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Rigid body with a free spinning rotor and nonlinear stability

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Dedicated to Professor Lajos Tamássy on his 75th birthday

Abstract. The dynamics and the stability of the rigid body with a free spinning rotor are discussed and some of their properties are pointed out.

1. Introduction

The study of the rigid body with a free spinning rotor has a long history dues to its deep connections with engineering and robotic problems. However its first successful application was carried out in the RCA Satcom I satellite in December 1975.

On the other hand, the first step in the program of understanding the geometry of its dynamics was made by Krishnaprasad [4]. He has proved that the system has a Lie–Poisson realization on the dual of the Lie algebra $so(3) \times \mathbb{R}$, and then using this realization he has proved the stability of some equilibrium states via Lasalle's principle (see also [1]).

The goal of our paper is to make a similar study, but using another Poisson structure which will substantially simplify the computations. This new realization of the rigid body with a free spinning rotor as an Hamilton-Poisson mechanical system on $\mathbb{R}^3$ will give us the possibility to discuss the

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stability of the equilibrium states $(0, M, 0)$ via the energy-Casimir method. Also we shall make some remarks concerning the numerical integration of the equations of motion via the Kahan's integrator and we shall point out some of its mechanical and geometrical properties.

2. Rigid body with a free spinning rotor

Consider a rigid body $\mathcal{B}$ carrying a free spinning rotor. Denote the system center of mass by O in the body frame and at this point place a set of orthonormal body axes $\{x_b^1, x_b^2, x_b^3\}$. Assume that our rotor belongs to the axis Ox_b^2, see Fig. 1.

The configuration space of the system is $so(3) \times S^1$ and on the dual of its Lie algebra, i.e. on $so(3)^* \times \mathbb{R}^* \approx \mathbb{R}^4$, the equations of motion can be written in the following form:

$$
\tag{2.1}
\begin{cases}
\dot{m}_1 = \left(\dfrac{1}{\lambda_3} - \dfrac{1}{\lambda_2}\right) m_2 m_3 + \dfrac{l_2}{\lambda_3} m_3 \\
\dot{m}_2 = \left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_3}\right) m_1 m_3 \\
\dot{m}_3 = \left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right) m_1 m_2 - \dfrac{l_2}{\lambda_1} m_1 \\
\dot{l}_2 = 0
\end{cases}
$$

where m_1, m_2, m_3 represent the components of the angular momentum of $\mathcal{B}$, l_2 is the angular momentum of the spinning rotor and λ_i's are the principal axes moments of inertia. We suppose as usually that

$$
\tag{2.2}
\lambda_1 > \lambda_2 > \lambda_3.
$$

Some geometrical and dynamical properties of this equations can be found in [2], [4], [5] and [8].

3. Hamilton-Poisson realization

It is easy to see that

$$
l_2 = B = \text{constant},
$$

Figure 1: The rigid body with a free spinning rotor

and so the equations (2.1) can be put in the equivalent form:

$$\begin{cases} \dot{m}_1 = \left(\dfrac{1}{\lambda_3} - \dfrac{1}{\lambda_2}\right) m_2 m_3 + \dfrac{B}{\lambda_3} m_3 \\ \dot{m}_2 = \left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_3}\right) m_1 m_3 \\ \dot{m}_3 = \left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right) m_1 m_2 - \dfrac{B}{\lambda_1} m_1. \end{cases} \tag{3.1}$$

Theorem 3.1. *The equations* (3.1) *can be realized as an Hamilton-Poisson mechanical system with the phase space* $\mathbb{R}^3$, *the Poisson structure given by the matrix*

$$\Pi = \begin{bmatrix} 0 & -m_3 & m_2 + B \\ m_3 & 0 & -m_1 \\ -m_2 - B & m_1 & 0 \end{bmatrix} \tag{3.2}$$

and the Hamiltonian H *given by:*

$$H(m_1, m_2, m_3) = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{m_3^2}{\lambda_3}\right). \tag{3.3}$$

Moreover, a Casimir of our configuration $(\mathbb{R}^3, \Pi)$ *is given by:*

$$C(m_1, m_2, m_3) = \frac{1}{2}\left[m_1^2 + (m_2 + B)^2 + m_3^2\right]. \tag{3.4}$$

Proof. Let $\{\cdot, \cdot\}$ be the canonical bracket on $C^\infty(\mathbb{R}^3, \mathbb{R})$ generated by the matrix Π, i.e.

