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Almost sure versions of some analogues of the invariance principle

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Abstract. Almost sure versions of some functional limit theorems for random step lines and random broken lines defined by sums of independent identically distributed random variables with replacements are obtained.

1. Introduction

The almost sure (a.s.) versions of the invariance principle have been studied in several papers (see, for example, Lacey and Philipp [1], Berkes and Dehling [14], Ibragimov [2], Rodzik and Rychlik [3]).

This paper deals with random step lines processes and random broken line processes defined by sums of independent identically distributed (i.i.d.) random variables multiplied by values of independent indicators defined on another probability space. These processes describe some models in which random variables are replaced with other ones. We prove a.s. versions of functional limit theorems in which these processes converge in distribution to p-stable random processes with independent increments or sums of such processes in Skorohod spaces and in spaces of continuous functions.

We have three different types of replacements:

1) summands are replaced randomly by zeros;

2) summands are replaced randomly by nonzero random variables;

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3) in each step we delete on average m summands and add nonrandomly m new summands.

In each case the result of the consecutive replacements is denoted by a sequence of random variables $X_k^{(in)}$, $i \in \{1,2,3\}$. Using $X_k^{(in)}$ we construct a step line random process $Z_n^{(i)}(x)$. From $Z_n^{(i)}(x)$ we construct a measure $Q_n^{(i)}(\omega_1, \omega)$, where $\omega \in \Omega$ and $\omega_1 \in \Omega$ are considered as parameters. (Here Ω is the space where the initial random variables are defined, and Ω_1 is the space where the indicators are defined.) Our main results state that for almost all ω_1 and ω, the measures $Q_n^{(i)}(\omega_1, \omega)$, $i \in \{1,2,3\}$, converge weakly to certain probability measures defined by some stochastic processes on the Skorohod space (Theorems 1, 2, and 5). Moreover, from the sequence of random variables $X_k^{(in)}$ we shall also construct a broken line process. This broken line process will be denoted by $\tilde{Z}_n^{(i)}(x)$ and the corresponding measure by $\tilde{Q}_n^{(i)}(\omega_1, \omega)$, $i \in \{1,2,3\}$. We prove that for almost all ω_1 and ω, the measures $\tilde{Q}_n^{(i)}(\omega_1, \omega)$, $i \in \{1,2,3\}$, converge weakly in the space of continuous functions to some probability measures defined by Wiener processes (Theorems 3, 4, and 6). To illustrate our theorems we shall list three special probabilities of replacements (Examples 1, 2, and 3).

The methods of the proofs of our theorems are the same as those of Lacey and Philipp [1]. The same method was used in Ibragimov [2] to prove an a.s. invariance principle in the case of the convergence to p-stable homogeneous random processes. We also apply particular cases of the functional limit theorems from Chuprunov [4]. Functional limit theorems in Chuprunov [4] are generalizations of the ones in Fazekas and Chuprunov [5]. We mention that in Rusakov [6], another type of replacement was considered and a functional limit theorem with an Ornstein–Uhlenbeck limit process was proved.

2. Results

Notation. We will denote by $\xrightarrow{d}$ the convergence in distribution and by $\xrightarrow{w}$ the weak convergence of measures. If it does not make any confusion we will use the same designation for a random process and for the random element defined by it.

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. $[c]$ and $\{c\}$ denote the integer part and the fractional part of the real number c.

Let $(\Omega, \mathfrak{A}, \mathbf{P})$ and $(\Omega_1, \mathfrak{A}_1, \mathbf{P}_1)$ be probability spaces. Let $\mathbf{E}$ and $\mathbf{E}_1$ be the expectations with respect to $\mathbf{P}$ and $\mathbf{P}_1$, respectively. Let I_A denote the indicator of the event A. Let $\delta_{(x)}$ be the point mass at $x \in \mathbb{R}$.

Let $D[0,1]$ and $D(\mathbb{R}^+)$ be the Skorohod spaces of functions defined on $[0,1]$ and $\mathbb{R}^+$, respectively. Let $C_b(\mathbb{R}^+)$ be the space of the bounded continuous functions defined on $\mathbb{R}^+$ and let $C_c(\mathbb{R}^+)$ be the subspace of $C_b(\mathbb{R}^+)$ which contains functions having limit at infinity.

Observe that $D(\mathbb{R}^+)$ is embedded into $D[0,1]$ by the following bijective mapping. Let $g : [0,1) \to \mathbb{R}^+$ be a continuous increasing one to one function. Then $x \in D(\mathbb{R}^+)$ corresponds $x \circ g \in D[0,1]$, where $x \circ g(t) = x(g(t))$, $t \in [0,1)$. The metric on $D(\mathbb{R}^+)$ we define as the preimage of the metric on $D[0,1]$ with respect to this embedding.

