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CR structures on real hypersurfaces of a complex space form

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Abstract. We define the generalized Tanaka connection for real hypersurfaces in Kählerian manifolds, and further we classify a real hypersurface of a complex space form whose shape operator field is parallel with respect to the generalized Tanaka connection.

0. Introduction

A complex n-dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $\widetilde{M}_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $E_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0$, $c = 0$ or $c < 0$. By virtue of Segre's work [9] we know that real hypersurfaces of $E_n\mathbb{C}$ with constant principal curvatures must lie on an open part of a sphere or a hyperplane, or a generalized cylinder. Takagi [11] classified the homogeneous real hypersurfaces of $P_n\mathbb{C}$ into six types. Cecil and Ryan [4] extensively investigated in [4] a real hypersurface whose structure vector $\xi = -JN$ for every unit normal vector N of the real hypersurface is a principal curvature vector, which is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map. By making use of these results and the mentioned work of Takagi, Kimura [5] proved the local classification theorem for

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real hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant and ξ is a principal curvature vector. Also, Bernt [2] classified locally real hypersurfaces with constant principal curvatures of $H_n\mathbb{C}$ under the condition that ξ is a principal curvature vector, where the structure vector field ξ is defined as in the case of $P_n\mathbb{C}$.

Let M be a real hypersurface of $\widetilde{M}_n(c)$. Then M has an almost contact metric structure (η, ϕ, g) induced from the Kählerian metric tensor $\widetilde{g}$ and complex structure tensor J of $\widetilde{M}_n(c)$ (see Sections 1 and 2). We denote by ∇, A the Levi–Civita connection with respect to g and the shape operator of M, respectively. It is well-known that there is no real hypersurface M in $\widetilde{M}_n(c)$, $c \neq 0$ with parallel shape operator field ($\nabla A = 0$).

In the present paper, we consider real hypersurfaces of Kählerian manifolds with the CR structures associated with almost contact metric structures. In general, real hypersurfaces in Kählerian manifolds have an integrable CR structures associated with the almost contact metric structures, but their associated Levi forms are not always hermitian and non-degenerate. On the other hand, Tanaka ([13]) defined a canonical affine connection on a pseudo-hermitian, non-degenerate, integrable CR manifold. For contact metric manifolds, their associated CR structures are pseudo-hermitian and strongly pseudo-convex, but they are not in general integrable. In this situation, Tanno ([14]) defined the generalized Tanaka connection for contact metric manifolds by relaxing the integrability condition of their associated CR structures.

Now, for a non-zero real number k we define the generalized Tanaka connection $\check{\nabla}^{(k)}$ for real hypersurfaces in Kählerian manifolds by the naturally extended one of Tanno's generalized Tanaka connection ($k = 1$) for contact metric manifolds. The generalized Tanaka connection $\check{\nabla}^{(k)}$ coincides with the Tanaka connection if real hypersurfaces satisfy $\phi A + A\phi = 2k\phi$ (see Proposition 3 in Section 2). In Section 3 we find that there are real hypersurfaces M of $\widetilde{M}_n(c)$ such that their almost contact metric structures are not contact metric structures, but their associated integrable CR structures are pseudo-hermitian, strongly pseudo-convex. And further we show that $\check{\nabla}^{(k)}$ defined on those M for some $k \neq 1$ coincides with the Tanaka connection (see Remark 1). The main purpose of the present paper is to classify the real hypersurfaces of $\widetilde{M}_n(c)$ whose shape operator is parallel with respect to the generalized Tanaka connection $\check{\nabla}^{(k)}$. More specifically, in Section 4, we prove the

Main Theorem. *Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$ with constant holomorphic sectional curvature c. If the shape operator A of M in $\widetilde{M}_n(c)$ is $\check{\nabla}^{(k)}$-parallel ($\check{\nabla}^{(k)}A = 0$), then ξ is a principal curvature vector. Furthermore,*

(I) *If $\widetilde{M}_n(c) = P_n\mathbb{C}$, then M is locally congruent to one of the following:*
  (1) *a tube of radius r over a totally geodesic $P_m\mathbb{C}$ ($0 \le m \le n-1$), where $0 < r < \frac{\pi}{2}$,*
  (2) *a tube of radius r over a complex quadric Q_{n-1},where $0 < r < \frac{\pi}{4}$.*

