus $n \equiv 2 \pmod 4$ so $n = 2k$, with k odd. Now $V_k^2 - 2 = C$. As it is known (see [9] for example) the only solution in integers of $X^2 - 2 = Y^3$ is $x = \pm 1$, $y = -1$ so $k = 1$, $n = 2$ (excluded). $\square$

We shall also require the following fact:

(6.3). *Let $S = \{5C, 25C, 110C, 20C, 50C, 100C\}$. Then $U_n \in S$ if and only if $n = 5$. In particular $U_n \neq 25C, 10C, 20C, 50C, 100C$.*

Proof. For $n \leq 5$, $U_n \in S$ exactly when $n = 5$. Let $n > 5$ be the smallest index such that $U_n \in S$. Since $5|U_n$ then 5 (the rank of appearance of 5) divides n. Let $n = 5k$, so $U_n = U_k f_5\left(U_k^2\right)$, where $f_5 \in \mathbb{Z}[X]$ with constant term ± 5. Let $d = \gcd\left(U_k, f_5\left(U_k^2\right)\right)$, so $d = 1$ or 5. Whether $d = 1$ or $d = 5$, we deduce that $U_k \in \{C, 2C, 4C\} \cup S$. By the preceding results and the minimality of n, we have $k = 1, 2, 3, 5, 6$ giving $n = 5$ (excluded) or $n = 10, 15, 25, 30$. But, by calculation, we see that $U_{10}, U_{15}, U_{25}, U_{30} \notin S$. $\square$

It is worth noting that since $5 \nmid V_n$ for all n, then $V_n \notin \{5C, 25C\}$ for all n. The same method may be used to determine the Fibonacci numbers and the Lucas numbers of the form aC, for some given integer $a > 1$. It is worth noting, as an illustration that the above results amount to the determination of solutions in integers of certain diophantine equations. For example: Solutions in positive integers:

$$\begin{aligned} X^2 - 5Y^6 &= \pm 1 && (x, y) = (2, 1) \\ X^2 - 45Y^6 &= \pm 4 && (x, y) = (7, 1) \\ X^2 - 20Y^6 &= \pm 1 && \text{no solution} \\ X^2 - 125Y^6 &= \pm 4 && (x, y) = (11, 1) \\ X^2 - 45Y^6 &= \pm 1 && \text{no solution} \\ X^2 - 405Y^6 &= \pm 4 && \text{no solution.} \end{aligned}$$

And similarly

$$X^6 - 5Y^2 = \pm 1 \qquad \text{no solution}$$

$$
\begin{array}{ll}
9X^6 - 5Y^2 = \pm 4 & (x,y) = (1,1) \\
4X^6 - 5Y^2 = \pm 1 & (x,y) = (1,1) \\
9X^6 - 5Y^2 = \pm 1 & \text{no solution} \\
81X^6 - 5Y^2 = \pm 4 & \text{no solution.}
\end{array}
$$

Now we shall consider sums or differences of Fibonacci numbers and of Lucas numbers. First, we show,

(6.4). *Let* $s \geq 1$, $k \geq 1$ *be integers and assume that* $d = \gcd(s,k) = 1, 2,$ *or* 3. *Then*

$$U_{s+2k} - (-1)^k U_s \neq C.$$

Proof. We have $C = U_{s+2k} - (-1)^k U_s = U_s V_{s+k}$.
Let $e = \gcd(U_s, V_{s+k})$. If $e = 1$ then $U_s = C$, $V_{s+k} = C$, which is impossible. If $e = 2$ then $\frac{U_s}{2}\frac{V_{s+k}}{2} = 2C$ and

$$\begin{cases} U_s = 4C \\ V_{s+k} = 2C \end{cases} \quad \text{or} \quad \begin{cases} = 2C \\ = 4C. \end{cases}$$

Then $2|U_s$, $2|V_{s+k}$ so $3|s$, $3(s+k)$ and $3\gcd(s,k)$ which is contrary to the hypothesis.

