Publ. Math. Debrecen
54 / 3-4 (1999), 403–415

On a class of iterations containing the Chebyshev and the Halley methods

By J. A. EZQUERRO (Logroño) and M. A. HERNÁNDEZ (Logroño)

Abstract. We investigate a parametrized set of cubically convergent iterative methods for solving nonlinear equations in a Banach space. The methods can be thought of as a weighted mean between the Chebyshev and the Halley methods, the weights being α and $1-\alpha$, where $\alpha \in (-15, 2)$. A Kantorovich-type convergence theorem and corresponding error bounds are provided. Finally, we decide that Halley's method is more suitable for solving a nonlinear equation than Chebyshev's method. Even more, we can consider other iterations more suitable than Halley's one.

1. Introduction

Let $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain Ω and X, Y Banach spaces. One of the main problems in numerical analysis is to solve the nonlinear equation

$$F(x) = 0. \tag{1}$$

The two most famous third-order iterations to solve equation (1) are the Chebyshev method [2], [4])and the Halley method ([1], [3], [6], [15]).

Two are the main goals of this paper. Firstly, getting a priori election criterion between Chebyshev's method and Halleys's method for solving (1). Secondly, constructing a new third-order uniparametric class of iterations such that we can use a priori more appropriate iterative methods than the two mentioned above. For that, we construct new parameter-based methods of order three which contain both methods for a specific

Mathematics Subject Classification: 47H17, 65J15.
Key words and phrases: iterative processes in Banach spaces, third-order methods, Chebyshev's method, Halley's method, majorizing sequences, error bounds.

choice of the parameter. Let us assume that $F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega_0$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X. According to the basic idea of continuation methods ([11], [12]), a homotopy $\alpha H_1(x) + (1-\alpha) H_0(x)$, where $\alpha \in [0,1]$, can be defined between two operators H_0 and H_1 ([7], [8]). Following this idea, the next uniparametric family of iterations is designed:

$$x_0 = x_{\alpha,0}, \quad x_{\alpha,n+1} = \alpha H_1(x_{\alpha,n}) + (1-\alpha) H_0(x_{\alpha,n}),\ n \geq 0, \tag{2}$$

where

$$H_0(x_n) = x_n - \left[I + \frac{1}{2} L_F(x_n)\right] F'(x_n)^{-1} F(x_n),$$

$$H_1(x_n) = x_n - \left[I + \frac{1}{2} L_F(x_n) \left(I - \frac{1}{2} L_F(x_n)\right)^{-1}\right] F'(x_n)^{-1} F(x_n)$$

and $\alpha \in [0,1]$. We have denoted by I the identity operator on X and by $L_F(x)$ the linear operator defined by

$$L_F(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x), \quad x \in X,$$

provided that $F'(x)^{-1}$ exists, see [9] to get more information about this operator.

Observe that iteration (2) reduces to the Chebyshev method for $\alpha = 0$ and the Halley method for $\alpha = 1$. We also see that we extend the domain of values for the parameter α to $(-15, 2)$, so that the second goal of the paper is reached.

We will show, under standard Newton–Kantorovich assumptions ([10], [14]), the family of iterative methods defined by (2) converges to a zero x^* of equation (1). We provide an existence-uniqueness theorem and give error bounds for $\alpha \in (-15, 2)$.

Finally, a numerical result is provided where our previous analysis do agree.

2. A Kantorovich theorem and corresponding error bounds

Following Argyros and Chen [2], we write iteration (2) as

$$y_{\alpha,n} = x_{\alpha,n} - F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}), \qquad n \geq 0, \tag{3}$$

$$x_{\alpha,n+1} = y_{\alpha,n} + \frac{1}{2}L_F(x_{\alpha,n})G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n}), \quad n \geq 0, \tag{4}$$

where $G_\alpha(x_{\alpha,n}) = I + \frac{\alpha}{2}L_F(x_{\alpha,n})J(x_{\alpha,n}),\ J(x_{\alpha,n}) = \left(I - \frac{1}{2}L_F(x_{\alpha,n})\right)^{-1}$ and $\alpha \in (-15, 2)$. It is also assumed that

(I) There exists a continous linear operator $F'(x_0)^{-1}$, $x_0 \in \Omega$. Moreover $\|F'(x_0)^{-1}\| \leq \beta$.