(II) *If $\widetilde{M}_n(c) = H_n\mathbb{C}$, then M is locally congruent to one of the following:*
  (1) *a horosphere in $H_n\mathbb{C}$,*
  (2) *a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $H_m\mathbb{C}$ ($0 \le m \le n-1$),*
  (3) *a tube of radius $r \in \mathbb{R}_+$ over a totally real hyperbolic space $H_n\mathbb{R}$.*

(III) *If $\widetilde{M}_n(c) = E_n\mathbb{C}$, then M is locally congruent to one of the following:*
  (1) *a sphere $S^{2n-1}(r)$ of radius $r \in \mathbb{R}_+$,*
  (2) *a plane $\mathbb{E}^{2n-1}$,*
  (3) *it a generalized cylinder $S^p(r) \times \mathbb{E}^q$ of radius $r \in \mathbb{R}_+$, where p is an odd number and $p + q = 2n - 1$.*

In this paper, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

1. Almost contact metric structures and the associated CR structures

First, we give a brief review of several fundamental concepts and formulas which we will need later on. An odd-dimensional Riemannian manifold M with metric tensor g is said to have an almost contact metric structure if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \qquad (1.1)$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

From (1.1) we get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \qquad (1.2)$$

We call (η, ϕ, g) an almost contact metric structure of M and $M = (M; \eta, \phi, g)$ an almost contact metric manifold. The tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = \mathfrak{D}_p \oplus \{\xi\}_p$ (direct sum), where we denote $\mathfrak{D}_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $\mathfrak{D} : p \to \mathfrak{D}_p$ defines a distribution orthogonal to ξ. For an almost contact metric manifold $M = (M; \eta, \phi, g)$, one may define naturally an almost complex structure on the product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ denotes the real line. If the almost complex structure is integrable, M is said to be normal. The integrability condition for the almost complex structure is the vanishing of the tensor $[\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ. Also, for an almost contact metric manifold M we define its fundamental 2-form Φ by $\Phi(X, Y) = g(\phi X, Y)$. If

$$\Phi = d\eta, \tag{1.3}$$

M is called a contact metric manifold. A normal contact metric manifold is called a Sasakian manifold. For more details about the general theory of almost contact metric manifolds, we refer to [3], [8], for example.

On the other hand, for an almost contact metric manifold $M = (M; \eta, \phi, g)$, the restriction $\bar{\phi} = \phi|\mathfrak{D}$ of ϕ to $\mathfrak{D}$ defines an almost complex structure to $\mathfrak{D}$. If the associated Levi form L, defined by $L(X, Y) = d\eta(X, \bar{\phi}Y)$, $X, Y \in \mathfrak{D}$, is hermitian, then $(\eta, \bar{\phi})$ is called a pseudo-hermitian CR structure and further, if its Levi form is non-degenerate (positive or negative definite, resp.), then $(\eta, \bar{\phi})$ is called a non-degenerate (strongly pseudo-convex, resp.) pseudo-hermitian CR structure. Moreover, if the following conditions are satisfied:

$$[\bar{\phi}X, \bar{\phi}Y] - [X, Y] \in \mathfrak{D} \tag{1.4}$$

and

$$[\bar{\phi}, \bar{\phi}](X, Y) = 0 \tag{1.5}$$

for all $X, Y \in \mathfrak{D}$, where $[\bar{\phi}, \bar{\phi}]$ is the Nijenhuis torsion of $\bar{\phi}$, then the pair $(\eta, \bar{\phi})$ is called a pseudo-hermitian, non-degenerate (strongly pseudo-convex, resp.), integrable CR structure associated with the almost contact metric structure (η, ϕ, g). In particular, for a contact metric manifold its associated CR structure is pseudo-hermitian, strongly pseudo-convex but is not in general integrable. For further details about CR structures, we refer for example to [1], [14].

