Publ. Math. Debrecen
41 / 1–2 (1992), 213–223

On the solutions of the functional equation: $f(f(y)^k x + f(x)^\ell y) \;=\; \lambda f(x) f(y)$

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Let E be a real Hausdorff topological vector space. We consider on $\mathbb{R}\times E$ the following binary law:

$$(\alpha,\beta)*(\alpha',\beta') = (\lambda\alpha\alpha', {\alpha'}^k\beta + \alpha^\ell\beta') \quad (\text{for} \quad (\alpha,\beta),(\alpha',\beta')\in\mathbb{R}\times E)$$

where λ is a fixed real number, k and ℓ are fixed integers. When we look for the subgroupoids of $(\mathbb{R}\times E,*)$ which depend faithfully on a set of parameters (cf. [1], [2]), we have to solve the following functional equation:

$$(1) \qquad f(f(y)^k x + f(x)^\ell y) = \lambda f(x) f(y) \quad (x,y\in E)$$

where f is a function mapping E into $\mathbb{R}$.

Let DB_1 be the set of all functions from $\mathbb{R}$ into $\mathbb{R}$ which are in class I of Baire and have the Darboux property.

In [1], all the solutions of (1) in the class of functions DB_1 are given in the case where λ is a non-negative real number. All the continuous solutions $f: E\to\mathbb{R}$ of (1) are also obtained.

In [3], J. Brzdęk obtained all the continuous solutions of (1) when λ is an arbitrary non zero real number and k, ℓ are distinct positive integers.

Here, we obtain first all the solutions of (1) in the class of functions DB_1 and then all the continuous solutions $f: E\to\mathbb{R}$ of (1) when λ is an arbitrary real number and k, ℓ are arbitrary integers. This result is an answer to a question asked by J. Brzdęk at the 27$^{\text{th}}$ International Symposium on Functional Equations in Bielsko–Biala in 1989 (cf. [4]).

1. Let us first recall the results obtained in [1] for some particular cases.

If $\lambda=0$ and if k and ℓ are arbitrary integers, the unique solution of (1) in the class of functions $f: E\to\mathbb{R}$ which have the property that, for every x in E, the function defined by: $f_x(t) = f(tx) \;\; (t\in\mathbb{R})$ belongs to DB_1, is $f\equiv 0$ (cf. [1]).

In the case where $k = \ell = 0$ and λ is an arbitrary non-zero real number, all the solutions of (1) are given by:

$$(i) \quad f = 0$$

$$(ii) \quad f(x) = \frac{1}{\lambda} e^{g(x)} \quad (x \in E)$$

where $g : E \to \mathbb{R}$ is an arbitrary additive function.

2. Before considering the case where $k = 0$, we give some property of the function ζ defined by:

$$\zeta(x, y) = f(y)^k x + f(x)^\ell y \quad (x, y \in E)$$

in the case where $E = \mathbb{R}$.

Lemma 1. *Let us suppose that λ is a non-zero real number. If f is a non identically zero solution of (1) in DB_1, the function ψ defined by: $\psi(x) = \zeta(x, x_0)$ $(x \in \mathbb{R})$ is continuous when x_0 is any non zero real number satisfying $f(x_0) \neq 0$.*

Proof of Lemma 1. Since f is in DB_1, the graph of the function $x_0 f(.)^\ell$ is connected (cf. [6]). Therefore, because the function:

$$\mathbb{R}^2 \to \mathbb{R}^2$$

$$(t, s) \to (t, f(x_0)^k t + s)$$

is continuous, the graph of the function ψ is also connected and, thus, ψ has the Darboux property. Moreover, ψ is one-to-one (cf. [3]). Therefore, ψ is continuous (cf. [6]).

So, let us consider now the functional equation (1) when $k = 0$, ℓ is a positive integer and λ is a non zero real number.

If f is a non identically zero solution of (1) in DB_1, there exists x_0 in $\mathbb{R} - \{0\}$ such that $f(x_0) \neq 0$. By Lemma 1, the function $\psi(x) = \zeta(x, x_0)$ $(x \in \mathbb{R})$ is continuous. We deduce that the function $g(x) = f(x)^\ell$ $(x \in \mathbb{R})$ is continuous. Moreover, g is a solution of:

$$g(x + g(x)y) = \lambda^\ell g(x)g(y) \quad (x, y \in \mathbb{R}) \tag{2}$$

which is similar to the Golab–Schinzel functional equation. By taking $x = y = 0$ in (2), we obtain either $g(0) = 0$ or $g(0) = \lambda^{-\ell}$. When $g(0) = 0$, we get $g \equiv 0$ as we can see by taking $y = 0$ in (2).

