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On the uniqueness of rings of coefficients in skew polynomial rings

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Abstract. Let R be a ring, let α be an automorphism of R, and let δ be an α-derivation of R. The ring R is said to be *strongly invariant in* a skew polynomial ring $R[X;\alpha,\delta]$ if for any isomorphism Ψ of $R[X;\alpha,\delta]$ to any skew polynomial ring $S[Y,\beta,\partial]$, there holds $\Psi(R)=S$. We consider what conditions imply that R is strongly invariant in T.

1. Introduction

Throughout this paper, all rings are associative with unit. Let α be an automorphism of a ring R. An α-derivation of R is any additive map $\delta : R \to R$ such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The skew polynomial ring $R[X;\alpha,\delta]$ is a ring of polynomials in X over R with the usual addition and with multiplication subject to the rule $Xa = \alpha(a)X + \delta(a)$ for all $a \in R$ (see [2, Definition, p. 10]). A ring T may be viewed as a skew polynomial ring over a subring T' if and only if there exists an isomorphism Φ from a skew polynomial ring $R[X,\alpha,\delta]$ to T such that $\Phi(R) = T'$. In fact, if there exists such an isomorphism Φ, then $\alpha' = \Phi\beta\Phi^{-1}$ is an automorphism of $T' = \Phi(R)$, $\delta' = \Phi\delta\Phi^{-1}$ is an α'-derivation of T', T is a free left T'-module with the basis $1, \Phi(X), \Phi(X)^2, \dots$ and $\Phi(X)a = \alpha'(a)\Phi(X) + \delta'(a)$ for all $a \in T'$. Therefore we obtain $T = T'[\Phi(X);\alpha',\delta']$. There may possibly be many different ways to represent T as a skew polynomial ring over a subring. For example, consider the first Weyl algebra $A_1(K)$ over a field K. This is an algebra over K generated by

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x, y with relation $xy - yx = 1$. We may write $A_1(K) = K[y]\left[x; 1, \frac{d}{dy}\right] = K[x]\left[y; 1, -\frac{d}{dx}\right]$. Hence two different subrings $K[y]$, $K[x]$ can become rings of coefficients of $A_1(K)$. In this paper, we consider what conditions imply R to be unique as a ring of coefficients of $T = R[X; \alpha, \delta]$.

2. Strongly invariant rings

To discuss the uniqueness of rings of coefficients in skew polynomial rings, we need the following two definitions.

Definition 1. A ring R is *strongly invariant in* a skew polynomial ring $R[X; \alpha, \delta]$ if for any isomorphism Ψ of $R[X; \alpha, \delta]$ to any skew polynomial ring $S[Y; \beta, \partial]$, there holds $\Psi(R) = S$.

Definition 2. A ring R is *reduced* if R contains no nonzero nilpotent elements. A reduced ring R with an automorphism α is α-*reduced* if, for any $r \in R$, $r\alpha(r) = 0$ implies $r = 0$.

We give an example of a reduced ring which is not α-reduced. Let K be a field, and let $R = K \oplus K$. Then R is reduced. Consider the automorphism α of R given by $\alpha(a, b) = (b, a)$. Then $(1, 0)\alpha(1, 0) = (1, 0)(0, 1) = (0, 0)$. Therefore R is not α-reduced.

Now we begin with the following lemma.

Lemma 1. *Let R be a ring, let α be an automorphism of R, and let δ be an α-derivation of R. Suppose that R is α-reduced and let $a, b \in R$.*

(1) *If $ab = 0$, then $\alpha^i(a)\alpha^j(b) = 0$ for any integers i, j.*

(2) *If $ab = 0$, then $\delta^i(a)\delta^j(b) = 0$ for any non-negative integers i, j.*

(3) *If $ab = 0$, then $aX^m bX^n = 0$ in $R[X; \alpha, \delta]$ for any nonnegative integers m, n.*

Proof. (1) Assume $ab = 0$. Then $b\alpha(a)\alpha(b\alpha(a)) = b\alpha(ab)\alpha^2(a) = 0$. Since R is α-reduced, we have $b\alpha(a) = 0$. Since R is reduced, $(\alpha(a)b)^2 = 0$ implies $\alpha(a)b = 0$. Similarly $(ba)^2 = 0$ implies $ba = 0$. Hence, by the same way as above, we obtain $a\alpha(b) = 0$. Using these repeatedly, we obtain $\alpha^i(a)\alpha^j(b) = 0$ for any non-negative integers i, j. Take a positive integer n and apply α^{-n} to this equation, we have $\alpha^{i-n}(a)\alpha^{j-n}(b) = 0$. This proves the claim.