2. The generalized Tanaka connection for real hypersurfaces

Let M be a real hypersurface of a Kählerian manifold $\widetilde{M} = (\widetilde{M}; J, \widetilde{g})$ and N a global unit normal vector on M. By $\widetilde{\nabla}$, A we denote the Levi–Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric of M induced from $\widetilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). We denote by V_λ the eigenspace associated with an eigenvalue λ. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi. \tag{2.1}$$

We easily see that the structure (η, ϕ, g) is an almost contact metric structure on M. From the condition $\widetilde{\nabla} J = 0$, the relations (2.1) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.2}$$

$$\nabla_X \xi = \phi AX. \tag{2.3}$$

By using (2.2) and (2.3), we see that a real hypersurface in a Kähler manifold always satisfies (1.4) and (1.5), the integrability condition of the associated CR structure. From (1.3) and (2.3) we have

Proposition 2. *Let $M = (M; \eta, \phi, g)$ be a real hypersurface of a Kählerian manifold. The almost contact metric structure of M is contact metric if and only if $\phi A + A\phi = \pm 2\phi$, where $\pm$ is determined by the orientation.*

The Tanaka connection ([13]) is the canonical affine connection defined on non-degenerate integrable CR manifold. S. Tanno ([14]) defined the generalized Tanaka connection for contact metric manifolds by the unique linear connection which coincides with the Tanaka connection if the associated CR structure is integrable. We define the generalized Tanaka connection for real hypersurfaces of Kählerian manifolds by the naturally extended one of S. Tanno's generalized Tanaka connection for contact metric manifolds.

Now we recall the generalized Tanaka connection $\check{\nabla}$ for contact metric manifolds;

$$\check{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y$$

for all vector fields X and Y.

Thus, by taking account of (2.3), the generalized Tanaka connection $\check{\nabla}^{(k)}$ for real hypersurfaces of Kählerian manifolds is naturally defined by

$$\check{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y, \tag{2.4}$$

where k is a non-zero real number. We put $F_X Y = g(\phi AX, Y)\xi - \eta(Y)$ $\phi AX - k\eta(X)\phi Y$. Then the torsion tensor $\check{T}^{(k)}$ is given by $\check{T}^{(k)}(X,Y) = F_X Y - F_Y X$. Also, by using (1.2), (1.3), (2.2), (2.3) and (2.4) we can see that

$$\check{\nabla}^{(k)}\eta = 0, \quad \check{\nabla}^{(k)}\xi = 0, \quad \check{\nabla}^{(k)} g = 0, \quad \check{\nabla}^{(k)}\phi = 0. \tag{2.5}$$

and

$$\check{T}^{(k)}(X,Y) = 2d\eta(X,Y)\xi, \quad X,\ Y \in \mathfrak{D}.$$

We note that the associated Levi form is
$L(X,Y) = \frac{1}{2} g\left((\bar{\phi}\bar{A} + \bar{A}\bar{\phi})X, \bar{\phi}Y\right)$, where we denote by $\bar{A}$ the restriction A to $\mathfrak{D}$. If M satisfies $\phi A + A\phi = 2k\phi$, then we see that the associated CR structure is pseudo-hermitian, strongly pseudo-convex and further satisfies $\check{T}^{(k)}(\xi, \phi Y) = -\phi \check{T}^{(k)}(\xi, Y)$, and hence the generalized Tanaka connection $\check{\nabla}$ coincides with the Tanaka connection (see [14]). That is, we have

Proposition 3. *Let $M = (M; \eta, \phi, g)$ be a real hypersurface of a Kählerian manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the associated CR-structure is pseudo-hermitian, strongly pseudo-convex, integrable, and further the generalized Tanaka connection $\check{\nabla}^{(k)}$ coincides with the Tanaka connection.*

3. Real hypersurfaces of a complex space form

Let $\widetilde{M} = \widetilde{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature c and M a real hypersurface of $\widetilde{M}$. Then we have the following Gauss and Codazzi equations:

$$
\begin{aligned}
(3.1)\ R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y \\
&+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \\
&+ g(AY,Z)AX - g(AX,Z)AY,
\end{aligned}
$$

$$(3.2)\quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}$$

for any tangent vector fields X, Y, Z on M.

