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On a class of QR-submanifolds of quaternion space forms

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Abstract. Let M be a QR-submanifolds (cf. A. Bejancu [2]) of a quaternion space form $N(c)$ such that every geodesic of M is a circle of $N(c)$. Then we prove that either $c=0$ or M is a totally real submanifold.

1. Introduction

Let N be a quaternion Kaehlerian manifold of real dimension $4n$. Then there exist on N a Riemannian metric g and a vector bundle V of tensors of type (1,1) with a local basis of Hermitian structures J_1, J_2, J_3 such that $J_1J_2=-J_2J_1=J_3$. Moreover, for each local section S of V and vector field X on N, $\widetilde{\nabla}_X \mathrm{S}$ is also a section of V, where $\widetilde{\nabla}$ is the Levi–Civita connection on N with respect to g. If the quaternion sectional curvature of N is a constant c then we say that N is a quaternion space form and denote it by $N(c)$. As it is well known (cf. K. Yano and M. Kon [10], p. 172) the curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ on $N(c)$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z=\frac{c}{4}\Big\{&g(Y,Z)X-g(X,Z)Y\\ &+\sum_{a=1}^{3}\{g(J_aY,Z)J_aX-g(J_aX,Z)J_aY+2g(X,J_aY)J_aZ\}\Big\}\end{aligned}\tag{1.1}$$

for any $X,Y,Z\in\Gamma(TN(c))$.

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Throughout the paper we use the following notations:

$F(N)$ – the algebra of smooth functions on N,

$\Gamma(E)$ – the $F(N)$-module of smooth sections of a vector bundle E over N. The same notations will be used for any other manifold.

Next, we consider an m-dimensional Riemannian manifold M isometrically immersed in N. Then M is said to be a *quaternion-real submanifold (QR-submanifold)* (cf. A. Bejancu [2]) if there exist a vector subbundle ν of the normal bundle $TM^{\perp}$ such that

$$J_a(\nu_x) = \nu_x \quad \text{and} \quad J_a(\nu_x^{\perp}) \subset T_xM,$$

for any $x \in M$ and $a \in \{1,2,3\}$, where $\nu^{\perp}$ is the complementary orthogonal vector bundle to ν in $TM^{\perp}$. If, in particular, $\nu = TM^{\perp}$ (resp. $\nu = \{0\}$), M is said to be a *quaternion submanifold* (cf. B.Y. Chen [4]) (resp. *anti-quaternion submanifold*, cf. J.S. Pak [8]).

Denote by s the rank of $\nu^{\perp}$, that is, each fibre of $\nu^{\perp}$ is of dimension s. Then consider the s-dimensional vector subspaces $D_{a,x} = J_a(\nu_x^{\perp})$, $a \in$\{1,2,3\} of T_xM which are mutually orthogonal. Thus we obtain a globally defined distribution

$$D^{\perp} : x \to D_x^{\perp} = D_{1,x} \oplus D_{2,x} \oplus D_{3,x},$$

on M. Also, we have

$$J_a(D_{a,x}) = \nu_x^{\perp} \quad \text{and} \quad J_a(D_{b,x}) = D_{c,x},$$

for any $x \in M$, where (a,b,c) is a cyclic permutation of $(1,2,3)$. Denote by D the complemetary orthogonal distribution to $D^{\perp}$ in TM. It is easy to see that D is a quaternion distribution on M, that is, $J_a(D_x) = D_x$ for any $x \in M$ and $a \in \{1,2,3\}$.

Finally, we recall from K. Nomizu and K. Yano [6] the notion of circle in N. A curve $C : x = x(t)$ with arc-length parameter t in N is said to be a *circle* if there exists a field of unit vectors Y_t along C, which, together with the unit tangent vectors X_t, satisfies the differential equations

$$\widetilde{\nabla}_t X_t = kY_t \quad \text{and} \quad \widetilde{\nabla}_t Y_t = -kX_t, \tag{1.2}$$

where k is a positive constant.

The main purpose of this note is to prove the following

Theorem 1. *Let M be a QR-submanifold of $N(c)$, such that each geodesic of M is a circle in $N(c)$. Then either $c = 0$ or $D = \{0\}$.*

2. Preliminaries

First, from the general theory of submanifolds we recall the formulas of Gauss and Weingarten (cf. B.Y. Chen [3], p. 39):

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(TM) \tag{2.1}$$

and

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad \forall X \in \Gamma(TM),\ \xi \in \Gamma(TM^\perp), \tag{2.2}$$

respectively, where ∇, $\nabla^\perp$, h and A_ξ are the Levi–Civita connection on M, the normal connection of M, the second fundamental form and the shape operator of M, respectively. Also, h and A_ξ are related by

$$g(h(X,Y),\xi) = g(A_\xi X, Y), \quad \forall X, Y \in \Gamma(TM),\ \xi \in \Gamma(TM^\perp). \tag{2.3}$$

The covariant derivative of h is defined by

$$\begin{gathered}(\nabla_X h)(Y,Z) = \nabla^\perp_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),\\ \forall X, Y, Z \in \Gamma(TM).\end{gathered} \tag{2.4}$$

Then the Codazzi equation is given by

$$\widetilde{R}(X,Y)Z\}^\perp = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \quad \forall X, Y, Z \in \Gamma(TM), \tag{2.5}$$

where $\{*\}^\perp$ means the normal part of $*$.