(2) Since R is reduced, $ab = 0$ implies $ba = 0$, and hence $b\alpha(a) = 0$ by (1). Since $0 = \delta(ab) = \alpha(a)\delta(b) + \delta(a)b$, $\{\alpha(a)\delta(b)\}^2 = -\delta(a)b\alpha(a)\delta(b) = 0$,

so that $\alpha(a)\delta(b) = 0$. Hence $a\delta(b) = 0$ by (1). Using $ba = 0$, we similarly obtain $\delta(a)b = 0$. Using these repeatedly, we can prove our claim.

(3) Using the rule $Xr = \alpha(r)X + \delta(r)$ for each $r \in R$, we can write $aX^m bX^n = c_{m+n}X^{m+n} + c_{m+n-1}X^{m+n-1} + \cdots + c_1X + c_0$. Then we see that $c_{m+n} = a\alpha^m(b)$, $c_{m+n-1} = \sum_{i=0}^{m-1} a\alpha^{m-i-1}\delta\alpha^i(a)$, and in general c_k is the sum of some terms of the form $a\alpha^{i_1}\delta^{j_1}\alpha^{i_2}\delta^{j_2}\cdots\alpha^{i_t}\delta^{j_t}(b)$ with $i_1 + j_1 + \cdots + i_t + j_t = m$. However, using (1) and (2), we see $a\alpha^{i_1}\delta^{j_1}\alpha^{i_2}\delta^{j_2}\cdots\alpha^{i_t}\delta^{j_t}(b) = 0$ for each $i_1, \dots, i_t, j_1, \dots, j_t$, and therefore $c_k = 0$ for $k = 0, 1, \dots, m+n$.

The following theorem improves [4, Proposition 3.4].

Theorem 2. *Let R be a ring, let α be an automorphism of R, and let δ be an α-derivation of R. If R is α-reduced, then the set of all units in $R[X;\alpha,\delta]$ equals the set of all units in R.*

Proof. Let $f(X) = \sum_{i=0}^m a_iX^i$ be a unit in $R[X,\alpha,\delta]$ and let $g(X) = \sum_{j=0}^n b_jX^j$ be its inverse. Then we can write $1 = f(X)g(X) = \sum_{k=0}^{m+n}(\sum_{i+j=k} a_iX^ib_jX^j) = c_{m+n}X^{m+n} + c_{m+n-1}X^{m+n-1} + \cdots + c_1X + c_0$. We prove that $f(X) \in R$. Suppose, on the contrary, that $m > 0$ and $a_m \neq 0$. We claim that $a_sb_t = 0$ for $s + t \geq m$. We can easily see that $c_{m+n} = a_m\alpha^m(b_n) = 0$. Thus we obtain $a_mb_n = 0$ by Lemma 1(1). This proves our claim for $s + t = m + n$. Let p be an integer such that $m+n > p \geq m$, and suppose that $a_sb_t = 0$ if $s+t > p$. We shall prove that $a_sb_t = 0$ when $s + t = p$. By Lemma 1(3), we have $\sum_{i+j=u} a_iX^ib_jX^j = 0$ for $u = m+n, m+n-1, \dots, p+1$. Hence we obtain

$$c_p = \sum_{i+j=p} a_i\alpha^i(b_j) = 0. \tag{1}$$

Since $a_sb_t = 0$ for $s + t > p$, $a_s\alpha^s(b_t) = 0$ for $s + t > p$ by Lemma 1(1), and hence $\alpha^s(b_t)a_s = 0$ for $s + t > p$ because R is reduced. Multipling the equation (1) on the right by a_p, we obtain

$$0 = \Big\{\sum_{i+j=p} a_i\alpha^i(b_j)\Big\}a_p = a_p\alpha^p(b_0)a_p.$$

Since R is reduced, $a_p\alpha^p(b_0) = 0$, so that $a_pb_0 = 0$ by Lemma 1(1). Now the equation (1) becomes