We now suppose that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$. Differentiating covariantly, and then by using (2.3) we have

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX,$$

and further by using (3.2) we obtain

$$(\nabla_\xi A)X = \frac{c}{4}\phi X + (X\alpha)\xi + \alpha\phi AX - A\phi AX$$

for any vector field X on M. From this, we get $X\alpha = (\xi\alpha)\eta(X)$, and hence we have

$$2A\phi AX - \frac{c}{2}\phi X = \alpha(\phi A + A\phi)X.$$

Thus we have the following (see also [7])

Proposition 4. *Let M be a real hypersurface of $\widetilde{M}_n(c)$. If we assume that ξ is a principal curvature vector and $AX = \lambda X$ for X orthogonal to ξ, then $(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X$.*

Proposition 4 and the results in [9], [10] and [15] produce

Theorem 2. *Let M be a real hypersurface of $\widetilde{M}_n(c)$. Suppose that M satisfies $\phi A + A\phi = 2k\phi$ for some non-zero constant k.*

(I) *If $\widetilde{M}_n(c) = P_n\mathbb{C}$, then M is locally congruent to one of the following:*
   (1) *a geodesic hypersphere, that is, a tube of radius r over $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*

(2) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$.*

(II) *If $\widetilde{M}_n(c) = H_n\mathbb{C}$, then M is locally congruent to one of the following:*
(1) *a horosphere in $H_n\mathbb{C}$,*
(2) *a geodesic hypersphere, that is, a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $H_{n-1}\mathbb{C}$,*
(3) *a tube of radius $r \in \mathbb{R}_+$ over a totally real hyperbolic space $H_n\mathbb{R}$.*

(III) *If $\widetilde{M}_n(c) = E_n\mathbb{C}$, then M is locally congruent to one of the following:*
(1) *a sphere $S^{2n-1}(r)$ of radius $r \in \mathbb{R}_+$,*
(2) *a generalized cylinder $S^{n-1}(r) \times \mathbb{E}^n$ of radius $r \in \mathbb{R}_+$.*

Remark 1. From Proposition 2, we see that the almost contact metric structure of M appearing in Theorem 2 is a contact metric structure only for the very special case which is determined by $k = \pm 1$, where $\pm$ depends on the orientation. More precisely, with the help of the tables in [2] and [12], we see that the almost contact metric structures are contact metric only for a geodesic hypersphere of radius $\frac{\pi}{4}$ in $P_n\mathbb{C}$, for a horosphere in $H_n\mathbb{C}$, for a unit sphere $S^{2n-1}(1)$ or a generalized cylinder $S^{n-1}(\frac{1}{2}) \times \mathbb{E}^n$ in $E_n\mathbb{C}$. Thus, together with Proposition 3, we see that the real hypersurfaces appearing in Theorem 2, except those just mentioned, do not have contact metric structures but their associated CR structures are pseudo-hermitian, strongly pseudo-convex, integrable, and further the generalized Tanaka connection $\check{\nabla}^{(k)}$ defined on them coincides with the Tanaka connection.

The following Theorems 3 and 4 are very useful for the proof of our Main Theorem in Section 4.

Theorem 3 ([6]). *Let M be a real hypersurface of $P_n\mathbb{C}$. Then the shape operator satisfies $g((\nabla_X A)Y, Z) = 0$ for any tangent vectors X, Y and Z which are orthogonal to ξ and ξ is a principal curvature vector if and only if M is locally congruent to one of the following:*

(1) *a tube of radius r over a totally geodesic $P_m\mathbb{C}$ $(0 \le m \le n-1)$, where $0 < r < \frac{\pi}{2}$,*
(2) *a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$.*

Theorem 4 ([10]). *Let M be a real hypersurface of $H_n\mathbb{C}$. Then the shape operator satisfies $g((\nabla_X A)Y, Z) = 0$ for any tangent vectors X, Y and Z which are orthogonal to ξ and ξ is a principal curvature vector if and only if M is locally congruent to one of the following:*

(1) *a horosphere in $H_n\mathbb{C}$,*
(2) *a tube of radius $r \in \mathbb{R}_+$ over a totally geodesic $H_m\mathbb{C}$ $(0 \le m \le n-1)$,*
(3) *a tube of radius $r \in \mathbb{R}_+$ over a totally real hyperbolic space $H_n\mathbb{R}$.*

4. Proof of the Main Theorem

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$, $c \neq 0$. We define a vector field U on M by $U = \nabla_\xi \xi$. Then, from (1.2) and (2.3) we easily observe that

$$\begin{aligned} &g(U,\xi) = 0, \quad g(U, A\xi) = 0, \\ &\| U \|^2 = g(U,U) = \beta - \alpha^2, \end{aligned} \tag{4.1}$$

where $\beta = g(A^2\xi, \xi)$. From (1.2), (2.3) and (4.1) we have at once the following