Throughout the paper, $A_{ij} = A_{ij}(n) \in \mathfrak{A}_1$, $i,j,n \in \mathbb{N}$, will be events which are independent for each fixed $n \in \mathbb{N}$, and $\mathbf{P}_1(A_{ij}(n)) = p_j(n) = p_j$ for all $n, i \in \mathbb{N}$. Let $I_{ij} = I_{ij}(n) = I_{A_{ij}(n)}(\omega_1)$, $\omega_1 \in \Omega_1$. We will suppose that

$$\tag{A} \begin{aligned} &\lim_{n\to\infty} \mathbf{E}_1 I_{i1}(n) I_{i2}(n) \dots I_{i[nx]}(n) \\ &= \lim_{n\to\infty} p_1(n) p_2(n) \dots p_{[nx]}(n) = f(x), \quad \text{for all } x \in \mathbb{R}^+, \end{aligned}$$

and f is a continuous function.

So (A) implies $f(0) = 1$ and f is decreasing.

To define and study our processes we need the following notation and conditions:

(Y) Let $1 < p \le 2$ and let Y, Y_i, $i \in \mathbb{N}$, be i.i.d. random variables defined on $(\Omega, \mathfrak{A}, \mathbf{P})$.

Suppose that

$$\tag{S} \frac{1}{a_n} \sum_{i=1}^{n} Y_i \xrightarrow{d} \gamma_p, \quad n \to \infty,$$

where $a_n \to \infty$, $n \to \infty$, and γ_p is a symmetric p-stable random variable. (That is the characteristic function of γ_p is of the form

$$\phi_p(t) = e^{-|at|^p}, \quad t \in \mathbb{R},$$

where $a > 0$.) Let $W_p(x)$, $x \in \mathbb{R}^+$, be a homogeneous p-stable random process with independent increments such that $W_p(1) = \gamma_p$.

Replacement by 0, convergence in $D(\mathbb{R}^+)$. First we will consider the case of replacement by 0. We start with the normalized sum $X_0^{(1n)} = \frac{1}{a_n}\sum_{i=1}^n Y_i$ of the random variables $Y_1, \dots, Y_n$. In each step we randomly replace summands by 0. In the k-th step we obtain

$$X_k^{(1n)} = \frac{1}{a_n}\sum_{i=1}^n I_{i1}(n)I_{i2}(n)\cdots I_{ik}(n)Y_i, \quad k = 1, 2, \dots . \tag{X1}$$

We will consider sequences of random processes and measures. As we shall use the same type of construction in several cases, (Zi) and (Qi) will denote that the process $Z_n^{(i)}$ and the measure $Q_n^{(i)}$ are defined by the random variables $X_k^{(in)}$, $i \in \{1, 2, 3\}$. For $i \in \{1, 2\}$ let

$$Z_n^{(i)}(x) = X_{[nx]}^{(in)}, \quad x \in \mathbb{R}^+,\ n \in \mathbb{N}. \tag{Zi}$$

The process $Z_n^{(i)}$ depends on $\omega \in \Omega$ and $\omega_1 \in \Omega_1$ and for fixed $\omega \in \Omega$ and $\omega_1 \in \Omega_1$ we will denote by $Z_n^{(i)}(\omega_1, \omega) \in D(\mathbb{R}^+)$ the trajectory of this process. For $i \in \{1, 2\}$ let

$$Q_n^{(i)}(\omega_1, \omega) = \frac{1}{\ln(n)}\sum_{k=1}^n \frac{1}{k}\delta_{(Z_k^{(i)}(\omega_1,\omega))}, \quad \omega_1 \in \Omega_1,\ \omega \in \Omega,\ n \in \mathbb{N}. \tag{Qi}$$

Theorem 1. *Suppose that (Y), (S) and (A) are satisfied. Then for the measures defined by (X1), (Z1) and (Q1) for almost all $\omega_1 \in \Omega_1$ it holds*

$$Q_n^{(1)}(\omega_1, \omega) \xrightarrow{w} W^{(1)}, \quad n \to \infty,\ \text{in } D(\mathbb{R}^+)$$

for almost all $\omega \in \Omega$, where $W^{(1)}(x) = W_p\big(f(x)\big)$, $x \in \mathbb{R}^+$.

Replacement by nonzero, convergence in $D(\mathbb{R}^+)$. To examine the case of the replacement by nonzero random variables, we need the following notation and conditions:

(Y') Let Y', Y_i', $i \in \mathbb{N}$, be i.i.d. random variables defined on $(\Omega, \mathfrak{A}, \mathbf{P})$ such that Y_i', Y_i, $i \in \mathbb{N}$, are independent random variables.