So, we consider now the case where $g(0) = \lambda^{-\ell}$. By taking $x = 0$ in (2), we get:

$$g(y) = g(\lambda^{-\ell} y) \quad (y \in \mathbb{R}) \tag{3}$$

and therefore:

$$g(y) = g(\lambda^{-n\ell} y) \quad (y \in \mathbb{R}) \tag{4}$$

for every positive integer n.

When $|\lambda|$ is different from 1, (4) implies: $g \equiv g(0) = \lambda^{-\ell}$ and therefore $f \equiv \frac{1}{\lambda}$.

When λ^ℓ is equal to 1, (2) is just the functional equation of Golab–Schinzel for which we know all the continuous solutions (cf. [2]).

When λ^ℓ is equal to -1 (i.e. $\lambda = -1$ and ℓ odd), (3) implies by changing y into $-y$ in (2):

$$g(x - g(x)y) = -g(x)g(y) \quad (x, y \in \mathbb{R})$$

This means that $-g$ is a continuous solution of the functional equation of Golab–Schinzel.

So, we obtain the following result:

Proposition 2. *When λ is an arbitrary real number and ℓ is a positive integer, all the solutions in the class of functions DB_1 of the following functional equation:*

$$f(x + f(x)^\ell y) = \lambda f(x) f(y) \quad (x, y \in \mathbb{R}) \tag{5}$$

are given by:

(i) $f \equiv 0$

and

(ii) *if* $\lambda \neq 0$ *and* $|\lambda| \neq 1$ $f \equiv \frac{1}{\lambda}$

(iii) *if* $\lambda = 1$ *and if* ℓ *is odd*
$f(x) = (1 + ax)^{1/\ell}$ $(x \in \mathbb{R})$
and $f(x) = \mathrm{Sup}(1 + ax, 0))^{1/\ell}$ $(x \in \mathbb{R})$

(iv) *if* $\lambda = 1$ *and if* ℓ *is even*
$f(x) = (\mathrm{Sup}(1 + ax, 0))^{1/\ell}$ $(x \in \mathbb{R})$

(v) *if* $\lambda = -1$ *and if* ℓ *is odd*
$f(x) = -(1 + ax)^{1/\ell}$ $(x \in \mathbb{R})$ *and*
$f(x) = -(\mathrm{Sup}(1 + ax, 0))^{1/\ell}$ $(x \in \mathbb{R})$

(vi) *if* $\lambda = -1$ *and if* ℓ *is even* $f(x) = -(\mathrm{Sup}(1 + ax, 0))^{1/\ell}$ $(x \in \mathbb{R})$

where a is an arbitrary non zero real number.

With the same method, we obtain all the continuous solutions $f : E \to \mathbb{R}$ of (1) when E is a real Hausdorff topological vector space, namely:

Proposition 3. *When λ is an arbitrary real number and ℓ is a positive integer, all the continuous solutions $f : E \to \mathbb{R}$ of the following functional equation:*

$$f(x + f(x)^\ell y) = \lambda f(x) f(y) \quad (x, y \in E) \tag{5}$$

are given by:

(i) $f \equiv 0$

and

(ii) *if* $\lambda \neq 0$ *and* $|\lambda| \neq 1$ $\quad f \equiv \frac{1}{\lambda}$

(iii) *if* $\lambda = 1$ *and if* ℓ *is odd*
$f(x) = (1 + \langle x, x^* \rangle)^{1/\ell}$ $(x \in E)$ *and*
$f(x) = (\mathrm{Sup}(1 + \langle x, x^* \rangle, 0))^{1/\ell}$ $(x \in E)$

(iv) *if* $\lambda = 1$ *and if* ℓ *is even*
$f(x) = (\mathrm{Sup}(1 + \langle x, x^* \rangle, 0))^{1/\ell}$ $(x \in E)$

(v) *if* $\lambda = -1$ *and if* ℓ *is odd*
$f(x) = -(1 + \langle x, x^* \rangle)^{1/\ell}$ $(x \in E)$ *and*
$f(x) = -(\mathrm{Sup}(1 + \langle x, x^* \rangle), 0))^{1/\ell}$ $(x \in E)$

(vi) *if* $\lambda = -1$ *and if* ℓ *is even*
$f(x) = -(\mathrm{Sup}(1 + \langle x, x^* \rangle, 0))^{1/\ell}$ $(x \in E)$

where x^* *is an arbitrary non zero element of the topological dual of* E.