Lemma 1. *$A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.*

Taking account of (2.4), we have

$$\begin{aligned} \left(\check{\nabla}^{(k)}_X A\right) Y &= \check{\nabla}^{(k)}_X AY - A\check{\nabla}^{(k)}_X Y \\ &= (\nabla_X A)Y + F_X AY - AF_X Y \\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi \\ &\quad - \eta(AY)\phi AX - k\eta(X)\phi AY \\ &\quad - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y \end{aligned} \tag{4.2}$$

for any vector fields X and Y on M. First, we prove that ξ is a principal curvature vector, i.e., $A\xi = \alpha\xi$. From (4.2) we see that the condition $\check{\nabla}^{(k)}_\xi A = 0$ implies that

$$\begin{aligned} (\nabla_\xi A)X = k(\phi AX - A\phi X) + \eta(AX)U \\ - g(AX, U)\xi + g(X, U)A\xi - \eta(X)AU, \end{aligned} \tag{4.3}$$

for any vector field X on M. From (4.1) and (4.3) we easily see that $\xi\alpha = 0$. The above equation (4.3) together with (3.2), yields

$$\begin{aligned} (\nabla_X A)\xi = k(\phi AX - A\phi X) - \frac{c}{4}\phi X \\ - g(U, AX)\xi + \eta(AX)U - \eta(X)AU + g(U, X)A\xi. \end{aligned} \tag{4.4}$$

With $U = \phi A\xi$ and from (2.2), (2.3) and (4.4) we have

$$
\begin{aligned}
(4.5)\quad \nabla_X U = \frac{c}{4}X &+ (\alpha - k)AX - k\phi A\phi X + \phi A\phi AX \\
&- \eta(A^2X)\xi + (\alpha + k)\eta(AX)\xi - \frac{c}{4}\eta(X)\xi - \eta(AX)A\xi \\
&+ g(U, X)U - \eta(X)\phi AU.
\end{aligned}
$$

Also, it follows from (4.2) and $(\check{\nabla}^{(k)}_X A)\xi = 0$ that

$$
(4.6)\qquad (\nabla_X A)\xi = g(AX, U)\xi + \alpha\phi AX + k\eta(X)U - A\phi AX
$$

for any vector field X on M. By using (2.3) it follows that

$$
\begin{aligned}
X\alpha &= g((\nabla_X A)\xi, \xi) - 2g(AX, U), \\
X\beta &= 2g((\nabla_X A)\xi, A\xi) - 2g(AX, \phi A^2\xi)
\end{aligned}
$$

for any vector field X on M. Together with (4.6) we can show the following

Lemma 2. *If M satisfies $\check{\nabla}^{(k)}A = 0$, then $\alpha\,(= g(A\xi, \xi))$ and $\beta\,(= g(A^2\xi, \xi))$ are constant.*

From (4.4) and (4.6) we get

$$
\begin{aligned}
& kg(\phi AX, Y) - kg(A\phi X, Y) - \frac{c}{4}g(\phi X, Y) - g(U, AX)\eta(Y) \\
(4.7)\qquad & \qquad + \eta(AX)g(U, Y) - \eta(X)g(AU, Y) + g(U, X)\eta(AY) \\
& = g(AX, U)\eta(Y) + \alpha g(\phi AX, Y) \\
& \qquad + k\eta(X)g(U, Y) - g(A\phi AX, Y)
\end{aligned}
$$

for any vector fields X and Y on M. We put $Y = \xi$ in (4.7) and by taking account of (1.1) and (1.2), we have

$$
(4.8)\qquad (\alpha + k)g(X, U) = 3g(AX, U)
$$

for any vector field X on M. The equation (4.8) yields that

$$
(4.9)\qquad 3AU = (\alpha + k)U.
$$

If we put $X = U$ and $Y = \phi U$ in (4.7), then by using (4.1) and (4.9) we have

$$\left(k^2 - \frac{3}{4}c - 3\beta + 2\alpha^2\right)(\beta - \alpha^2) = -(\alpha - 2k)g(A\phi U, \phi U). \tag{4.10}$$

