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A section theorem in *W*-space with applications

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Abstract. The purpose of this paper is to construct a W-space by a function without the condition of convexities, and to prove an interesting section theorem in a W-space. As applications, a minimax theorem of Joó type and some related results are obtained.

1. Introduction

The minimax problems have been studied extensively by many authors (for example, FAN [11], CHANG [7], HA [8], STACHÓ [9] and KOMORNIK [10]) under some conditions of the convexity of the function. Recently, Joó obtained a minimax theorem not involving convexities of the function on the interval space [1–2]. Most recently, CHANG et al. in [3–5] introduced the concept of a W-space which is a topological space equipped with the family of its nonempty connected subsets. In this paper we construct a W-space by means of a function not involving convexities and linearities, on which an interesting section theorem is established. Using this result, Joó's minimax theorem is generalized, and a fixed point theorem and some related results are obtained.

First of all we state two definitions needed in this paper.

Definition (CHANG [5, Definition 2.1]). Let X be a Hausdorff topological space and $\{C_A\}$ a family of nonempty connected subsets of X indexed by finite subsets A of X such that $A \subset C_A$; then we call $(X, \{C_A\})$ a W-space. A subset K of a W-space is convex if for every $u, v \in K$ we have $C_{\{u,v\}} \subset K$.

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Definition. Let X be a topological space. A function $f : X \to R$ is called upper semicontinuous (abbreviated as u.s.c.) on X if the set $\{x \in X : f(x) < c\}$ is open in X for any $c \in R$.

2. A section theorem

Let E, F be Hausdorff topological spaces, $X \subset E$, $Y \subset F$ be nonempty subsets, and $f : X \times Y \to R$ a function. Let

$$H_y^c := \{x \in X : f(x,y) < c\} \quad \forall y \in Y \text{ and } c \in (-\infty, +\infty]$$

and

$$c_0 := \inf\{c \in (-\infty, +\infty] : H_y^c \neq \emptyset,\ \forall y \in Y\}.$$

Here, $c_0 \in [-\infty, +\infty]$. If $c_0 = +\infty$, then $H_y^{c_0} = X$ for any $y \in Y$.

In this paper we are interested in $c_0 \in [-\infty, +\infty)$. In the sequel, we always assume that $c_0 \in [-\infty, +\infty)$, and that $\mathcal{F}(Y)$ be the family of all nonempty finite subsets of Y.

We are now in a position to prove our main result.

Theorem 2.1. *Let X be a nonempty compact subset of a Hausdorff topological space E, Y a Hausdorff topological space and $f : X \times Y \to R$ a function. If the following conditions are satisfied:*

(i) *any intersection of the system $\Omega = \{B_c^x : c \in R,\ x \in X\}$ is connected where $B_c^x := \{y \in Y : f(x,y) \geq c\}$;*

(ii) *the function $x \mapsto f(x,y)$ is continuous for every fixed $y \in Y$;*

(iii) *the function $y \mapsto f(x,y)$ is upper semicontinuous for every fixed $x \in X$;*

(iv) *for any $A \in \mathcal{F}(Y)$ and any fixed real number $c > c_0$, the set $\bigcap\{H_y^c : y \in A\}$ is connected,*

then there exists a $x_0 \in X$ such that

$$\{x_0\} \times Y \subset P(c) := \{(x,y) \in X \times Y : f(x,y) \leq c\}.$$

Proof. The proof will proceed through a series of claims.

Claim 1. The space Y is equipped with the topological structure of the W-space. In fact, for any fixed $x \in X$ and $A = \{y_1, \dots, y_n\} \in \mathcal{F}(Y)$, we take

$$\bar{c} = \min\{f(x, y_1), \dots, f(x, y_n)\}$$

such that $A \subset B^x_{\bar{c}}$. If define

$$C_A = \bigcap \{B^x_c \in \Omega : A \subset B^x_c\} \tag{2.1}$$

then C_A is a nonempty, connected closed subset in Y and $A \subset C_A$. Therefore, $(Y, \{C_A\})$ becomes a W-space.