(II) $\|F''(x)\| \leq M$ for $x \in \Omega$.

(III) $\|F''(x) - F''(y)\| \leq N\|x - y\|$ for $x, y \in \Omega$.

(IV) $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$.

(V) The equation

$$p(t) \equiv \frac{k}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta} = 0 \tag{5}$$

where

(V_1) $k^2 \geq \frac{3(1-\alpha)}{3-\alpha}M^2 + \frac{N}{3\beta}$ if $\alpha \in (-15, -1)$,

(V_2) $k^2 \geq \frac{2}{3-\alpha}\left[(2-\alpha)M^2 + \frac{2N}{3\beta}\right]$ if $\alpha \in [-1, 0)$,

(V_3) $k^2 \geq M^2 + \frac{N}{3\beta}$ if $\alpha = 0$,

(V_4) $k^2 \geq \frac{4}{3(2-\alpha)}\left[(2+\alpha)M^2 + \frac{2N}{3\beta}\right]$ if $\alpha \in (0, 1)$,

(V_5) $k^2 \geq M^2 + \frac{2N}{3\beta}$ if $\alpha = 1$,

(V_6) $k^2 \geq \frac{2}{3(2-\alpha)}\left[3(1+\alpha)M^2 + (3+\alpha)\frac{N}{3\beta}\right]$ if $\alpha \in (1, 2)$,

has two positive roots t^* and t^{**} ($t^* \leq t^{**}$). Equivalently, $2k\beta\eta \leq 1$.

For each $\alpha \in (-15, 2)$, we define the scalar sequence $\{t_{\alpha,n}\}$ by

$$t_0 = t_{\alpha,0} = 0, \quad s_{\alpha,n} = t_{\alpha,n} - \frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})},\ n \geq 0, \tag{6}$$

$$(7)\quad \begin{aligned} t_{\alpha,n+1} &= \Phi_\alpha(t_{\alpha,n}) \\ &= s_{\alpha,n}+\frac{1}{2}L_p(t_{\alpha,n})\left(1+\frac{\alpha L_p(t_{\alpha,n})}{2-L_p(t_{\alpha,n})}\right)(s_{\alpha,n}-t_{\alpha,n}), \quad n\geq 0, \end{aligned}$$

where p is the polynomial defined in (5).

We will show that the sequences $\{t_{\alpha,n}\}$ and $\{x_{\alpha,n}\}$ are well-defined, converge respectively to a solution t^* of (5) and a solution x^* of (1), and $\{t_{\alpha,n}\}$ is a majorizing sequence of $\{x_{\alpha,n}\}$ (see [10], [14]).

Let us denote

$$\overline{B(x,r)}=\{x'\in X; \|x'-x\|\leq r\} \quad \text{and} \quad B(x,r)=\{x'\in X; \|x'-x\|<r\}.$$

Theorem 2.1. *Let us assume that assumptions* (I)–(V) *hold and* $\overline{B(y_{\alpha,0},t^*-\eta)}\subset\Omega$. *Then, the sequences defined by* (3) *and* (4) *are well-defined for all* $n\geq 0$ *and* $\alpha\in(-15,2)$, *converges to a solution* $x^*\in\overline{B(x_0,t^*)}$ *of equation* (1), *and* $x_{\alpha,n},y_{\alpha,n}\in\overline{B(x_0,t^*)}$, *for each* $n\geq 0$. *The solution* x^* *is unique in* $B(x_0,t^{**})\cap\Omega$. *Moreover the following error estimates are satisfied for all* $n\geq 0$:

$$\|x^*-x_{\alpha,n}\|\leq t^*-t_{\alpha,n} \quad \text{and} \quad \|x^*-y_{\alpha,n}\|\leq t^*-s_{\alpha,n}.$$

We first prepare the following lemmas. The first one contains sufficient conditions for the convergence of the sequences defined by (6) and (7).

Lemma 2.2. *Let* p *be the polynomial defined in* (5). *Then the process defined by* (6) *and* (7) *is increasing and converges cubically to* t^* *for all* $\alpha\in(-15,2)$.