Now, suppose M is a QR-submanifold of a quaternion Kaehlerian manifold N. Then according to C.L. Bejan [1] the second fundamental form of M satisfies

$$g(h(X,Y),\xi) = 0, \quad \forall X, Y \in \Gamma(D),\ \xi \in \Gamma(\upsilon). \tag{2.6}$$

Let $C : x = x(t)$ be a geodesic of M, that is, $\nabla_t X_t = 0$. Thus from (2.1) we deduce

$$\widetilde{\nabla}_t X_t = h(X_t, X_t),$$

which, together with (2.2), implies

$$\widetilde{\nabla}^2_t X_t = -A_{h(X_t,X_t)} X_t + \nabla^\perp_t (h(X_t, X_t)). \tag{2.7}$$

Suppose now C is a circle in N. Then by using (2.1) and (1.2) we deduce

$$\widetilde{\nabla}_t^2 X_t = -k^2 X_t. \tag{2.8}$$

Comparing (2.7) and (2.8) we infer

$$A_{h(X_t,X_t)} X_t = k^2 X_t, \tag{2.9}$$

and

$$\nabla_t^{\perp}(h(X_t, X_t) = 0. \tag{2.10}$$

Consider $t = 0$. Using (2.3) and (2.9) we obtain

$$g(h(X,X), h(X,X)) = k^2$$

for any unit tangent vector X at a point $x \in M$. Thus there exists a non-zero function λ such that

$$g(h(X,X), h(X,X)) = \lambda^2, \tag{2.11}$$

for any unit vector field X on M. According to O'Neill [7], the immersion of M in $N(c)$ is λ-isotropic. From (2.11) we deduce

$$g(h(X,X), h(X,X)) = \lambda^2 g(X,X) g(X,X), \quad \forall X \in \Gamma(TM), \tag{2.12}$$

which is equivalent to

$$\begin{aligned} &g(h(X,Y), h(Z,U)) + g(X,Z), h(U,Y)) + g(X,U), h(Y,Z)) \\ &= \lambda^2\{g(X,Y)g(Z,U) + g(X,Z)g(U,Y) + g(X,U)g(Y,Z)\}, \end{aligned} \tag{2.13}$$

for any $X, Y, Z, U \in \Gamma(TM)$. Also, from (2.10) and (2.4), taking into account that C is a geodesic, we deduce:

$$(\nabla_X h)(X,X) = 0, \quad \forall X \in \Gamma(TM), \tag{2.14}$$

which is equivalent to

$$\begin{gathered} (\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) + (\nabla_Z h)(X,Y) = 0, \\ \forall X, Y, Z \in \Gamma(TM). \end{gathered} \tag{2.15}$$

Remark 1. K. Nomizu [5] obtained both (2.11) and (2.14) for a Kaehler submanifold of a complex projective space. Actually, it is easy to see that the above calculations hold good for a Riemannian submanifold whose geodesics are circles in the ambient space.

3. Proof of the Theorem

First, differentiating (2.11) and taking account of (2.14) we deduce $X(\lambda) = 0$ for any X$\in \Gamma(TM)$ that is, λ is locally constant on M. Then we take $Y = Z \in \Gamma(D)$ and X$\in \Gamma(D)$ in (2.15) and obtain

$$(\nabla_X h)(Y,Y) + 2(\nabla_Y h)(X,Y) = 0.$$

On the other hand, by using (1.1) in (2.5) we derive

$$(\nabla_X h)(Y,Y) - (\nabla_Y h)(X,Y) = 0, \quad \forall X,Y \in \Gamma(D).$$

Hence we have

$$(\nabla_X h)(Y,Y) = 0, \quad \forall X,Y \in \Gamma(D). \tag{3.1}$$

Next, we take $Y = Z = U \in \Gamma(D)$, $X = J_1\eta$ for $\eta \in \Gamma(\nu^\perp)$ in (2.13) and differentiate (2.13) with respect to $W \in \Gamma(D)$. Then taking into account that λ is locally constant and by using (3.1) we obtain

$$g((\nabla_W h)(J_1\eta, Y), h(Y,Y)) = 0, \quad \forall\, Y, W \in \Gamma(D),\ \eta \in \Gamma(\nu^\perp). \tag{3.2}$$

By using again (1.1) and (2.5) we deduce

$$(\nabla_W h)(J_1\eta, Y) - (\nabla_{J_1\eta} h)(W,Y) = \frac{c}{4} g(J_1W, Y)\eta,$$

and

$$(\nabla_W h)(J_1\eta, Y) - (\nabla_Y h)(W, J_1\eta) = \frac{c}{2} g(J_1W, Y)\eta.$$

Adding the last two relations and taking account of (2.15) we get

$$(\nabla_W h)(J_1\eta, Y) = \frac{c}{4} g(J_1W, Y)\eta,$$

which together with (3.2) implies

$$\frac{c}{4} g(J_1W, Y) g(\eta, h(Y,Y)) = 0, \quad \forall\, Y, W \in \Gamma(D),\ \eta \in \Gamma(\nu^\perp). \tag{3.3}$$

Finally, we suppose $c \neq 0$ and $D \neq \{0\}$. Then we take $Y = J_1$W in (3.3) and by linearity obtain:

$$g(h(X,Y),\eta) = 0, \quad \forall X, Y \in \Gamma(D),\ \eta \in \Gamma(\nu^{\perp}). \tag{3.4}$$

As a consequence of (2.6) and (3.4) we deduce $h(X,Y) = 0$ for any $X, Y \in \Gamma(D)$, which contradict (2.11). This completes the proof of the theorem.

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