$$\{f, g\} = (\nabla f)^t \Pi (\nabla g),$$

for each $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$. Now, it can be put in the equivalent form:

$$\{f, g\} = \nabla C \cdot (\nabla f \times \nabla g),$$

and then using a general result due to Marsden and Ratiu [6], it follows that it is a Poisson bracket on $\mathbb{R}^3$. C is obviously a Casimir of this configuration and moreover we can easily check that

$$\dot{m}_i = \{m_i, H\}, \quad i = 1, 2, 3$$

and this leads us to the desired result. □

Remark 3.1. Since H and C are constants of motion it follows that the dynamics takes place at the intersection of the ellipsoid:

$$\frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{m_3^2}{\lambda_3}\right) = \text{constant}$$

with the sphere

$$m_1^2 + (m_2 + B)^2 + m_3^2 = \text{constant}.$$

It is a natural problem to try to decide if the above Hamilton–Poisson realization of (3.1) is an unique one. The answer is given in the following:

Theorem 3.2. *The equations* (3.1) *may be realized as an Hamilton–Poisson mechanical system in an infinite number of different ways.*

Proof. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha\delta - \beta\gamma = 1$, and $C_{\alpha\beta}, H_{\gamma\delta} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given respectively by:

$$C_{\alpha\beta} = \alpha H + \beta C; \;\; H_{\gamma\delta} = \gamma H + \delta C.$$

Then a long but straightforward computation shows us that our system (3.1) is an Hamilton–Poisson one with the phase space $\mathbb{R}^3$, the Poisson bracket $\{\cdot,\cdot\}_{\alpha\beta}$ given by

$$\{f,g\}_{\alpha\beta} = \nabla C_{\alpha\beta} \cdot (\nabla f \times \nabla g),$$

for each $f,g \in C^\infty(\mathbb{R}^3,\mathbb{R})$ and the Hamiltonian $H_{\gamma\delta}$. Moreover, a Casimir of our configuration $(\mathbb{R}^3, \{\cdot,\cdot\}_{\alpha\beta})$ is given by $C_{\alpha\beta}$. □

4. Stability

We shall study now the stability of the equilibrium state $(0, M, 0)$ of our system (3.1). The following arise naturally:

Case 1: $M, B > 0$. Consider first the system linearized about $(0, M, 0)$. Its eigenvalues are given by the solutions of:

$$t\left[t^2 + M^2 \frac{\lambda_1 - \frac{M+B}{M}\lambda_2}{\lambda_1\lambda_2} \cdot \frac{\lambda_3 - \frac{M+B}{M}\lambda_2}{\lambda_2\lambda_3}\right] = 0. \tag{4.1}$$

Since $\lambda_1 > \lambda_2 > \lambda_3$ (see (2.2)) and

$$\lambda_3 < \frac{M+B}{M}\lambda_2$$

the equation (4.1) has two imaginary eigenvalues and one zero eigenvalue if and only if

$$\lambda_1 < \frac{M+B}{M}\lambda_2. \tag{4.2}$$

Under this restriction is the equilibrium state $(0, M, 0)$ nonlinear (or Lyapunov) stable? We shall prove that it is via the energy-Casimir method [3]. Recall that the energy-Casimir method requires finding a constant of motion for the system, H, usually the energy, and a family if constants of motion $\mathcal{C}$, such that for some $C \in \mathcal{C}$, $H + C$ has a critical point at the equilibrium of interest. Often the C's are taken to be Casimirs. Definiteness of $\delta^2(H + C)$, the second variation of $H + C$, at the critical point is sufficient to prove stability.

Consider the energy-Casimir function H_φ given by:

$$H_\varphi(m_1, m_2, m_3) = H(m_1, m_2, m_3) + \varphi\left(\frac{1}{2}(m_1^2 + (m_2 + B)^2 + m_3^2)\right),$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$. Now, the first variation of H_φ is given by:

$$\delta H_\varphi = \frac{m_1}{\lambda_1}\delta m_1 + \frac{m_2}{\lambda_2}\delta m_2 + \frac{m_3}{\lambda_3}\delta m_3 + \dot{\varphi}[m_1\delta m_1 + (m_2 + B)\delta m_2 + m_3\delta m_3].$$

This equals zero at the equilibrium of interest $(0, M, 0)$ if

$$\dot{\varphi}\left(\frac{1}{2}(M + B)^2\right) = -\frac{M}{(M + B)\lambda_2}.$$