Suppose that

$$\frac{1}{a_n}\sum_{i=1}^n Y_i' \xrightarrow{d} \gamma_p', \quad n \to \infty, \tag{S'}$$

where the sequence $(a_n)_{n=1}^{\infty}$ is from condition (S) and γ'_p is a p-stable random variable with characteristic function

$$\phi'_p(t) = e^{-|a't|^p}, \quad t \in \mathbb{R},$$

where $0 \leq a'$. Let $W'_p(x)$, $x \in \mathbb{R}^+$, be a homogeneous p-stable random process with independent increments such that $W'_p(1) = \gamma'_p$. Suppose that the random processes W_p and W'_p are independent.

We start with the summands Y_i, $1 \leq i \leq n$. Let $X_0^{(2n)} = \frac{1}{a_n}\sum_{i=1}^n Y_i$. In each step we randomly change Y_i to Y'_i. At the k-th step we get

(X2)
$$X_k^{(2n)}$$
$$= \frac{1}{a_n}\left(\sum_{i=1}^n I_{i1}(n)I_{i2}(n)\cdots I_{ik}(n)Y_i + \sum_{i=1}^n (1 - I_{i1}(n)I_{i2}(n)\cdots I_{ik}(n))Y'_i\right),$$

$k = 1, 2, \ldots$. We will consider the random processes $Z_n^{(2)}$ and the measures $Q_n^{(2)}$ defined by (Z2) and (Q2), respectively.

Theorem 2. *Suppose that (Y), (S), (Y'), (S') and (A) are satisfied. Then for the measures defined by (X2), (Z2) and (Q2) for almost all $\omega_1 \in \Omega_1$ one has*

$$Q_n^{(2)}(\omega_1, \omega) \xrightarrow{w} W^{(2)}, \quad n \to \infty, \text{ in } D(\mathbb{R}^+)$$

for almost all $\omega \in \Omega$, where $W^{(2)}(x) = W_p(f(x)) + W'_p(1) - W'_p(f(x))$, $x \in \mathbb{R}^+$.

Convergence in $C_c(\mathbb{R}^+)$. We will consider analogues to Theorem 1 and Theorem 2 in $C_c(\mathbb{R}^+)$. We will assume that $p = 2$ and

(G)
$$\frac{1}{a_n}\sum_{i=1}^n Y_i \xrightarrow{d} \gamma(\sigma), \quad n \to \infty,$$

and

(G')
$$\frac{1}{a_n}\sum_{i=1}^n Y'_i \xrightarrow{d} \gamma(\sigma'), \quad n \to \infty,$$

where $\gamma(\sigma)$ and $\gamma(\sigma')$ are centered Gaussian random variables with variances σ^2 and σ'^2, respectively. Denote by W and W' independent standard Wiener processes. We will consider the following random processes:

$$(\tilde{\mathrm{Z}}i) \qquad \tilde{Z}_n^{(i)}(x) = X_{[nx]}^{(in)} + \{nx\}\left(X_{[nx]+1}^{(in)} - X_{[nx]}^{(in)}\right), \quad x \in \mathbb{R}^+,$$

and the following measures:

$$(\tilde{\mathrm{Q}}i) \qquad \tilde{Q}_n^{(i)}(\omega_1,\omega) = \frac{1}{\ln(n)} \sum_{k=1}^{n} \frac{1}{k} \delta_{(\tilde{Z}_k^{(i)}(\omega_1,\omega))}, \quad \omega_1 \in \Omega_1,\ \omega \in \Omega,$$

$n \in \mathbb{N}$, $i \in \{1,2\}$.

Theorem 3. *Suppose that (Y), (A), and (G) are satisfied. Then for the measures defined by (X1), ($\tilde{Z}$1) and ($\tilde{Q}$1) for almost all $\omega_1 \in \Omega_1$ one has*

$$\tilde{Q}_n^{(1)}(\omega_1,\omega) \xrightarrow{w} \tilde{W}^{(1)}, \quad n \to \infty,\ \textit{in } C_c(\mathbb{R}^+)$$

for almost all $\omega \in \Omega$, where $\tilde{W}^{(1)}(x) = \sigma W(f(x))$, $x \in \mathbb{R}^+$.