Proof. As $p(t_0)>0$, it is clear that $t_0-t^*\leq 0$. By applying the Mean Value theorem, we obtain

$$t_{\alpha,1}-t^*=\Phi'_\alpha(\varepsilon_0)(t_0-t^*),$$

for some $\varepsilon_0\in(t_0,t^*)$. Moreover it follows that

$$\Phi'_\alpha(t)=\frac{3L_p(t)^2}{2(2-L_p(t))^2}\left[2(2-\alpha)+(1-\alpha)L_p(t)(L_p(t)-4)\right]\geq 0$$

in $[t_0,t^*]$, since $L_p(t)\leq\frac{1}{2}$ (see [1]). In consequence, $t_{\alpha,1}\leq t^*$.

On the other hand,

$$t_{\alpha,1} - t_0 = -\frac{p(t_0)}{p'(t_0)}\left[1 + \frac{1}{2}L_p(t_0)\left(1 + \frac{\alpha L_p(t_0)}{2 - L_p(t_0)}\right)\right] \geq 0.$$

Now we apply mathematical induction to obtain $t_{\alpha,n} \leq t^*$ and $t_{\alpha,n} \geq t_{\alpha,n-1}$ for all $n \geq 1$, since $(t_{\alpha,n-1}, t^*) \subset (t_0, t^*)$.

Therefore the sequences defined by (6) and (7) converges to $r \in [t_0, t^*]$. From

$$1 + \frac{1}{2}L_p(r)\left(1 + \frac{\alpha L_p(r)}{2 - L_p(r)}\right) > 0,$$

it follows that $r = t^*$. Besides

$$\Phi_\alpha(t^*) = \Phi'_\alpha(t^*) = \Phi''_\alpha(t^*) = 0,$$

and consequently, the convergence is cubical. □

Next we give the following representation for $F(x_{\alpha,n})$.

Lemma 2.3. *We assume that the iterations $x_{\alpha,n}$ generated by (3) and (4) belong to Ω and $F'(x_{\alpha,n})^{-1}$ exists for all $n \geq 0$. Then we have for $n \geq 0$:*

$$\begin{aligned}
F(x_{\alpha,n}) = &\int_0^1 F''(y_{\alpha,n-1} + t(x_{\alpha,n} - y_{\alpha,n-1}))(x_{\alpha,n} - y_{\alpha,n-1})^2(1-t)\,dt \\
&+ \int_0^1 F''(x_{\alpha,n-1} + t(y_{\alpha,n-1} - x_{\alpha,n-1})) \\
&\quad \times (x_{\alpha,n} - y_{\alpha,n-1})(y_{\alpha,n-1} - x_{\alpha,n-1})\,dt \\
&+ \int_0^1 F''(x_{\alpha,n-1} + t(y_{\alpha,n-1} - x_{\alpha,n-1})) \\
&\quad \times (I - G_\alpha(x_{\alpha,n-1}))(y_{\alpha,n-1} - x_{\alpha,n-1})^2(1-t)\,dt \\
&+ \int_0^1 [F''(x_{\alpha,n-1} + t(y_{\alpha,n-1} - x_{\alpha,n-1})) - F''(x_{\alpha,n-1})] \\
&\quad \times G_\alpha(x_{\alpha,n-1})(y_{\alpha,n-1} - x_{\alpha,n-1})^2(1-t)\,dt.
\end{aligned}$$

Proof. Observe that

$$
\begin{aligned}
F(x_{\alpha,n}) &= F(x_{\alpha,n}) - F(y_{\alpha,n-1}) - F'(y_{\alpha,n-1})(x_{\alpha,n} - y_{\alpha,n-1}) \\
&\qquad + F(y_{\alpha,n-1}) + F'(y_{\alpha,n-1})(x_{\alpha,n} - y_{\alpha,n-1}) \\
&= \int_{y_{\alpha,n-1}}^{x_{\alpha,n}} F''(x)(x_{\alpha,n} - x)\,dx \\
&\qquad + F(y_{\alpha,n-1}) + F'(y_{\alpha,n-1})(x_{\alpha,n} - y_{\alpha,n-1})
\end{aligned}
$$