Claim 2. For any $u, v \in Y$, we have that $y \in C_{\{u,v\}}$ if and only if

$$H^c_y \subset H^c_u \cup H^c_v.$$

Suppose that there exists a $x_0 \in H^c_{y_0}$ for some $y_0 \in C_{\{u,v\}}$ and $x_0 \notin H^c_u \cup H^c_v$. This implies $y_0 \notin B^{x_0}_c$ and $u, v \in B^{x_0}_c$. From (2.1) we have that $y_0 \notin C_{\{u,v\}}$. It is a contradiction. On the contrary, we need to prove that $y_0 \notin C_{\{u,v\}}$ implies $H^c_{y_0} \nsubseteq H^c_u \cup H^c_v$. If $y_0 \in Y \setminus C_{\{u,v\}}$ then there exists a $B^{x_0}_c$ such that $y_0 \notin B^{x_0}_c$ and $u, v \in B^{x_0}_c$. It follows that $x_0 \in H^c_{y_0}$ and $x_0 \notin H^c_u \cup H^c_v$, i.e., $H^c_{y_0} \nsubseteq H^c_u \cup H^c_v$. The Claim 2 is true.

We now assume that c is a fixed real number and $c \in (c_0, +\infty)$.

Claim 3. The system $\{H^c_y \subset X : y \in Y\}$ has the finite intersection property.

We use induction. Suppose that for any n elements of $\{H^c_y \subset X : y \in Y\}$ their intersction is nonempty, next we prove that for any $n+1$ elements of $\{H^c_y \subset X : y \in Y\}$ their intersection is also nonempty where $n \geq 2$. If this is not the case, then there exists a subset $\{u, v, y_2, \dots, y_n\} \subset Y$ such that

$$H^c_u \cap H^c_v \cap H = \emptyset,$$

where $H = \bigcap_{i=2}^n H^c_{y_i}$. Letting $H^*_y = H^c_y \cap H$ for each $y \in Y$, by the assumption of induction and condition (iv) we know that H^*_u and H^*_v both are nonempty connected subsets of X and

$$H^*_u \cap H^*_v = \emptyset. \tag{2.2}$$

By virtue of Claim 2, we have

$$H_y^* \subset H_u^* \cup H_v^* \qquad \forall y \in C_{\{u,v\}}.$$

Letting

$$S_u = \{y \in C_{\{u,v\}} : H_y^* \subset H_u^*\},$$
$$S_v = \{y \in C_{\{u,v\}} : H_y^* \subset H_v^*\}$$

we see that S_u and S_v both are nonempty convex subsets of Y and $S_u \cap S_v = \emptyset$. Futhermore, we prove

$$S_u \cup S_v = C_{\{u,v\}}.$$

Obviously, $S_u \cup S_v \subset C_{\{u,v\}}$. On the contrary, if there exists a $y \in C_{\{u,v\}}$ such that $y \notin S_u \cup S_v$ then the connected set H_y^* could be represented as union of two open, nonempty and disjoint subsets:

$$H_y^* = (H_y^* \cap H_u^*) \cup (H_y^* \cap H_v^*),$$

which is impossible.

Consequently, the sets S_u and S_v are disjoint, nonempty convex subsets of Y and their union is $C_{\{u,v\}}$. From the (2.1), (2.2) and the condition (iii) we know that the equality

$$\begin{aligned} S_u &= \{y \in C_{\{u,v\}} : H_y^* \subset H_u^*\} \\ &= \{y \in C_{\{u,v\}} : H_y^* \cap H_v^* = \emptyset\} \\ &= \{y \in C_{\{u,v\}} : \forall x \in H_v^*,\ f(x,y) \geq c\} \\ &= \bigcap_{x \in H_v^*} \{y \in C_{\{u,v\}} : f(x,y) \geq c\} \end{aligned}$$

implies that S_u is a closed subset of Y. A similar equality holds for S_v. It is in contradiction with the connectivity of $C_{\{u,v\}}$. The Claim 3 is proved.