Let us finally mention that J. Brzdęk studied in detail the functional equation (5) in the case $\lambda = 1$ in his Doctor Thesis (cf. [5]).

3. Let us consider now the general case where λ is a non zero real number and k, ℓ are positive integers.

In [3], J. Brzdęk proved that, if $k \neq \ell$, all the continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of (1) are: $f \equiv 0$ and $f \equiv \frac{1}{\lambda}$. It is not difficult to see that this result is also valid for the class of functions DB_1. In fact, the proofs in [3] use only the Darboux property of f, the continuity of the functions: $x \to \zeta(x, y)$ and $y \to \zeta(x, y)$ (given by Lemma 1 when f belongs to DB_1) and the continuity of the function $g(x) = f(x)^\ell$ $(x \in \mathbb{R})$. Now, if f is a non identically zero solution of (1) in DB_1, the continuity of g is deduced from the fact that, if x_0 is a non zero real number satisfying $f(x_0) \neq 0$, the function $\zeta(x, x_0) = f(x_0)^k x + f(x)^\ell x_0$ is continuous by Lemma 1.

So, we have to study now the case where $k = \ell$ is a positive integer and λ is a non zero real number.

If $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1), the function: $g(x) = f(x)^\ell$ $(x \in \mathbb{R})$ satisfies the following functional equation:

$$g(g(y)x + g(x)y) = \lambda^\ell g(x)g(y) \quad (x, y \in \mathbb{R}) \tag{6}$$

Moreover, if f belongs to DB_1, g is continuous as we have noticed it previously.

Now, all the continuous solutions $g : \mathbb{R} \to \mathbb{R}$ of (6) are known when λ^ℓ is a positive real number (cf. [2]). Let us recall the result:

Lemma 4. *All the continuous solutions* $g : \mathbb{R} \to \mathbb{R}$ *of the functional equation:*

$$g(g(y)x + g(x)y) = \mu\, g(x)g(y) \quad \text{where } \mu > 0 \tag{7}$$

are given by:

$$\text{(i)}\ \ g \equiv 0 \qquad\qquad \text{(ii)}\ \ g \equiv \frac{1}{\mu}$$

and, in the case $\mu = 2$ only:

$$\text{(iii)}\quad g(x) = \alpha x \quad (x \in \mathbb{R})$$
$$\text{(iv)}\quad g(x) = \mathrm{Sup}(\alpha x, 0) \quad (x \in \mathbb{R})$$

where α is an arbitrary non zero real number.

Let us find now all the continuous solutions $g : \mathbb{R} \to \mathbb{R}$ of (7) when μ is a negative real number. We shall prove the following result:

Lemma 5. *All the continuous solutions $g : \mathbb{R} \to \mathbb{R}$ of the functional equation:*

$$g(g(y)x + g(x)y) = \mu\, g(x) g(y) \quad \text{where} \ \ \mu < 0 \tag{7}$$

are given by:

$$\text{(i)}\ \ g \equiv 0 \qquad\qquad \text{(ii)}\ \ g \equiv \frac{1}{\mu}.$$

Proof of Lemma 5. By taking $x = y = 0$ in (7), we have either $g(0) = \frac{1}{\mu}$ or $g(0) = 0$.

a) *Let us first consider the case $g(0) = \frac{1}{\mu}$*

By taking $y = 0$ in (7), we get:

$$g(x) = g\left(\frac{x}{\mu}\right) \quad \text{for every } x \text{ in } \mathbb{R}. \tag{8}$$

Therefore, we have for every x in $\mathbb{R}$ and for every positive integer n:

$$g(x) = g\left(\frac{x}{\mu^n}\right) = g(x\mu^n)$$

If $\mu \neq -1$, we see, as n goes to $+\infty$ and by using the continuity of g at 0, that: $g(x) = g(0) = \frac{1}{\mu}$ for every x in $\mathbb{R}$.