Theorem 4. *Suppose that (Y), (Y'), (G), (G'), and (A) are satisfied. Then for the measures defined by (X2), ($\tilde{Z}$2), and ($\tilde{Q}$2) for almost all $\omega_1 \in \Omega_1$ one has*

$$\tilde{Q}_n^{(2)}(\omega_1,\omega) \xrightarrow{w} \tilde{W}^{(2)}, \quad n \to \infty,\ \textit{in } C_c(\mathbb{R}^+)$$

for almost all $\omega \in \Omega$, where $\tilde{W}^{(2)}(x) = \sigma W(f(x)) + \sigma'(W'(1) - W'(f(x)))$, $x \in \mathbb{R}^+$.

3. Preliminary results

Our proofs are based on the following particular cases of Theorems 1–4 from Chuprunov [4].

Proposition 1. *Suppose that (Y), (S) and (A) are satisfied. Then for the random processes defined by (X1) and (Z1) for almost all $\omega_1 \in \Omega_1$ it holds $Z_n^{(1)} \xrightarrow{d} W^{(1)}$, $n \to \infty$, in $D(\mathbb{R}^+)$.*

Proposition 2. *Suppose that (Y), (S), (Y'), (S') and (A) are satisfied. Then for the random processes defined by (X2) and (Z2) for almost all $\omega_1 \in \Omega_1$ one has $Z_n^{(2)} \xrightarrow{d} W^{(2)}$, $n \to \infty$, in $D(\mathbb{R}^+)$.*

Proposition 3. *Suppose that (Y), (G), and (A) are satisfied. Then for the random processes defined by (X1) and ($\tilde{Z}$1) for almost all $\omega_1 \in \Omega_1$ one has $\tilde{Z}_n^{(1)} \xrightarrow{d} \tilde{W}^{(1)}$, $n \to \infty$, in $C_c(\mathbb{R}^+)$.*

Proposition 4. *Suppose that (Y), (G), (Y'), (G'), and (A) are satisfied. Then for the random processes defined by (X2) and ($\tilde{Z}$2) for almost all $\omega_1 \in \Omega_1$ one has $\tilde{Z}_n^{(2)} \xrightarrow{d} \tilde{W}^{(2)}$, $n \to \infty$, in $C_c(\mathbb{R}^+)$.*

In [4], Proposition 3 and Proposition 4 were proved for $C_b(\mathbb{R}^+)$ instead of $C_c(\mathbb{R}^+)$. However, one can obtain the above propositions with minor modifications in the proofs presented in [4].

We will use the following law of large numbers (see, for example, Móri [8]).

Lemma 1. *Let ξ_i, $i \in \mathbb{N}$, be centered second order random variables with the property: there are $C > 0$ and $\beta > 0$ such that $|\mathbf{E}\xi_l\xi_k| < C\left(\frac{l}{k}\right)^\beta$ for all $l, k \in \mathbb{N}$, $l \le k$. Then it holds*

$$\frac{1}{\ln(n)} \sum_{k=1}^{n} \frac{1}{k}\xi_k \to 0, \quad n \to \infty, \text{ a.s.}$$

Recall some results from the theory of measures on metric spaces. Let $(\mathbf{B}, \rho)$ be a separable complete metric space and $BL(\mathbf{B})$ be the space of the continuous bounded functions $g : \mathbf{B} \to \mathbb{R}$ with $\|f\|_{BL} = \|g\|_\infty + \|g\|_L < \infty$. Here

$$\|g\|_L = \sup_{x \neq y} \frac{|g(x) - g(y)|}{\rho(x, y)}.$$

We will use the following known lemma in the case $\mathbf{B} = D(\mathbb{R}^+)$ or $\mathbf{B} = C_c(\mathbb{R}^+)$.

Lemma 2. *Let μ be a finite Borel measure on $\mathbf{B}$. There exists a countable set $M \subset BL(\mathbf{B})$ (depending from μ) such that for all finite Borel measures μ_n, $n \in \mathbb{N}$, on $\mathbf{B}$ we have: $\mu_n \xrightarrow{w} \mu$, $n \to \infty$, in $\mathbf{B}$, if and only if for all $g \in M$*

$$\int_{\mathbf{B}} g(x) d\mu_n(x) \to \int_{\mathbf{B}} g(x) d\mu(x), \quad n \to \infty. \tag{1}$$

The proof of Lemma 2 follows from that of Theorem 11.3.3 in Dudley [9].

We will use the following inequalities.

From the definition of the metric in $D[0,1]$ (see Billingsley, [10]) it follows that

$$\rho_{D[0,1]}(x,y) \le \|x-y\|_\infty, \quad x,y \in D[0,1],$$

and

$$\rho_{D(\mathbb{R}^+)}(x,y) \le \|x-y\|_\infty, \quad x,y \in D(\mathbb{R}^+). \tag{2}$$

Let η, η_i, $i \in \mathbb{N}$, be independent identically distributed random variables.