If $d = \gcd(s, s+k)$ with s/d even, $s+k/d$ odd, then
$V_d = \gcd(U_s, V_{s+k})$. If $d = 1$ this leads to $e = 1$, already seen. If $d = 2$ then $V_2 = 3$, hence $\frac{U_s}{3}\frac{V_{s+k}}{3} = 3C$. Therefore,

$$\begin{cases} U_s = 3C \\ V_{s+k} = 9C \end{cases} \quad \text{or} \quad \begin{cases} = 9C \\ = 3C. \end{cases}$$

The first case is not possible by (5.4) while the second is impossible by (5.3).

If $d = 3$ then $V_3 = 4$ so $\frac{U_s}{4}\frac{V_{s+k}}{4} = 4C$ and therefore,

$$\begin{cases} U_s = 2C \\ V_{s+k} = 4C \end{cases} \quad \text{or} \quad \begin{cases} = 4C \\ = 2C. \end{cases}$$

By (6.1), (6.2), in the first case $s = 3$, $s+k = 3$, which is impossible. The second case is also impossible by (6.2). $\square$

(6.5). *Let $s \geq 1$, $k \geq 1$ be integers with* $\gcd(s,k) = 1, 2,$ *or* 3. *Then* $U_{s+2k} + (-1)^k U_s = C$ *if and only if* $(s,k) = (1,1)$ *or* $(5,1)$.

Proof. We have $C = U_{s+2k} + (-1)^k U_s = U_{s+k}V_k$.
Let $e = \gcd(U_{s+k}, V_k)$. If $e = 1$ then $U_{s+2k} = C$, $V_k = C$, which implies that $k = 1$, $s + k = 2$ or 6 so $s = 1$ or 5. Let $e = 2$, so $\frac{U_{s+k}}{2}\frac{V_k}{2} = 2C$, hence

$$\begin{cases} U_{s+k} = 4C \\ V_k = 2C \end{cases} \quad \text{or} \quad \begin{cases} = 2C \\ = 4C. \end{cases}$$

However, by (6.1), $U_{s+k} \neq 4C$ while by (6.2) if $V_k = 4C$ then $k = 3$ and if $U_{s+k} = 2C$ then $s + k = 3$, which is incompatible.

Let $d = \gcd(s,k)$ and assume that $(s+k)/d$ is even while k/d is odd. Then $e = V_d$. If $d = 1$ then $e = 1$, already considered. Let $d = 2$ so $V_2 = 3$ and $\frac{U_{s+k}}{3}\frac{V_k}{3} = 3C$. Thus

$$\begin{cases} U_{s+k} = 9C \\ V_k = 3C \end{cases} \quad \text{or} \quad \begin{cases} = 3C \\ = 9C. \end{cases}$$

However, by (6.1) and (6.2), $U_{s+k} \neq 9C$ and $V_k \neq 9C$, so all cases are impossible.

If $d = 3$ then $V_3 = 4$ and we have $\frac{U_{s+k}}{4}\frac{V_k}{4} = 4C$, leading to

$$\begin{cases} U_{s+k} = 2C \\ V_k = 4C \end{cases} \quad \text{or} \quad \begin{cases} = 4C \\ = 2C. \end{cases}$$

In the first case, by (6.1) and (6.2) $s + k = 3$, $k = 3$, which is impossible. By (6.2) the second case is impossible. □

Next we consider similar results for Lucas numbers:

(6.6). *Let $s \geq 1$, $k \geq 1$ be integers such that* $\gcd(s,k) = 1, 2,$ *or* 3. *Then*

$$V_{s+2k} - (-1)^k V_s \neq C.$$

Proof. We have $C = V_{s+2k} - (-1)^k V_s = 5U_{s+k}U_k$ hence $U_{s+k}U_k = 25C$. Let $d = \gcd(s,k)$. Then $\gcd(U_{s+k}, U_k) = U_d$. If $d = 1$ or 2 then $U_d = 1$ so $U_{s+k} = 25C$ or $U_s = 25C$, which is impossible by (6.3).