and

$$
\begin{aligned}
F(y_{\alpha,n-1}) &= \int_{x_{\alpha,n-1}}^{y_{\alpha,n-1}} F''(x)(y_{\alpha,n-1} - x)\,dx + F(x_{\alpha,n-1}) \\
&\qquad + F'(x_{\alpha,n-1})(y_{\alpha,n-1} - x_{\alpha,n-1}) \\
&= \int_0^1 F''(x_{\alpha,n-1} + t(y_{\alpha,n-1} - x_{\alpha,n-1}))(y_{\alpha,n-1} - x_{\alpha,n-1})^2(1-t)dt.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&F'(y_{\alpha,n-1})(x_{\alpha,n} - y_{\alpha,n-1}) \\
&\quad = \int_{x_{\alpha,n-1}}^{y_{\alpha,n-1}} F''(x)(x_{\alpha,n} - y_{\alpha,n-1})\,dx + F'(x_{\alpha,n-1})(x_{\alpha,n} - y_{\alpha,n-1}) \\
&\quad = \int_{x_{\alpha,n-1}}^{y_{\alpha,n-1}} F''(x)(x_{\alpha,n} - y_{\alpha,n-1})\,dx \\
&\qquad - \frac{1}{2} F''(x_{\alpha,n-1}) G_\alpha(x_{\alpha,n-1})(y_{\alpha,n-1} - x_{\alpha,n-1})^2.
\end{aligned}
$$

Now the proof concludes immediately. □

Lemma 2.4. *The sequence $\{t_{\alpha,n}\}$ defined by (6) and (7) is a majorizing sequence of the sequence $\{x_{\alpha,n}\}$ defined by (3) and (4). More precisely:*

$$\|x_{\alpha,n+1} - x_{\alpha,n}\| \le t_{\alpha,n+1} - t_{\alpha,n}, \quad n \ge 0.$$

Proof. For each $n \ge 0$, we prove that

(i_n) $x_{\alpha,n} \in \overline{B(x_0, t_{\alpha,n})}$,

(ii_n) $\|F'(x_{\alpha,n})^{-1}\| \le -\dfrac{1}{p'(t_{\alpha,n})}$, (iii$_n$) $\|y_{\alpha,n} - x_{\alpha,n}\| \le s_{\alpha,n} - t_{\alpha,n}$,

(iv$_n$) $y_{\alpha,n} \in \overline{B(x_0, s_{\alpha,n})}$,

(v$_n$) $\|x_{\alpha,n+1} - y_{\alpha,n}\| \le t_{\alpha,n+1} - s_{\alpha,n}$.

These can be proved directly by induction on n. Taking into account (I)–(V), we deduce easily (i$_0$)–(v$_0$). Now we assume that (i$_k$)–(v$_k$) are true for $k = 1, \dots, n-1$. Item (i$_n$) follows immediately. To see (ii$_n$), we note that

$$I - F'(x_0)^{-1}F'(x_{\alpha,n}) = \int_0^1 F'(x_0)^{-1}F''(x_0 + t(x_{\alpha,n} - x_0))(x_{\alpha,n} - x_0)\,dt.$$

Then

$$\|I - F'(x_0)^{-1}F'(x_{\alpha,n})\| \le \beta k \|x_{\alpha,n} - x_0\| \le \beta k t^* < 1,$$

and by the Banach lemma [10], $F'(x_{\alpha,n})^{-1}$ exists and

$$\begin{aligned}\|F'(x_{\alpha,n})^{-1}\| &\le \frac{\|F'(x_0)^{-1}\|}{1 - \|I - F'(x_0)^{-1}F'(x_{\alpha,n})\|} \\ &\le \frac{\beta}{1 - \beta k\|x_{\alpha,n} - x_0\|} \le \frac{-1}{p'(t_{\alpha,n})}.\end{aligned}$$

So (ii$_n$) is also true.

Let $\alpha \in (0,1)$. By Lemma 2.3, the Altman lemma ([1], [15]) and taking into account that $\|L_F(x_{\alpha,n-1})\| \le L_p(t_{\alpha,n-1})$, $\|J(x_{\alpha,n-1})\| \le \frac{2}{2 - L_p(t_{\alpha,n-1})}$ and $\|G_\alpha(x_{\alpha,n-1})\| = 1 + \frac{\alpha L_p(t_{\alpha,n-1})}{2 - L_p(t_{\alpha,n-1})}$, we deduce that