Let η, η_i be such that $\mathbf{E}|\eta| < \infty$. Then for all $n \in \mathbb{N}$ (see Shorack and Wellner [11, p. 858]) we have

$$\mathbf{E}\left(\max_{j\le n}\Big|\sum\nolimits_{i=1}^{j} \eta_i\Big|\right) \le 8\mathbf{E}\Big|\sum\nolimits_{i=1}^{n} \eta_i\Big|, \quad n \in \mathbb{N}. \tag{3}$$

Let η be such that $\frac{1}{a_n}\sum_{i=1}^{n}\eta_i \xrightarrow{d} \gamma_p$, $n \to \infty$. Then there exist constants $C_1(\eta) > 0$ and $\beta > 0$, depending on η and (a_n) only, such that

$$\frac{\mathbf{E}|\sum_{i=1}^{l}\eta_i|}{|a_n|} \le C_1(\eta)\left(\frac{l}{n}\right)^{\beta}, \quad l \le n,\ l,n \in \mathbb{N}. \tag{4}$$

To prove (4) observe the following. Let $l \le n$. By Araujo and Giné [12, p. 91],

$$\mathbf{E}\frac{|\sum_{i=1}^{n}\eta_i|}{|a_n|} \to \mathbf{E}|\gamma_p|, \quad n \to \infty.$$

Therefore,

$$\mathbf{E}\frac{|\sum_{i=1}^{l}\eta_i|}{|a_n|} \le \sup_{j\in\mathbb{N}} \frac{\mathbf{E}|\sum_{i=1}^{j}\eta_i|}{|a_j|}\frac{|a_l|}{|a_n|} = C_p\frac{|a_l|}{|a_n|},$$

where $C_p < \infty$. Since $a_n = n^{1/p}h(n)$, $n \in \mathbb{N}$, where h is a slowly varying function (see Araujo and Giné [12, p. 90]), for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $|a_l|/|a_n| < C_\epsilon\,(l/n)^{1/p-\epsilon}$, $l,n \in \mathbb{N}$, $l \le n$ (to prove it one can use Lemma 1.3 in Seneta [13]). So (4) is valid with $C_1(\eta) = C_pC_\epsilon$ and $\beta = 1/p-\epsilon$, where $0 < \epsilon < 1/p$.

Let $k, l \in \mathbb{N}$, $l < k$. Consider the random processes

$$r_{kl}^{(1)}(x) = \frac{1}{a_k} \sum_{i=l+1}^{k} I_{i1}(k) I_{i2}(k) \cdots I_{i[kx]}(k) Y_i,$$

$$r_{kl}^{(2)}(x) = \frac{1}{a_k} \left(\sum_{i=l+1}^{k} I_{i1}(k) I_{i2}(k) \cdots I_{i[kx]}(k) Y_i \right.$$
$$\left. + \sum_{i=l+1}^{k} \left(1 - I_{i1}(k) I_{i2}(k) \cdots I_{i[kx]}(k)\right) Y_i' \right),$$

and

$$\tilde{r}_{kl}^{(i)}(x) = r_{kl}^{(i)}(x) + \{kx\} \left(r_{kl}^{(i)}(x + 1/k) - r_{kl}^{(i)}(x) \right).$$

$x \in \mathbb{R}^+$, $i \in \{1, 2\}$.

Lemma 3. *Let $i \in \{1, 2\}$, $l < k$. Let the conditions of Theorem i be valid. Then there exist $C^* > 0$ and $\beta > 0$ depending on i, Y, Y', and (a_n) only, such that*

$$\mathbf{E} \left\| Z_k^{(i)} - r_{kl}^{(i)} \right\|_\infty \le C^* \left(\frac{l}{k} \right)^\beta, \tag{5}$$

for all $\omega_1 \in \Omega_1$. Inequality (5) is valid for $\tilde{Z}_k^{(i)}$ and $\tilde{r}_{kl}^{(i)}$, too.

Proof. Let $i = 1$. Using (3) and (4) we obtain

$$\begin{aligned} \mathbf{E} \left\| Z_k^{(1)} - r_{kl}^{(1)} \right\|_\infty &= \frac{1}{|a_k|} \mathbf{E} \left(\sup_{x \in \mathbb{R}^+} \left| \sum_{i=1}^{l} I_{i1} I_{i2} \ldots I_{i[kx]} Y_i \right| \right) \\ &\le \frac{1}{|a_k|} \mathbf{E} \left(\max_{j \le l} \left| \sum_{i=1}^{j} Y_i \right| \right) \le 8 \frac{1}{|a_k|} \mathbf{E} \left| \sum_{i=1}^{l} Y_i \right| \le 8 C_1(Y) \left(\frac{l}{k} \right)^\beta. \end{aligned} \tag{6}$$