Then

$$\delta^2 H_\varphi(0, M, 0) = \left[\frac{1}{\lambda_1} - \frac{M}{(M+B)\lambda_2}\right](\delta m_1)^2 + \left[\frac{1}{\lambda_2} - \frac{M}{(M+B)\lambda_2}\right](\delta m_2)^2 + \left[\frac{1}{\lambda_3} - \frac{M}{(M + B)\lambda_2}\right](\delta m_3)^2 + \ddot{\varphi}\left(\frac{1}{2}(M + B)^2\right)(M + B)^2(\delta m_2)^2$$

or equivalent

$$\delta^2 H_\varphi(0, M, 0) = -\frac{M}{M + B}\frac{1}{\lambda_1\lambda_2}\left(\lambda_1 - \frac{M + B}{M}\lambda_2\right)(\delta m_1)^2 + \frac{B}{M + B}\frac{1}{\lambda_2}(\delta m_2)^2 - \frac{M}{M + B}\frac{1}{\lambda_2\lambda_3}\left(\lambda_3 - \frac{M + B}{M}\lambda_2\right)(\delta m_3)^2 + \ddot{\varphi}\left(\frac{1}{2}(M + B)^2\right)(M + B)^2(\delta m_2)^2.$$

Now, under the restriction (4.2) and choosing φ such that

$$\ddot{\varphi}\left(\frac{1}{2}(M + B)^2\right) \geq 0,$$

it is easy to see that the second variation is positive definite and we have nonlinear stability.

Case 2: $M, B < 0$. We obtain the same result as in the Case 1. Therefore we have proved:

Theorem 4.1. *Let $M \cdot B > 0$. Then the equilibrium state $(0.M, 0)$ is nonlinear stable if and only if*

$$\lambda_1 < \frac{M+B}{M}\lambda_2.$$

Remark 4.1. As a consequence it follows that $\left(0, B\frac{\lambda_2}{\lambda_1-\lambda_2}, 0\right)$ is a bifurcation point of the dynamics (3.1).

In the cases:

3. $M > 0,\ B < 0,\ M > |B|$;
4. $M > 0,\ B < 0,\ M < |B|$;
5. $M < 0,\ B > 0,\ M < |B|$;
6. $M < 0,\ B > 0,\ M > |B|$

similar arguments as in the Case 1 lead us to:

Theorem 4.2. *Let $M \cdot B < 0$. Then the equilibrium state $(0, M, 0)$ is nonlinear stable if and only if*

$$\lambda_3 > \frac{M+B}{M}\lambda_2.$$

Remark 4.2. As a consequence it follows that $(0, B\lambda_2/(\lambda_3 - \lambda_2), 0)$ is a bifurcation point of the dynamics (3.1).

Case 7: $M = 0, B \in \mathbb{R}$. Then using as the Lyapunov function the Hamiltonian (3.3) we can deduce immediately that the equilibrium state $(0, 0, 0)$ is nonlinear stable.

Case 8: $B = 0$, $M \in \mathbb{R}$, $M \neq 0$. Then our system represents in fact the dynamics of a free rigid body and so the equilibrium state $(0, M, 0)$ is unstable, see for details [6] and [7].

5. Numerically integration

In this last section we want to discuss the numerical integration of the equations (3.1) using Kahan's integrator (see [9] for some physical motivations and historical remarks). In our case Kahan's integrator can

Figure 2: 4th order Runge-Kutta integrator for the rigid body with a free spinning rotor with: $\lambda_1 = 1$; $\lambda_2 = \frac{1}{2}$; $\lambda_3 = \frac{1}{3}$

Figure 3: Kahan's integrator for the rigid body with a free spinning rotor with: $\lambda_1 = 1$; $\lambda_2 = \frac{1}{2}$; $\lambda_3 = \frac{1}{3}$

be written in the following form:

$$\begin{cases} m_1^{n+1} - m_1^n = \dfrac{h}{2}\left(\dfrac{1}{\lambda_3} - \dfrac{1}{\lambda_2}\right)(m_2^{n+1}m_3^n + m_3^{n+1}m_2^n) + \dfrac{Bh}{2\lambda_3}(m_3^{n+1} + m_3^n) \\ m_2^{n+1} - m_2^n = \dfrac{h}{2}\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_3}\right)(m_1^{n+1}m_3^n + m_3^{n+1}m_1^n) \\ m_3^{n+1} - m_3^n = \dfrac{h}{2}\left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right)(m_1^{n+1}m_2^n + m_2^{n+1}m_1^n) - \dfrac{Bh}{2\lambda_1}(m_1^{n+1} + m_1^n) \end{cases}$$

Using now MAPLEV we can prove the following

Theorem 5.1. *The following statements are equivalent:*

(i) *Kahan's integrator is a Poisson integrator;*

(ii) *Kahan's integrator is energy preserving;*

(iii) *Kahan's integrator is Casimir preserving;*

(iv) $\lambda_1 = \lambda_3$.

In the particular case $B = 0$ we refined some results announced previously in [9].

A comparison between Kahan's integrator and Runge-Kutta integrator with 4-steps is given in Figures 2 and 3. It is clear that the results are almost the same. However Kahan's integrator has the advantage that it can be easier implemented.

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