$$\begin{aligned}\|F(x_{\alpha,n})\| \le{} & \frac{M}{2}\|x_{\alpha,n} - y_{\alpha,n-1}\|^2 + M\|x_{\alpha,n} - y_{\alpha,n-1}\|\|y_{\alpha,n-1} - x_{\alpha,n}\| \\ & + \frac{M}{2}\|I - G_\alpha(x_{\alpha,n-1})\|\,\|y_{\alpha,n-1} - x_{\alpha,n-1}\|^2 \\ & + \frac{N}{6}\|G_\alpha(x_{\alpha,n-1})\|\,\|y_{\alpha,n-1} - x_{\alpha,n-1}\|^3 \\ \le{} & \frac{M}{2}\|x_{\alpha,n} - y_{\alpha,n-1}\|^2 \\ & - \left[\frac{\alpha}{2}M^2\|J(x_{\alpha,n-1})\| + \left(M^2 + \frac{N}{3\beta}\right)\|G_\alpha(x_{\alpha,n-1})\|\right] \\ & \times \frac{\|y_{\alpha,n-1} - x_{\alpha,n-1}\|^3}{2p'(t_{\alpha,n-1})}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M}{2}(t_{\alpha,n} - s_{\alpha,n-1})^2 \\
&\quad - \left[\alpha M^2 + \left(M^2 + \frac{N}{3\beta}\right)(2 - (1-\alpha)L_p(t_{\alpha,n-1}))\right] \\
&\quad \times \frac{(s_{\alpha,n-1} - t_{\alpha,n-1})^3}{2p'(t_{\alpha,n-1})(2 - L_p(t_{\alpha,n-1}))} \\
&\leq \frac{M}{2}(t_{\alpha,n} - s_{\alpha,n-1})^2 \\
&\quad - \left[(2+\alpha)M^2 + \frac{2N}{3\beta}\right] \frac{(s_{\alpha,n-1} - t_{\alpha,n-1})^3}{3p'(t_{\alpha,n-1})} \\
&\leq \frac{k}{2}(t_{\alpha,n} - s_{\alpha,n-1})^2 \\
&\quad - \frac{k^2}{2}\left(1 - \alpha + \frac{\alpha}{2 - L_p(t_{\alpha,n-1})}\right) \frac{(s_{\alpha,n-1} - t_{\alpha,n-1})^3}{p'(t_{\alpha,n-1})}.
\end{aligned}
$$

Therefore

$$
\|F(x_{\alpha,n})\| \leq p(t_{\alpha,n}), \tag{8}
$$

and consequently

$$
\|y_{\alpha,n} - x_{\alpha,n}\| \leq \|F'(x_{\alpha,n})^{-1}\| \, \|F(x_{\alpha,n})\| \leq \frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})} = s_{\alpha,n} - t_{\alpha,n}.
$$

Thus (iii$_n$) is true. Now items (iv$_n$) and (v$_n$) follow immediately. That completes the induction for the case $\alpha \in (0,1)$.

The cases $\alpha = 0$, $\alpha = 1$, $\alpha \in (-15, 0)$ and $\alpha \in (1, 2)$ follow analogously to the previous one (see [2] and [5] for $\alpha = 0$ and $\alpha = 1$ repectively). □

Proof of Theorem 2.1. From Lemma 2.4 it follows that the sequence $\{t_{\alpha,n}\}$ majorizes the sequence $\{x_{\alpha,n}\}$. In consequence, the convergence of $\{t_{\alpha,n}\}$ implies the convergence of $\{x_{\alpha,n}\}$ to a limit x^*. Letting $n \to \infty$ in (8), we obtain $F(x^*) = 0$. Furthermore

$$
\begin{aligned}
\|x_{\alpha,n} - y_{\alpha,0}\| &\leq \|x_{\alpha,n} - y_{\alpha,n-1}\| + \|y_{\alpha,n-1} - x_{\alpha,n-1}\| + \cdots + \|x_{\alpha,1} - y_{\alpha,0}\| \\
&\leq (t_{\alpha,n} - s_{\alpha,n-1}) + (s_{\alpha,n-1} - t_{\alpha,n-1}) + \cdots + (t_{\alpha,1} - s_{\alpha,0}) \\
&= t_{\alpha,n} - \eta \leq t^* - \eta,
\end{aligned}
$$

and similarly

$$
\|y_{\alpha,n} - y_{\alpha,0}\| \leq s_{\alpha,n} - \eta \leq t^* - \eta.
$$

Next, for $m \geq 0$,

$$\|x_{\alpha,n+m} - x_{\alpha,n}\| \leq t_{\alpha,n+m} - t_{\alpha,n}, \quad \|x_{\alpha,n+m} - y_{\alpha,n}\| \leq t_{\alpha,n+m} - s_{\alpha,n}$$

and letting $m \to \infty$ we obtain

$$\|x^* - x_{\alpha,n}\| \leq t^* - t_{\alpha,n} \quad \text{and} \quad \|x^* - y_{\alpha,n}\| \leq t^* - s_{\alpha,n},\ n \geq 0.$$