Let $\overline{H_y^c} = \{x \in X : f(x,y) \leq c\ \}$ for each $y \in Y$. From the Claim 3 the system $\{\overline{H_y^c} : y \in Y\}$ has also the finite intersection property. In view of the compactness of X and the condition (ii) we have $\bigcap_{y \in Y} \overline{H_y^c} \neq \emptyset$. This implies that there exists an $x_0 \in X$ such that $x_0 \in \overline{H_y^c}$ for all $y \in Y$, i.e., $\{x_0\} \times Y \subset P(c)$.

This completes the proof. □

3. Some applications

The results as follows are all sightly applications of Theorem 2.1.

I) A fixed point problem

Theorem 3.1. *Assume that the conditions of Theorem 2.1 are satisfied and that* $X = Y$. *Define a multivalued mapping* $F : Y \to 2^X$ *by* $F(y) = \overline{H_y^c}$ *for a fixed* $c \in (c_0, +\infty)$. *Then* F *has a fixed point.*

Proof. By virtue of Theorem 2.1, there exists a $y_0 \in Y$ such that $\{y_0\} \times Y \subset P(c)$. In particular, $f(y_0, y_0) \leq c$, i.e., $y_0 \in \overline{H_{y_0}^c} = F(y_0)$. This proves F has a fixed point. $\square$

II) Some minimax problems

Theorem 3.2. *If the conditions of Theorem 2.1 are satisfied then*

$$\sup_{y \in Y} \min_{x \in X} f(x, y) = \min_{x \in X} \sup_{y \in Y} f(x, y).$$

Proof. We always have $\sup_{y \in Y} \min_{x \in X} f(x, y) \leq \min_{x \in X} \sup_{y \in Y} f(x, y)$. To prove opposite inequality

$$\sup_{y \in Y} \min_{x \in X} f(x, y) \geq \min_{x \in X} \sup_{y \in Y} f(x, y), \tag{3.3}$$

we denote $c^* = \sup_{y \in Y} \min_{x \in X} f(x, y)$. If $c^* = +\infty$, then (3.3) is ture, therefore, we can assume that $c^* < +\infty$. It is obvious that the definition of c_0 implies $c^* \geq c_0$. For any fixed real number $c > c^*$, by virtue of Theorem 2.1 there exists a $x_0 \in X$ such that $f(x_0, y) \leq c$ for all $y \in Y$. It follows that $\min_{x \in X} \sup_{y \in Y} f(x, y) \leq c$ for any $c > c^*$, i.e.,

$$\min_{x \in X} \sup_{y \in Y} f(x, y) \leq c^*.$$

The proof is completed. $\square$

Remark. Theorem 3.2 is based on the topological connectedness without the convexities of the function and the linearity of the space.

Theorem 3.3. *Assume that the conditions of Theorem 2.3 are satisfied, if, in addition, Y is compact, then there exists a saddle point of f in $X \times Y$.*

Proof. From the Theorem 3.2 and the compactness of Y we have

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

Hence, there exists a point $(x_0, y_0) \in X \times Y$ such that

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} f(x_0, y) \geq f(x_0, y) \qquad \forall y \in Y,$$

$$\max_{y \in Y} \min_{x \in X} f(x, y) = \min_{x \in X} f(x, y_0) \leq f(x, y_0) \qquad \forall x \in X.$$

Thus, $f(x_0, y) \leq f(x, y_0)$ for any $y \in Y$ and $x \in X$. In particular,

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0) \qquad \forall y \in Y, x \in X,$$

i.e., (x_0, y_0) is a saddle point of f in $X \times Y$. □

Remark. Theorem 3.3 improves Theorem 3.10.4 in [6] in the following ways:

1) the topological spaces may not be linear;
2) the function f may not be convex.

Theorem 3.4. *Assume that the conditions of Theorem 2.3 are satisfied. If $X = Y$ then there exists a $y_0 \in Y$ such that Ky Fan minimax inequality*

$$\max_{y \in Y} f(y_0, y) \leq \max_{y \in Y} f(y, y)$$

does hold.

Proof. If denote $c^* = \max_{y \in Y} f(y, y)$, the proof is similar to the one as stated in Theorem 3.2, and we omit it here. □

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