If $\mu = -1$, (8) becomes: $g(x) = g(-x)$ for every x in $\mathbb{R}$. By taking $y = -x$ in (7), we get:

$$g(g(-x)x - g(x)x) = -g(x)g(-x) \quad \text{for every } x \text{ in } \mathbb{R}$$

or, with (8):

$$g(0) = -1 = -g(x)^2 \quad \text{for every } x \text{ in } \mathbb{R}.$$

Using the continuity of g, we deduce:

$$g(x) = -1 \quad \text{for every } x \text{ in } \mathbb{R}.$$

b) *Let us now consider the case* $g(0) = 0$.

Let g be a non identically zero continuous solution of (7). By taking $y = x$ in (7), we see that the set of all real numbers x such that $g(x) < 0$ is a non empty open subset of $\mathbb{R}$. The continuity of g implies then that $g(\mathbb{R})$ is an interval of $\mathbb{R}$ containing an interval of the form $]\alpha, 0]$.

Let us suppose that g is bounded below by b, with $-\infty < b < 0$. We have: $g(x) > b$ for every x in $\mathbb{R}$. By taking $y = x$ in (7), we get:

$$|g(x)| < \sqrt{\frac{b}{\mu}} \quad \text{for every } x \text{ in } \mathbb{R}.$$

We have now:

$$g(x) > -\sqrt{\frac{b}{\mu}} \quad \text{for every } x \text{ in } \mathbb{R}.$$

By taking again $y = x$ in (7), we get:

$$|g(x)| < \frac{|b|^{1/4}}{|\mu|^{\frac{1}{2}+\frac{1}{4}}} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

Iterating this argument, we get for every positive integer n:

$$|g(x)| < \frac{|b|^{\frac{1}{2^n}}}{|\mu|^{\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}}} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

As n goes to $+\infty$, we obtain:

$$|g(x) \leq \frac{1}{|\mu|} \qquad \text{for every } x \text{ in } \mathbb{R}. \tag{9}$$

Let us define: $\phi(x, y) = g(y)x + g(x)y \quad (x, y \in \mathbb{R})$. Since g is bounded and not identically zero, we have: $\phi(s, \mathbb{R}) = \mathbb{R}$ for every real number s such that $g(s) \neq 0$. So, for every real number x, there exists a real number t such that $\phi(s, t) = x$. In view of the Darboux property of g, we may choose x and s in $\mathbb{R}$ such that: $0 < |g(s)| < |g(x)|$. By using (7) and (9), we obtain:

$$0 < |g(s)| < |g(x)| = |g(\phi(s, t))| = |\mu|\ |g(s)|\ |g(t)| \leq |g(s)|$$

which is a contradiction.

We deduce that g is not bounded below. Therefore, $g(\mathbb{R})$ *is an interval of* $\mathbb{R}$ *which contains* $]-\infty, 0]$.

So, there exists a non zero real number x_0 such that $g(x_0) = \frac{1}{\mu}$. Let us denote: $h(x) = \phi(x, x_0) = \frac{x}{\mu} + x_0 g(x)$ $(x \in \mathbb{R})$. h is continuous and satisfies the following functional equation

$$h(h(x)) = \left(1 + \frac{1}{\mu}\right) h(x) - \frac{1}{\mu} x \quad (x \in \mathbb{R}) \tag{10}$$

Now, all the continuous solutions of (10) are known (cf. [7]). Namely, we have the following result (cf. [7]):

Proposition 6. *All the continuous solutions of the functional equation:*

$$h(h(x)) = (1 + \gamma) h(x) - \gamma x \quad (x \in \mathbb{R}) \tag{11}$$

where γ is a negative given real number, are given by:

a) if $\gamma \neq -1$ (i) $h(x) = \gamma x + \delta$ *with $\delta \in \mathbb{R}$ arbitrary*

(ii) $h(x) \equiv x$

b) if $\gamma = -1$ (i) $h(x) \equiv x$

(ii) $$h(x) = \begin{cases} \phi(x) & \text{for } x \in \,]-\infty, c] \\ \phi^{-1}(x) & \text{for } x \in [c, +\infty[\end{cases}$$

where c is an arbitrary real number and ϕ is any strictly decreasing and continuous function mapping $]-\infty, c]$ onto $[c, +\infty[$.