Finally, to see the uniqueness, let us assume that there exists another solution ζ^* of the equation (1) in $B(x_0, t^{**}) \cap \Omega$. Following Argyros and Chen [2], we consider the equation

$$0 = F(x^*) - F(\zeta^*) = \int_0^1 F'(x^* + t(\zeta^* - x^*))(\zeta^* - x^*)\, dt.$$

Now it suffices to see that the operator $\int_0^1 F'(x^*+t(\zeta^*-x^*))\, dt$ is invertible in order to obtain $x^* = \zeta^*$. In fact,

$$\begin{aligned}
&\|F'(x_0)^{-1}\| \int_0^1 \|F'(x^* + t(\zeta^* - x^*)) - F'(x_0)\|\, dt \\
&\leq \beta M \int_0^1 \|x^* + t(\zeta^* - x^*) - x_0\|\, dt \\
&\leq \beta M \int_0^1 (t\|\zeta^* - x_0\| + (1-t)\|x^* - x_0\|)\ dt \\
&< \frac{\beta M}{2}(t^* + t^{**}) \leq 1
\end{aligned}$$

and the proof is complete. □

After that, we obtain error bounds for the sequence $\{t_{\alpha,n}\}$ defined by (6) and (7). We a priori compare the iterations of class (4) by these error bounds. Following Ostrowski [13] we derive the following result.

Theorem 2.5. *Let p be the polynomial defined in* (5) *and we assume that p has two positive roots t^* and t^{**} ($t^* \leq t^{**}$). Let $\{t_{\alpha,n}\}$ be the sequence given by* (6) *and* (7)*. Then*

(a) *If $t^* < t^{**}$, let $\omega_\alpha = \sqrt{2-\alpha}$, $\theta = \dfrac{t^*}{t^{**}}$ and $\vartheta_\alpha = \omega_\alpha \theta$. Besides if* (a$_1$) *$\alpha \in (-15, 1)$ and $k\beta\eta < \dfrac{2\omega_\alpha}{(1+\omega_\alpha)^2}$, then*

$$\frac{(t^{**}-t^*)\theta^{3^n}}{1-\theta^{3^n}} < t^* - t_{\alpha,n} < \frac{(t^{**}-t^*)\vartheta_\alpha^{3^n}}{\omega_\alpha - \vartheta_\alpha^{3^n}}, \quad n \geq 0,$$

(a$_2$) *$\alpha = 1$, then*

$$t^* - t_{\alpha,n} = \frac{(t^{**}-t^*)\theta^{3^n}}{1-\theta^{3^n}}, \quad n \geq 0,$$

(a$_3$) *$\alpha \in (1, 2)$, then*

$$\frac{(t^{**}-t^*)\vartheta_\alpha^{3^n}}{\omega_\alpha - \vartheta_\alpha^{3^n}} < t^* - t_{\alpha,n} < \frac{(t^{**}-t^*)\theta^{3^n}}{1-\theta^{3^n}}, \quad n \geq 0,$$

where $\theta_\alpha < 1$ and $\vartheta_\alpha < 1$.

(b) *If $t^* = t^{**}$, then*

$$t^* - t_{\alpha,n} = t^* \left(\frac{9-\alpha}{24}\right)^n, \quad n \geq 0.$$

Proof. Let us write $\rho_{\alpha,n} = t^* - t_{\alpha,n}$ and $\sigma_{\alpha,n} = t^{**} - t_{\alpha,n}$. So

$$p(t_{\alpha,n}) = \frac{k}{2}\rho_{\alpha,n}\sigma_{\alpha,n}, \quad p'(t_{\alpha,n}) = -\frac{k}{2}(\rho_{\alpha,n}+\sigma_{\alpha,n})$$

and

$$L_p(t_{\alpha,n}) = \frac{2\rho_{\alpha,n}\sigma_{\alpha,n})}{(\rho_{\alpha,n}+\sigma_{\alpha,n})^2}.$$