Therefore (5) is proved for $i = 1$. Let $i = 2$. Using (3), (4) and (6) we obtain

$$\mathbf{E} \left\| Z_k^{(2)} - r_{kl}^{(2)} \right\|_\infty \le \mathbf{E} \left\| Z_k^{(1)} - r_{kl}^{(1)} \right\|_\infty + \frac{1}{a_k} \mathbf{E} \left| \sum_{i=1}^{l} Y_i' \right|$$

$$+\frac{1}{a_k}\mathbf{E}\left(\sup_{x\in\mathbb{R}^+}\left|\sum_{i=1}^{l} I_{i1}I_{i2}\cdots I_{i[kx]}Y_i'\right|\right)$$

$$\leq 8C_1(Y)\left(\frac{l}{k}\right)^{\beta} + C_1(Y')\left(\frac{l}{k}\right)^{\beta} + 8C_1(Y')\left(\frac{l}{k}\right)^{\beta} \leq C^*\left(\frac{l}{k}\right)^{\beta}.$$

In the case $i=2$ inequality (5) is proved.

Now observe that $\mathbf{E}\|\tilde{Z}_k^{(i)} - \tilde{r}_{kl}^{(i)}\|_\infty \leq 3\mathbf{E}\|Z_k^{(i)} - r_{kl}^{(i)}\|_\infty$, $i \in \{1,2\}$. Therefore the second assertion follows from the first one.

4. Proofs

We will prove our theorems simultaneously. Let $i \in \{1,2\}$. First we will suppose that $\mathbf{B} = D(\mathbb{R}^+)$.

Proof of Theorem i. Let Ω' be a subset of Ω_1 such that for all $\omega_1 \in \Omega'$ $Z_n^{(i)} \xrightarrow{d} W^{(i)}$, $n \to \infty$. By Proposition i, $\mathbf{P}_1(\Omega') = 1$. Let $\omega_1 \in \Omega'$. Let M be the set from Lemma 2 and $g \in M$.

Consider the random variables $\xi_k = g(Z_k^{(i)}) - \mathbf{E}g(Z_k^{(i)})$, $k \in \mathbb{N}$. Let $k, l \in \mathbb{N}$, $l < k$. Using (2), (3) and the independence of $r_{kl}^{(i)}$ and ξ_l, we have

$$\begin{aligned}
|\mathbf{E}\xi_k\xi_l| &= \left|\mathbf{E}\left(g\left(Z_k^{(i)}\right) - \mathbf{E}g\left(Z_k^{(i)}\right)\right)\left(g\left(Z_l^{(i)}\right) - \mathbf{E}g\left(Z_l^{(i)}\right)\right)\right| \\
&= \left|\mathbf{E}\left(g\left(Z_k^{(i)}\right) - g\left(r_{kl}^{(i)}\right)\right)\left(g\left(Z_l^{(i)}\right) - \mathbf{E}g\left(Z_l^{(i)}\right)\right)\right. \\
&\quad \left. +\mathbf{E}\left(g\left(r_{kl}^{(i)}\right) - \mathbf{E}g\left(Z_k^{(i)}\right)\right)\left(g\left(Z_l^{(i)}\right) - \mathbf{E}g\left(Z_l^{(i)}\right)\right)\right| \\
&= \left|\mathbf{E}\left(g\left(Z_k^{(i)}\right) - g\left(r_{kl}^{(i)}\right)\right)\left(g\left(Z_l^{(i)}\right) - \mathbf{E}g\left(Z_l^{(i)}\right)\right)\right| \\
&\leq 2\|g\|_{BL}\left|\mathbf{E}\left(g\left(Z_k^{(i)}\right) - g\left(r_{kl}^{(i)}\right)\right)\right| \\
&\leq 2\|g\|_{BL}^2\mathbf{E}\left\|Z_k^{(i)} - r_{kl}^{(i)}\right\|_\infty \leq 2C^*\|g\|_{BL}^2\left(\frac{l}{k}\right)^{\beta}.
\end{aligned}$$

Also observe, that

$$\mathbf{E}(\xi_k)^2 = \mathbf{E}\left(g\left(Z_k^{(i)}\right) - \mathbf{E}g\left(Z_k^{(i)}\right)\right)^2 \leq 4\|g\|_{BL}^2, \quad k \in \mathbb{N}.$$