If $d = 3$ then $U_3 = 2$ and $\frac{U_{s+k}}{2}\frac{U_k}{2} = 50C$ so

$$\begin{cases} U_{s+k} = 100C \\ U_k = 2C \end{cases} \quad \text{or} \quad \begin{cases} = 50C \\ = 4C \end{cases} \quad \begin{cases} = 4C \\ = 50C \end{cases} \quad \begin{cases} = 2C \\ = 100C. \end{cases}$$

By (6.3) all cases are impossible. □

(6.7). *Let* $s \geq 1$, $k \geq 1$ *be integers with* $\gcd(s,k) = 1, 2,$ *or* 3. *Then*

$$V_{s+2k} + (-1)^k V_s \neq C.$$

PROOF. We have $C = V_{s+2k} + (-1)^k V_s = V_{s+k} V_k$.
Let $e = \gcd(V_{s+k}, V_k)$. If $e = 1$ then $V_{s+k} = C$, $V_k = C$, so $s + k = k = 1$, which is impossible. If $e = 2$ then $\frac{V_{s+k}}{2}\frac{V_k}{2} = 2C$, hence

$$\begin{cases} V_{s+k} = 2C \\ V_k = 4C \end{cases} \quad \text{or} \quad \begin{cases} = 4C \\ = 2C. \end{cases}$$

By (6.2) both cases are impossible, since $V_n \neq 2C$ for all n. Now let $d = \gcd(s,k)$, with k/d, $(s+k)/d$ odd. Then $e = V_d$. If $d = 1$, this case has already been considered. If $d = 2$, $V_2 = 3$. Then $\frac{V_k}{3}\frac{V_{s+k}}{3} = 3C$, hence $V_k = 3C$ or $9C$ and respectively $V_{s+k} = 9C$ or $3C$. However, this is impossible because $V_n \neq 9C$ for all n.

If $d = 3$ then $V_3 = 4$ so $\frac{V_k}{4}\frac{V_{s+k}}{4} = 4C$ and $V_k = 2C$ or $V_{s+k} = 2C$, which is impossible. □

As particular cases of the preceding results, we deduce:

(6.8).

a) $U_n \neq C \pm 1$ *for all* n *except* $n = 3$.

b) $U_n \neq C \pm 2$ *for all odd* n.

c) $U_n \neq C \pm 4$ *for all even* n.

d) $V_n \neq C \pm 1$ *for all odd* n.

e) $V_n \neq C \pm 3$ *for all even* n.

f) $V_n \neq C \pm 4$ *for all odd* n.

PROOF. This is just a special case of (6.4), (6.5), (6.6), and (6.7). □

As a supplement, we show:

(6.9). *$V_{2n} = C \pm 1$ if and only if $n = 2$ (giving $V_4 = 7$).*

Proof. Let $C \pm 1 = V_{2n} = V_n^2 - 2(-1)^n$ so $V_n^2 - 2(-1)^n \mp 1 = C$. According to the parity of n and sign this leads to the Mordell equations

$X^2 - 3 = Y^3$ (only solution in positive integers $x = 2$, $y = 1$)

$X^2 + 3 = Y^3$ (no solution)

$X^2 - 1 = Y^3$ (only solution in positive integers $x = 3$, $y = 2$) and

$X^2 + 1 = Y^3$ (no solution).

It follows that $V_n = 2$ (impossible) or $V_n = 3$ so $n = 2$ and $V_4 = 7 = C - 1$. $\square$

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PAULO RIBENBOIM
DEPARTMENT OF MATHEMATICS AND STATISTICS
QUEEN'S UNIVERSITY
KINGSTON, ONTARIO K7L 3N6
CANADA

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