Now from (6) and (7), it follows that

$$\rho_{\alpha,n} = \rho_{\alpha,n-1}^3 \frac{(\rho_{\alpha,n-1}+\sigma_{\alpha,n-1})^3 + (1-\alpha)\sigma_{\alpha,n-1}^3}{(\rho_{\alpha,n-1}+\sigma_{\alpha,n-1})^3[(\rho_{\alpha,n-1}+\sigma_{\alpha,n-1})^2 - \rho_{\alpha,n-1}\sigma_{\alpha,n-1}]} \tag{9}$$

and

$$\sigma_{\alpha,n} = \rho_{\alpha,n-1}^3 \frac{(\rho_{\alpha,n-1}+\sigma_{\alpha,n-1})^3 + (1-\alpha)\rho_{\alpha,n-1}^3}{(\rho_{\alpha,n-1}+\sigma_{\alpha,n-1})^3\,[(\rho_{\alpha,n-1}+\sigma_{\alpha,n-1})^2 - \rho_{\alpha,n-1}\sigma_{\alpha,n-1}]}.$$

For $t^* < t^{**}$, we denote the ratio of $\rho_{\alpha,n}$ and $\sigma_{\alpha,n}$ by $\gamma_{\alpha,n}$. So

$$\gamma_{\alpha,n} = \gamma_{\alpha,n-1}^3 \frac{(1+\gamma_{\alpha,n-1})^3 + (1-\alpha)}{(1+\gamma_{\alpha,n-1})^3 + (1-\alpha)\gamma_{\alpha,n-1}^3} = \gamma_{\alpha,n-1}^3 \phi_\alpha(\gamma_{\alpha,n-1}).$$

Taking into account that the function ϕ_α is nonincreasing for all $\alpha \in (-15, 1)$, we obtain

$$\gamma_{\alpha,n} < (2-\alpha)\gamma_{\alpha,n-1}^3 < \cdots < (2-\alpha)^{\frac{3^n-1}{2}} \gamma_{\alpha,0}^{3^n}$$

and

$$\gamma_{\alpha,n} > \gamma_{\alpha,n-1}^3 > \cdots > \gamma_{\alpha,0}^{3^n}.$$

Hence (a$_1$) is true, since $\vartheta_\alpha < 1$ as a consequence of $k\beta\eta < \dfrac{2\omega_\alpha}{(1+\omega_\alpha)^2}$. The cases (a$_2$) and (a$_3$) follow similarly to the previous one.

If $t^* = t^{**}$, then $\rho_{\alpha,n} = \sigma_{\alpha,n}$ and, by (9), we obtain

$$\rho_{\alpha,n} = \rho_{\alpha,n-1}\left(\frac{9-\alpha}{24}\right).$$

Now (b) follows by applying recurrence. □

Remark. From the last result we can get our two initiated goals. Firstly, although the operational cost of the Chebyshev method is lesser than the Halley method, we follow from Theorem 2.5 that the latter one is a priori the more appropriate, while the difference of the operational cost between both methods is valued from the expression of F. Secondly, some iterations are obtained, (4) for $\alpha \in (1,2)$, with the same operational cost as Halley's method but a priori more appropriate.

Next we illustrate these commentaries with a numerical example.

3. Example

Consider the system of equations $F(x,y) = 0$, where

$$F(x,y) = (x^2 - y - 2,\ y^3 - x^2 + y + 1).$$

The numerical results (see tables 1, 2 and 3) do agree quite well with our a priori previous analysis which showed that iteration (4) for $\alpha \in$

(1, 2), for instance $\alpha = 3/2$, is the most appropriate iteration for solving $F(x, y) = 0$.

n	$x^* - x_n$	$y^* - y_n$
0	4.267949192431122	2.000000000000000
1	0.987389032648569	0.727937814357567
2	0.098670840456724	0.138820258896789
3	0.001198513291186	0.003038870277247
4	$1.548883374005515 \times 10^{-8}$	$4.634827149840334 \times 10^{-8}$
5	$5.406728245765557 \times 10^{-23}$	$1.659393238500346 \times 10^{-23}$