The solution $h(x) \equiv x$ $(x \in \mathbb{R})$ of (10) gives:

$$g(x) = (1 - \frac{1}{\mu}) \frac{x}{x_0} \qquad (x \in \mathbb{R})$$

which does not satisfy (7).

The solution: $h(x) = \frac{1}{\mu} x + \delta$ $(x \in \mathbb{R})$ of (10) leads to a constant function g. This is not possible since we have supposed that $g(0) = 0$ and g is not identically zero.

So, we have necessarily $\mu = -1$ and

$$h(x) = \begin{cases} \phi(x) & \text{for } x \in \,]-\infty, c] \\ \phi^{-1}(x) & \text{for } x \in [c, +\infty[\end{cases} \tag{12}$$

where ϕ is a strictly decreasing and continuous function mapping $]-\infty, c]$ onto $[c, +\infty[$. The function: $x \to h(x) - x$ is continuous and strictly decreasing on $\mathbb{R}$. Therefore, it vanishes at most once. From $h(c) = c$ and $h(0) = 0$, we deduce: (13) $c = 0$. By taking $y = x_0$ in (7), we get:

$$g(h(x)) = g(x) \quad \text{for every } x \text{ in } \mathbb{R}. \tag{14}$$

Therefore, we may suppose that x_0 is a positive real number. By taking $y = h(x)$ in (7) and using (14), we get:

$$g\big(x\,g(x) + g(x)\,(x_0\,g(x) - x)\big) = -g(x)^2$$

or

$$(15)\qquad g(x_0\,g(x)^2) = -g(x)^2 \quad (x \in \mathbb{R})$$

Since $g(\mathbb{R})$ is an interval which contains $]-\infty, 0]$, the set $\{x_0 g(x)^2; x \in \mathbb{R}\}$ is the interval $[0, +\infty[$. (15) implies:

$$(16)\qquad g(x) = -\frac{x}{x_0} \quad \text{for every } x \text{ in } [0, +\infty[.$$

For x in $]-\infty, 0]$, we may use (14) and the fact that $h(x)$ belongs to $[0, +\infty[$. (16) implies then:

$$g(x) = g(h(x)) = -\frac{h(x)}{x_0} = -g(x) + \frac{x}{x_0} \quad \text{for } x \text{ in } \;]-\infty, 0]$$

or

$$g(x) = \frac{x}{2x_0} \quad \text{for every } x \text{ in } \;]-\infty, 0].$$

It is now easy to check that the function defined by:

$$g(x) = \begin{cases} -\dfrac{x}{x_0} & \text{for } x \geq 0 \\ \dfrac{x}{2x_0} & \text{for } x \leq 0 \end{cases} \quad \text{does not satisfy } \; (7).$$

Therefore, there does not exist a non identically zero continuous solution of (7) satisfying $g(0) = 0$.

This ends the proof of Lemma 5.

From Lemma 4 and Lemma 5, we obtain the possible expressions of $g(x) = f(x)^\ell$ $(x \in \mathbb{R})$ when f is a solution of (1) in DB_1 in the case $k = \ell$. Using also the result of J. Brzdęk in [3], we deduce then all the solutions of (1) in DB_1.

Theorem 7. *When λ is a non zero real number and k, ℓ are positive integers, all the solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation (1) in the class of functions DB_1 are given by:*

a) if $k \neq \ell$ or if $k = \ell$ and $\lambda^\ell \neq 2$

$$(i)\;\; f \equiv 0 \qquad (ii)\;\; f \equiv \frac{1}{\lambda}$$

b) if $k = \ell$ is an odd integer and $\lambda = 2^{1/\ell}$

$$(i)\ \ f \equiv 0 \qquad (ii)\ \ f \equiv \frac{1}{\lambda} \qquad (iii)\ \ f(x) = \mu x^{1/\ell} \quad (x \in \mathbb{R})$$
$$(iv)\ \ f(x) = \mathrm{Sup}(\mu x^{1/\ell}, 0) \quad (x \in \mathbb{R})$$

where μ is an arbitrary non zero real number.

c) if $k = \ell$ is an even integer and $\lambda = 2^{1/\ell}$

$$(i)\ \ f \equiv 0 \qquad (ii)\ \ f \equiv \frac{1}{\lambda} \qquad (iii)\ \ f(x) = (\mathrm{Sup}(\mu x, 0))^{1/\ell} \quad (x \in \mathbb{R})$$

where μ is an arbitrary non zero real number.

d) if $k = \ell$ is an even integer and $\lambda = -2^{1/\ell}$

$$(i)\ \ f \equiv 0 \qquad (ii)\ \ f \equiv \frac{1}{\lambda} \qquad (iii)\ \ f(x) = -(\mathrm{Sup}(\mu x, 0))^{1/\ell} \quad (x \in \mathbb{R})$$

where μ is an arbitrary non zero real number.

