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On weakly symmetric Riemannian spaces

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Abstract. The object of this paper is to present the modified form of weakly symmetric Riemannian spaces introduced by TAMÁSSY and BINH [1] with an illustrative example.

1. Introduction

The notions of weakly symmetric and weakly projective symmetric spaces were introduced by TAMÁSSY and BINH [1]. A non-flat Riemannian space V_n $(n > 2)$ is called a weakly symmetric space if the curvature tensor R_{hijk} satisfies the condition:

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + c_i R_{hljk} + d_j R_{hilk} + e_k R_{hijl} \tag{1.1}$$

where a, b, c, d, e are 1-forms (non-zero simultaneously) and the comma ',' denotes covariant differentiation with respect to the metric tensor of the space.

The 1-forms a, b, c, d, e are called the associated 1-forms of the space and an n-dimensional space of this kind is denoted by $(WS)_n$. It may be mentioned in this connection that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric space studied by CHAKI and MONDAL [2], the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo-symmetric space. A reduction in generalized pseudo-symmetric spaces has been obtained by Chaki and Mondal. But in this paper $(WS)_n$ is investigated and a reduction in $(WS)_n$ is obtained in a simpler form.

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In Section 2 it is shown that the 1-forms c and e are identical with b and d, respectively. Then the defining condition of a $(WS)_n$ can always be expressed in the following form:

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + b_i R_{hljk} + d_j R_{hilk} + d_k R_{hijl}. \tag{1.2}$$

In Section 3 an example of a weakly symmetric space has been given.

2. Associated 1-forms of a $(WS)_n$

In this section it will be shown that the five associated 1-forms a, b, c, d, e of a $(WS)_n$ cannot be all different.

Interchanging h and i in (1.1) we get

$$R_{ihjk,l} = a_l R_{ihjk} + b_i R_{lhjk} + c_h R_{iljk} + d_j R_{ihlk} + e_k R_{ihjl}. \tag{2.1}$$

Now, adding (1.1) and (2.1) we obtain

$$(b_h - c_h) R_{lijk} + (b_i - c_i) R_{lhjk} = 0$$

or

$$A_h R_{lijk} + A_i R_{lhjk} = 0 \tag{2.2}$$

where $A_h = b_h - c_h$. We want to show that $A_h = 0$ $(h = 1, \dots, n)$. Suppose on the contrary that there exists a fixed index q for which $A_q \neq 0$. Putting $h = l = q$ in (2.2) we get $A_q R_{qijk} = 0$ which implies that $R_{qijk} = 0$ for all l, j, k. Next, putting $i = q$ in (2.2) we obtain $A_h R_{lqjk} + A_q R_{lhjk} = 0$ which implies that $R_{hljk} = 0$ for all l, h, j, k, since $R_{qijk} = 0$ for all i, j, k and $A_q \neq 0$. Then the space is flat contradicting our hypothesis. Hence $A_h = 0$ for all h, which implies that

$$b_h = c_h \quad \text{for all } h. \tag{2.3}$$

Similarly, interchanging j and k in (1.1) and proceding as before we get

$$d_h = e_h \quad \text{for all } h. \tag{2.4}$$

From (2.3) and (2.4) we see that the associated 1-forms a, b, c, d, e are not all different, beacuse $b = c$ and $d = e$. In virtue of this we can state the following

Theorem 1. *The defining equation of* $(WS)_n$ *can always be expressed in the following form:*

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + b_i R_{hljk} + d_j R_{hilk} + d_k R_{hijl}.$$

3. Example of a $(WS)_n$

In this section we give an example of a $(WS)_n$.

Let each Latin index run over $1, 2, \ldots, n$ and each Greek index over $2, 3, \ldots, n-1$.

We define the metric g in the coordinate space R^n $(n \geq 4)$ by the formula

$$ds^2 = \varphi (dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{3.1}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants and φ is independent of x^n. Thus R_n becomes a Riemannian space V_n.