So by Lemma 1, it holds

$$\frac{1}{\ln(n)}\sum_{k=1}^{n}\frac{1}{k}\int_{\mathbf{B}} g(x)d\delta_{(Z_k^{(i)}(\omega_1,\omega))}(x)-\frac{1}{\ln(n)}\sum_{k=1}^{n}\frac{1}{k}\mathbf{E}g\left(Z_k^{(i)}\right)$$
$$=\frac{1}{\ln(n)}\sum_{k=1}^{n}\frac{1}{k}\xi_k\to 0,\quad n\to\infty,$$

for almost all $\omega\in\Omega$. Observe that $\mathbf{E}g(Z_n^{(i)})\to\mathbf{E}g(W^{(i)})$, $n\to\infty$. Therefore, $\frac{1}{\ln(n)}\sum_{k=1}^{n}\frac{1}{k}\mathbf{E}g(Z_k^{(i)})\to\mathbf{E}g(W^{(i)})$, $n\to\infty$. The set M is countable. Thus, for almost all $\omega\in\Omega$, for all $g\in M$

$$\frac{1}{\ln(n)}\sum_{k=1}^{n}\frac{1}{k}\int_{D(\mathbb{R}^+)} g(x)d\delta_{(Z_k^{(i)}(\omega_1,\omega))}(x)\to\mathbf{E}g(W^{(i)}),\quad n\to\infty.$$

By Lemma 2, this implies Theorem i. The proof is complete.

The proofs of Theorems 3–4 are the same as those of Theorem i, $i\in\{1,2\}$.

5. Examples and additional results

Examples. Let $m\in\mathbb{N}$ be fixed. We will suppose that one of the following three conditions is valid.

(A1) $\quad\mathbf{P}(A_{ij}(n))=1-\dfrac{m}{n}\quad$ for all $i,j\in\mathbb{N}$ $(n>m)$.

(A2) $\quad\mathbf{P}(A_{ij}(n))=1-\dfrac{m}{n+(j-1)m}\quad$ for all $i,j\in\mathbb{N}$ $(n>m)$.

(A3) $\quad\mathbf{P}(A_{ij}(n))=1-\dfrac{m}{n-(j-1)m},\quad$ if $n>jm$,

$\quad\mathbf{P}(A_{ij}(n))=0\quad$ if $n\le jm,\ i,j\in\mathbb{N}$.

We mention the meaning of conditions (A1) and (A3). Condition (A1) corresponds to the case when at the l-th step we replace m summands in the sums $X_{l-1}^{(1n)}$ by zero regardless of whether a summand equals zero or not. On the other hand, under (A3), after each replacement, the number of Y_i in $X_{l-1}^{(1n)}$ decreases by about m elements. So (A3) corresponds to the case when at the l-th step we replace about m nonzero summands in $X_{l-1}^{(1n)}$ by zero.

Example 1. Let (A1) be valid. Then for all $x \in \mathbb{R}^+$ it holds

$$p_1(n)p_2(n)\cdots p_{[nx]}(n) = \left(1 - \frac{m}{n}\right)^{[xn]} \to e^{-mx}, \quad n \to \infty.$$

So, Theorems 1–4 are valid with $f(x) = e^{-mx}$.

Example 2. Let (A2) be valid. Then for all $x \in \mathbb{R}^+$ it holds

$$\begin{aligned} p_1(n)p_2(n)\cdots p_{[nx]}(n) &= \frac{n + ([nx]-2)m}{n + ([nx]-1)m} \frac{n + ([nx]-3)m}{n + ([nx]-2)m} \cdots \frac{n-m}{n} \\ &= \frac{n-m}{n + ([nx]-1)m} \to \frac{1}{1+mx}, \quad n \to \infty. \end{aligned}$$

So, Theorems 1–4 are valid with $f(x) = \frac{1}{1+mx}$.

Example 3. Let (A3) be valid. Let $0 \le x < \frac{1}{m}$. Then we obtain

$$\begin{aligned} p_1(n)p_2(n)\cdots p_{[nx]}(n) &= \frac{n - [nx]m}{n - ([nx]-1)m} \frac{n - ([nx]-1)m}{n - ([nx]-2)m} \cdots \frac{n-m}{n} \\ &= \frac{n - [nx]m}{n} \to 1 - mx, \quad n \to \infty, \ x < \frac{1}{m}. \end{aligned}$$

So, Theorems 1–4 are valid with $f(x) = (1 - mx) I_{[0, \frac{1}{m})}(x)$.