As in [1], we may obtain now all the continuous solutions $f : E \to \mathbb{R}$ of (1) when E is a real Hausdorff topological vector space.

Theorem 8. *Let E be a real Hausdorff topological vector space. When λ is a non zero real number and k, ℓ are positive integers, all the continuous solutions $f : E \to \mathbb{R}$ of the functional equation (1) are given by:*

a) if $k \neq \ell$ or if $k = \ell$ and $\lambda^\ell \neq 2$

$$(i)\ \ f \equiv 0 \qquad (ii)\ \ f \equiv \frac{1}{\lambda}$$

b) if $k = \ell$ is an odd integer and $\lambda = 2^{1/\ell}$

$$(i)\ \ f \equiv 0 \quad (ii)\ \ f \equiv \frac{1}{\lambda} \quad (iii)\ \ f(x) = (\langle x, x^* \rangle)^{1/\ell} \quad (x \in E)$$
$$(iv)\ \ f(x) = \mathrm{Sup}((\langle x, x^* \rangle)^{1/\ell}, 0) \quad (x \in E)$$

where x^* is a non zero element of the topological dual E^* of E.

c) if $k = \ell$ is an even integer and $\lambda = 2^{1/\ell}$

$$(i)\ \ f \equiv 0 \quad (ii)\ \ f \equiv \frac{1}{\lambda} \quad (iii)\ \ f(x) = (\mathrm{Sup}(\langle x, x^* \rangle, 0))^{1/\ell} \quad (x \in E)$$

where x^* is a non zero element of E^*.

d) if $k = \ell$ is an even integer and $\lambda = -2^{1/\ell}$

$$(i)\ \ f \equiv 0 \quad (ii)\ \ f \equiv \frac{1}{\lambda} \quad (iii)\ \ f(x) = -(\mathrm{Sup}(\langle x, x^* \rangle, 0))^{1/\ell} \quad (x \in E)$$

where x^* is a non zero element of E^*.

Proof of Theorem 8. Let $f : E \to \mathbb{R}$ be a continuous solution of (1). Then, $f(0)$ is either 0 or $\frac{1}{\lambda}$. For every $x \neq 0$ in E, we consider the function $f_x : \mathbb{R} \to \mathbb{R}$ defined by: $f_x(t) = f(tx)$ $(t \in \mathbb{R})$. It is easy to see that f_x is a continuous solution of (1). By Theorem 7, if $k \neq \ell$ or if $k = \ell$ and $\lambda^\ell \neq 2$, f_x is a constant function for every $x \neq 0$ in E. Therefore, we have: $f_x \equiv f_x(0) = f(0) = f_x(1) = f_x$ for every $x \neq 0$ in E. So, f is identically equal either to 0 or to $\frac{1}{\lambda}$.

Now, if $k = \ell$ and $\lambda^\ell = 2$, the function $g : E \to \mathbb{R}$ defined by: $g(x) = f(x)^\ell$ $(x \in E)$ is a continuous solution of the following functional equation:

$$g(g(y)x + g(x)y) = 2g(x)g(y) \quad (x, y \in E) \tag{18}$$

Now, all the continuous solutions $g : E \to \mathbb{R}$ of (18) are known (cf. [2] Theorem 15) and are given by:

$$\text{(i)}\quad g \equiv 0 \qquad \text{(ii)}\quad g \equiv \frac{1}{2} \qquad \text{(iii)}\quad g(x) = \langle x, x^* \rangle \quad (x \in E)$$
$$\text{(iv)}\quad g(x) = \text{Sup}(\langle x, x^* \rangle, 0) \quad (x \in E)$$

where x^* is a non zero element of the topological dual E^* of E. (We should notice here that, in a private communication, K. Baron observed that Theorem 15 of [2] stated for a real Hausdorff locally convex topological vector space is also true for a general real Hausdorff topological vector space).

We deduce then the continuous solutions of (1) given in b), c) and d).

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(Received June 20, 1991)