In the metric considered, the only non-vanishing components of the Christoffel symbols and the curvature tensor R_{hijk} are (see [3])

$$\left\{ \begin{matrix} \beta \\ 11 \end{matrix} \right\} = -\frac{1}{2} k^{\beta\alpha} \varphi_{\cdot\alpha}, \quad \left\{ \begin{matrix} n \\ 11 \end{matrix} \right\} = \frac{1}{2} \varphi_{\cdot 1}, \quad \left\{ \begin{matrix} n \\ 1\alpha \end{matrix} \right\} = \frac{1}{2} \varphi_{\cdot\alpha},$$

and

$$R_{1\alpha\beta 1} = \frac{1}{2} \varphi_{\cdot\alpha\beta} \tag{3.2}$$

where $(\cdot)$ denotes the partial differentiation, and $[k^{\beta\alpha}]$ is the inverse matrix. Here we consider $k_{\alpha\beta}$ as $\delta_{\alpha\beta}$ and $\varphi = K_{\alpha\beta} x^\alpha x^\beta e^{x^1}$. In this case $\varphi = K_{\alpha\beta} x^\alpha x^\beta e^{x^1}$ reduces to

$$\varphi = \sum_{\alpha=2}^{n-1} x^\alpha x^\alpha e^{x^1}. \tag{3.3}$$

Hence

$$\varphi_{\cdot\alpha\alpha} = 2e^{x^1} \quad \text{and } \varphi_{\cdot\alpha\beta} = 0 \text{ for } \alpha \neq \beta. \tag{3.4}$$

It follows from (3.2) and (3.4) that the only non-zero components of R_{hijk} are

$$R_{1\alpha\alpha 1} = e^{x^1}. \tag{3.5}$$

Also we can easily show that the only non-zero components of $R_{hijk,l}$ are

$$R_{1\alpha\alpha 1,1} = e^{x^1}. \tag{3.6}$$

Let

$$a_i = \begin{cases} \frac{1}{2} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad b_i = \begin{cases} \frac{1}{3} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$d_i = \begin{cases} \frac{1}{6} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

In order to verify the relation (1.2) in our V_n, it is sufficient to check the following relations:

(A) $R_{1\alpha\alpha 1,1} = a_1 R_{1\alpha\alpha 1} + b_1 R_{1\alpha\alpha 1} + b_\alpha R_{11\alpha 1} + d_\alpha R_{1\alpha 11} + d_1 R_{1\alpha\alpha 1}$

(B) $R_{11\alpha 1,\alpha} = a_\alpha R_{11\alpha 1} + b_1 R_{\alpha 1\alpha 1} + b_1 R_{1\alpha\alpha 1} + d_\alpha R_{11\alpha 1} + d_1 R_{11\alpha\alpha}$

(C) $R_{1\alpha 11,\alpha} = a_\alpha R_{1\alpha 11} + b_1 R_{\alpha\alpha 11} + b_\alpha R_{1\alpha 11} + d_1 R_{1\alpha\alpha 1} + d_1 R_{1\alpha 1\alpha}$.

As for any case other than (A), (B) and (C), the components of R_{hijk} and $R_{hijk,l}$ vanish identically, and the relation (1.2) holds trivially.

From (3.5), (3.6) and (3.7) we get the following relation for the right-hand side (r.h.s.) and the left-hand side (l.h.s.) of (A):

$$\text{r.h.s. of (A)} = (a_1 + b_1 + d_1) R_{1\alpha\alpha 1} = 1.e^{x^1} = R_{1\alpha\alpha 1,1} = \text{l.h.s. of (A)}.$$

Now the r.h.s. of (B) $= b_1(R_{\alpha 1\alpha 1} + R_{1\alpha\alpha 1}) = 0$, and the by antisymmetric property of R_{hijk} also the l.h.s. of (B) vanishes. By a similar argument as in (B), it can be shown that the relation (C) is also true. Hence R^n equipped with the metric g given in (3.1) is a weakly symmetric space. □

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