Table 1: Error estimates by Chebyshev's method.

n	$x^* - x_n$	$y^* - y_n$
0	4.267949192431122	2.000000000000000
1	0.806253724829364	0.581818181818181
2	0.046398081903572	0.057032561548124
3	0.000061852501410	0.000113501450570
4	$4.338278564215300 \times 10^{-13}$	$9.746283420810687 \times 10^{-13}$
5	$2.536565808060148 \times 10^{-37}$	$6.171999046992419 \times 10^{-37}$

Table 2: Error estimates by Halley's method.

n	$x^* - x_n$	$y^* - y_n$
0	4.267949192431122	2.000000000000000
1	0.715686070919761	0.508758365548489
2	0.027768776212209	0.028954040361223
3	$5.834964892073052 \times 10^{-6}$	$4.665800522137905 \times 10^{-6}$
4	$3.433527622447421 \times 10^{-17}$	$1.692931591379338 \times 10^{-17}$
5	$3.560125611263405 \times 10^{-51}$	$8.086618920577318 \times 10^{-52}$

Table 3: Error estimates by iteration (4) with $\alpha = 3/2$.

References

[1] M. Altman, Concerning the Method of Tangent Hyperbolas for Operator Equations, *Bull. Acad. Pol. Sci., Serie Sci. Math., Ast. et Phys.* **9** (1961), 633–637.

[2] I. K. Argyros and D. Chen, Results on the Chebyshev Method in Banach Spaces, *Proyecciones* **12, 2** (1993), 119–128.

[3] V. Candela and A. Marquina, Recurrence Relations for Rational Cubic Methods I: The Halley Method, *Computing* **44** (1990), 169–184.

[4] V. Candela and A. Marquina, Recurrence Relations for Rational Cubic Methods II: The Chebyshev Method, *Computing* **45** (1990), 355–367.

[5] D. Chen, On the Convergence of the Halley's Method for Nonlinear Equation of One Variable, *Tamkang Journal of Mathematics* **24, 4** (1993), 461–467.

[6] D. Chen, I. K. Argyros and Q. S. Qian, A Note on the Halley's method in Banach Spaces, *Appl. Math. Comput.* **58** (1993), 215–224.

[7] J. A. Ezquerro, J. M. Gutiérrez and M. A. Hernández, A Construction Procedure of Iterative Methods with Cubical, *Appl. Math. Comput.* **85** (1997), 181–199.

[8] J. A. Ezquerro, J. M. Gutiérrez and M. A. Hernández, A Construction Procedure of Iterative Methods with Cubical Convergence II: Another Convergence Approach (*to appear in* Appl. Math. Comput).

[9] J. M. Gutiérrez, M. A. Hernández and M. A. Salanova, Accesibility of Solutions by Newton's Method, *Intern. J. Computer Math.* **57** (1995), 239–247.

[10] L. V. Kantorovich and G. P. Akilov, Functional Analysis, *Pergamon Press*, *Oxford*, 1982.

[11] D. Kincaid and W. Cheney, Numerical Analysis: Mathematics of Scientific Computing, *Brooks/Cole Publishing Company*, *E.U.A.*, 1991.

[12] J. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, *Academic Press*, *New York*, 1970.

[13] A. M. Ostrowski, Solution of Equations in Euclidean and Banach Spaces, *Academic Press*, *New York*, 1943.

[14] W. C. Rheinboldt, A Unified Convergence Theory for a Class of Iterative Processes, *SIAM J. Numer. Anal.* **5** (1968), 42–63.

[15] T. Yamamoto, On the Method of Tangent Hyperbolas in Banach Spaces, *J. Comput. Appl. Math.* **21** (1988), 75–86.

J. A. EZQUERRO
DEPARTMENT OF MATHEMATICS AND COMPUTATION
UNIVERSITY OF LA RIOJA
C/ LUIS DE ULLOA S/N
26004 LOGROÑO
SPAIN

E-mail: jezquer@dmc.unirioja.es

M. A. HERNÁNDEZ
DEPARTMENT OF MATHEMATICS AND COMPUTATION
UNIVERSITY OF LA RIOJA
C/ LUIS DE ULLOA S/N
26004 LOGROÑO
SPAIN

E-mail: mahernan@dmc.unirioja.es

(Received October 31, 1997; accepted April 16, 1998)