Further, from (3.2) and (4.6) we get

$$(\nabla_\xi A)X = \frac{c}{4}\phi X + g(AX, U)\xi + \alpha\phi AX + k\eta(X)U - A\phi AX \tag{4.11}$$

for any vector field X on M. It follows from (4.11) that

$$\begin{aligned}\frac{c}{2}g(\phi X, Y) + g(AX, U)\eta(Y) - g(AY, U)\eta(X) + \alpha g(\phi AX, Y)\\ - \alpha g(\phi AY, X) + k\eta(X)g(U, Y) - k\eta(Y)g(U, X)\\ - g(A\phi AX, Y) + g(A\phi AY, X) = 0,\end{aligned} \tag{4.12}$$

for all vector fields X and Y on M. By putting $X = A\xi$ and $Y = U$ in (4.12) and by taking account of (1.1), (1.2) and (4.9), we obtain

$$(\alpha - 2k)g(A\phi U, \phi U) = -\left(\alpha^2 + k\alpha + \frac{3}{2}c\right)(\beta - \alpha^2). \tag{4.13}$$

Thus, from (4.10) and (4.13) we obtain

$$\left(3\beta - \alpha^2 + k\alpha - k^2 + \frac{9}{4}c\right)(\beta - \alpha^2) = 0. \tag{4.14}$$

In view of Lemma 1 and Lemma 2, we now suppose that $\beta - \alpha^2 \neq 0$. Then from (4.14) we get

$$3(\beta - \alpha^2) = k^2 - \alpha k - 2\alpha^2 - \frac{9}{4}c. \tag{4.15}$$

From (4.15) we see that the inequality $k^2 - \alpha k - 2\alpha^2 - \frac{9}{4}c > 0$ must hold independently of k for a fixed α, hence we have $\alpha^2 + c < 0$. At the same time, it must hold that $-2\alpha^2 - k\alpha + k^2 - \frac{9}{4}c > 0$, for any α for a fixed k, which yields $k^2 - 2c < 0$. We have a contradiction. So, by Lemma 1 we conclude that $A\xi = \alpha\xi$ on M.

Thus, we see that (4.12) is reduced to

$$\frac{c}{2}\phi X + \alpha(\phi A + A\phi)X - 2A\phi AX = 0. \tag{4.16}$$

The equation (4.2), together with $A\xi = \alpha\xi$, shows that

$$g((\nabla_X A)Y, Z) = 0 \tag{4.17}$$

for any vector fields X, Y and Z orthogonal to ξ.

Now, we classify a real hypersurface of a complex space form $\widetilde{M}_n(c)$ which satisfying $\check{\nabla}^{(k)}A = 0$ according as $c > 0$, $c < 0$ or $c = 0$ respectively.

(I) $\widetilde{M}_n(c) = P_n\mathbb{C}$;

By (4.17) and Theorem 3, we see that M is locally congruent to one of real hypersurfaces (1) or (2) in Theorem 3. Conversely, by using (2.5) we can see that a real hypersurface M of case (1) or (2) satisfies $(\check{\nabla}^{(k)}A)\xi = 0$. Further M satisfies $(\check{\nabla}^{(k)}_\xi A)X = 0$ for any vector field X orthogonal to ξ. In fact, from (4.2) and by taking account of (3.2) and $A\xi = \alpha\xi$, we get

$$\left(\check{\nabla}^{(k)}_\xi A\right)X = \alpha\phi AX - A\phi AX + \frac{c}{4}\phi X - k(\phi AX - A\phi X)$$

for any vector field X orthogonal to ξ. Assume $X \in V_\lambda$. Then together with (4.16) we have

$$\left(\check{\nabla}^{(k)}_\xi A\right)X = \frac{\alpha - 2k}{2\lambda - \alpha}\left(\lambda^2 - \alpha\lambda - \frac{c}{4}\right)\phi X. \tag{4.18}$$

because of $2\lambda - \alpha \neq 0$. In view of the table in [12], we see that a real hypersurface of case (1) satisfies $\lambda^2 - \alpha\lambda - \frac{c}{4} = 0$. Also, we see that for a real hypersurface of case (2) $\alpha(= 2\cot 2r)$ is non-zero constant. So, from (4.18) we see that M of case (1) or (2) satisfies $(\check{\nabla}^{(k)}_\xi A)X = 0$ for any vector field X orthogonal to ξ.