Additional results. Now, we shall consider third method of replacement. Let $X_0^{(3n)} = \frac{1}{a_n} \sum_{i=1}^n Y_i$. At the k-th step we randomly delete summands from $X_{k-1}^{(3n)}$ and add nonrandomly m new summands. Thus, we obtain

$$\text{(X3)} \qquad X_k^{(3n)} = \frac{1}{a_n} \Bigg(\sum_{i=1}^{n} I_{i1}(n) I_{i2}(n) \cdots I_{ik}(n) Y_i + \sum_{j=1}^{k-1} \sum_{i=n+(j-1)m+1}^{n+jm} I_{i(j+1)}(n) I_{i(j+2)}(n) \cdots I_{ik}(n) Y_i' + \sum_{i=n+(k-1)m+1}^{n+km} Y_i' \Bigg),$$

$k = 1, 2, \ldots$. There are $n + (l-1)m$ summands in $X_{l-1}^{(3n)}$. So, (A2) corresponds to the case when at the l-th step we replace m summands in $X_{l-1}^{(3n)}$ by new Y_i' regardless of whether a summand equals zero or not. Condition (A1) corresponds to the case when at the l-th step we replace m nonzero summands in $X_{l-1}^{(3n)}$ by new Y_i'.

We will consider the following random processes

$$Z_n^{(3)}(x) = X_{[nx]}^{(3n)}, \quad x \in [0,1], \tag{Z3}$$

and

$$\tilde{Z}_n^{(3)}(x) = X_{[nx]}^{(3n)} + \{nx\}\left(X_{[nx]+1}^{(3n)} - X_{[nx]}^{(3n)}\right), \quad x \in [0,1), \tag{Z̃3}$$

$$\tilde{Z}_n^{(3)}(1) = X_n^{(3n)}, \quad n \in \mathbb{N}.$$

Theorem 5. *Suppose that (Y), (Y'), (S), and (S') are satisfied.*

(1) Let (A1) be valid. Then for the measures defined by (X3), (Z3), and (Q3) for almost all $\omega_1 \in \Omega_1$ one has

$$Q_n^{(3)}(\omega_1, \omega) \xrightarrow{w} W^{(31)}, \quad n \to \infty, \text{ in } D[0,1]$$

for almost all $\omega \in \Omega$, where $W^{(31)}(x) = W_p\big(e^{-mx}\big) + W_p'\big(1 - e^{-mx}\big)$, $x \in [0,1]$.

(2) Let (A2) be valid. Then for the measures defined by (X3), (Z3) and (Q3) for almost all $\omega_1 \in \Omega_1$ one has

$$Q_n^{(3)}(\omega_1, \omega_1) \xrightarrow{w} W^{(32)}, \quad n \to \infty, \text{ in } D[0,1]$$

for almost all $\omega \in \Omega$, where $W^{(32)}(x) = W_p\left(\frac{1}{1+mx}\right) + W_p'\left(\frac{1}{2}\frac{(1+mx)^2-1}{1+mx}\right)$, $x \in [0,1]$.

Theorem 6. *Suppose that (Y), (Y'), (G) and (G') are satisfied.*

(1) Let (A1) be valid. Then for the measures defined by (X3), ($\tilde{Z}3$) and ($\tilde{Q}3$) for almost all $\omega_1 \in \Omega_1$ one has

$$\tilde{Q}_n^{(3)}(\omega_1, \omega) \xrightarrow{w} \tilde{W}^{(31)}, \quad n \to \infty, \text{ in } C[0,1]$$

for almost all $\omega \in \Omega$, where $\tilde{W}^{(31)}(x) = \sigma W(e^{-mx}) + \sigma' W'(1 - e^{-mx})$, $x \in [0,1]$.

(2) Let (A2) be valid. Then for the measures defined by (X3), ($\tilde{Z}3$) and ($\tilde{Q}3$) for almost all $\omega_1 \in \Omega_1$ one has

$$\tilde{Q}_n^{(3)}(\omega_1, \omega) \xrightarrow{w} \tilde{W}^{(32)}, \quad n \to \infty, \text{ in } C[0,1]$$

for almost all $\omega \in \Omega$*, where* $\tilde{W}^{(32)}(x) = \sigma W\left(\frac{1}{1+mx}\right) + \sigma' W'\left(\frac{1}{2}\frac{(1+mx)^2-1}{1+mx}\right)$, $x \in [0,1]$.

The proofs of Theorem 5 and Theorem 6 are the same as the ones of Theorems 1–4 (see Chuprunov and Fazekas [7]).

Remark 1. Consider the case when the basic probability spaces coincide. Let $I_{ij}(n)$, Y_i and Y_i' be defined on the same probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ and for each fixed $n \in \mathbb{N}$, let $I_{ij}(n)$, Y_i, Y_i', $i, j \in \mathbb{N}$, be independent random variables. Then Theorems 1–6 hold. Of course, in this case the phrase "for almost all $\omega_1 \in \Omega_1$" should be omitted. For details see Chuprunov and Fazekas [7].

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