(II) $\widetilde{M}_n(c) = H_n\mathbb{C}$;

By (4.17) and Theorem 4, we see that M is locally congruent to one of real hypersurfaces (1), (2) or (3) in Theorem 4. Conversely, by using (2.5) we can see that a real hypersurface M of case (1), (2) or (3) satisfies $(\check{\nabla}^{(k)}A)\xi = 0$. Further M satisfies $(\check{\nabla}^{(k)}_\xi A)X = 0$ for any vector field X

orthogonal to ξ. In fact, from (4.2) taking account of (3.2) and $A\xi = \alpha\xi$, we get

$$\left(\check{\nabla}^{(k)}_{\xi} A\right) X = \alpha\phi AX - A\phi AX + \frac{c}{4}\phi X - k(\phi AX - A\phi X) \tag{4.19}$$

for any vector field X orthogonal to ξ. Assume $X \in V_\lambda$. If M is locally congruent to (1) horosphere, then the table in [2] says that M has two principal curvatures λ and $\alpha(= 2\lambda)$ with multiplicity $2n-2$ and 1, respectively. So, from (4.19) we see that M satisfies $(\check{\nabla}^{(k)}_{\xi} A)X = 0$. If M is of the case (2) or (3), then from the table in [2] we see that $\alpha \neq 2\lambda$ and from (4.16) and (4.19) we have

$$\left(\check{\nabla}^{(k)}_{\xi} A\right) X = \frac{\alpha - 2k}{2\lambda - \alpha}\left(\lambda^2 - \alpha\lambda - \frac{c}{4}\right)\phi X.$$

With the help of the table in [2], we see that a real hypersurface of case (1) satisfies $\lambda^2 - \alpha\lambda - \frac{c}{4} = 0$. Also, we see that for a real hypersurface of case (2) $\alpha(= 2\tanh 2r)$ is non-zero constant. So, from (4.19) we see that a real hypersurface of case (2) or (3) satisfies $(\check{\nabla}^{(k)}_{\xi} A)X = 0$ for any vector field X orthogonal to ξ.

(III) $\widetilde{M}_n(c) = E_n\mathbb{C}$;

We know that $A\xi = \alpha\xi$ and α is constant. Now, we fix a point $p \in M$ and we assume that in a sufficiently small neigborhood of p, $AY = \lambda Y$, where λ is a smooth function and Y is a unit vector field which is orthogonal to ξ. Then from (4.17) we see that $X\lambda = g((\nabla_X A)Y, Y) = 0$ for any vector field X orthogonal to ξ, and further by using (2.3), (3.2) and $c = 0$ we see that $\xi\lambda = g((\nabla_\xi A)Y, Y) = g((\nabla_Y A)\xi, Y) = 0$. Thus we can see that all principal curvatures are constant, and by virtue of Segre's result ([9]) we see that M has at most two distinct principal curvatures, say α and λ. Further, from (4.16) we see that

$$\lambda(\lambda - \alpha) = 0 \quad \text{or} \quad \alpha(\lambda - \alpha) = 0. \tag{4.20}$$

From (4.20), and by taking account of Proposition 4 and Segre's work, we can see that M is locally congruent to one of the following: (1) S^{2n-1}, (2) $\mathbb{E}^{2n-1}$, (3) $S^{n-1} \times E^n$ or $S^p \times \mathbb{E}^q$, where p is an odd number and

$p+q=2n-1$. On the other hand, from (4.2) and taking into account (3.2) and $A\xi=\alpha\xi$, we get

$$\left(\check{\nabla}^{(k)}_{\xi} A\right) X = \alpha\phi AX - A\phi AX - k(\phi AX - A\phi X), \tag{4.21}$$

for any vector field X orthogonal to ξ. We assume that $X \in V_\lambda$. If $2\lambda = \alpha$, then from (4.16) we have $\lambda = \alpha = 0$. Otherwise, $2\lambda \neq \alpha$, and in this case we have

$$(\alpha - 2k)\lambda(\lambda - \alpha) = 0. \tag{4.22}$$

Thus, since $k \neq 0$, together with (4.22) we conclude that M is locally congruent to one of the following: (1) S^{2n-1}, (2) $\mathbb{E}^{2n-1}$, (3) $S^p \times \mathbb{E}^q$, where p is an odd number and $p+q=2n-1$. Furthermore, we easily see that a real hypersurface of cases (1), (2) or (3) satisfies $g((\nabla_X A)Y, Z) = 0$ for any vector fields X, Y and Z orthogonal to ξ.

Summing up all the cases, we have our Main Theorem.

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