$$\sum_{\substack{i+j=p \\ j\geq 1}} a_i\alpha^i(b_j) = 0. \tag{2}$$

Multipling the equation (2) on the right by a_{p-1}, we have $a_{p-1}\alpha^{p-1}(b_1) \times a_{p-1} = 0$. Hence $a_{p-1}\alpha^{p-1}(b_1) = 0$, so that $a_{p-1}b_1 = 0$. Continuing this process, we have $a_ib_j = 0$ for all i, j with $i + j = p$. Thus we have proved $a_sb_t = 0$ for $s + t \geq m$. In particular, we have $a_mb_n = a_mb_{n-1} = \cdots = a_mb_0 = 0$. Thus $a_mX^mg(X) = 0$ by Lemma 1(3), and hence $(\sum_{i=0}^{m-1} a_iX^i)g(X) = 1$. Therefore we obtain $\sum_{i=0}^{m-1} a_iX^i = (\sum_{i=0}^{m-1} a_iX^i)g(X)f(X) = f(X) = \sum_{i=0}^{m} a_iX^i$. This implies $a_m = 0$, a contradiction. This completes the proof.

As a consequence of Theorem 2, we obtain the following corollary.

Corollary 3. *Let R be a ring, let α be an automorphism of R, and let δ be an α-derivation of R. Suppose that R is α-reduced and that R is generated by its units. Then $\Psi(R) = R$ for any automorphism Ψ of $R[X; \alpha, \delta]$.*

A ring is called an integral domain if the product of nonzero elements is always nonzero. For example, a division ring is an integral domain.

Corollary 4. *If R is an integral domain generated by its units, then R is strongly invariant in $R[X; \alpha, \delta]$ for any automorphism α and for any α-derivation δ.*

Proof. Let S be a ring with an automorphism β and with a β-derivation ∂, and assume that $\Psi : R[X; \alpha, \delta] \to S[Y, \beta; \partial]$ is an isomorphism. Since S also is an integral domain, the set of all units in $S[Y; \beta, \partial]$ equals the set of all units in S by Theorem 2. Hence, by hypothesis, we have $\Psi(R) \subseteq S$. Clearly $\Psi(X) \notin S$, and so we can write $\Psi(X) = s_kY^k + \cdots + s_1Y + s_0$ with some $s_0, \ldots, s_k(\neq 0) \in S$ and some $k > 0$. We have to prove $\Psi(R) = S$. Suppose, on the contrary, that $\Psi(R) \subsetneqq S$ and take an element $s \in S - \Psi(R)$. Then there is $f(X) = r_nX^n + r_{n-1}X^{n-1} + \cdots + r_1X + r_0 \in R[X; \alpha, \delta]$ with $n > 0$ and some $r_0, \ldots, r_n(\neq 0) \in R$ such that $\Psi(f(X)) = s$. Then $\Psi(r_n)\Psi(X)^n + \cdots + \Psi(r_1)\Psi(X) + (\Psi(r_0) - s) = 0$. Since the coefficient of Y^{nk} is zero, we obtain $\Psi(r_n)s_k\beta^k(s_k)\beta^{2k}(s_k) \cdots \beta^{(n-1)k}(s_k) = 0$. Since $r_n \neq 0$ and $s_k \neq 0$, this is a contradiction. Consequently we obtain $\Psi(R) = S$.

An integral domain R is called a local domain if $R/J(R)$ is a division ring, where $J(R)$ denotes the Jacobson radical of R. It is easy to see that a local domain R is generated by its units. Hence, by Corollary 4, R is strongly invariant in any skew polynomial ring $R[X; \alpha, \delta]$. We give an example of a commutative local domain with a non-trivial automorphism α and with a non-trivial α-derivation.

Example. Let $K[[x]]$ denote the ring of formal power series over a field K, and α the automorphism of $K[[x]]$ defined by $\alpha(f(x)) = f(-x)$ for all $f(x) \in K[[x]]$. We define a map $\delta : K[[x]] \to K[[x]]$ by

$$\delta\left(\sum_{i=0}^{\infty} a_i x^i\right) = -\sum_{i=0}^{\infty} a_{2i+1} x^{2i}.$$

We can easily see that δ is an α-derivation of $K[[x]]$. Since $K[[x]]$ is a local domain, it is generated by its units. By Corollary 4, $K[[x]]$ is strongly invariant in $K[[x]][Y;\alpha,\delta]$.

Recall that R is said to be *von Neumann regular* if, for each element a of R, there exists an element x of R such that $a = axa$. A reduced von Neumann regular ring is called a *strongly regular ring*. It is well-known that a von Neumann regular ring R is strongly regular if and only if every idempotent of R is central.

Lemma 5. *Let R be a ring, let α be an automorphism of R, and let δ be an α-derivation of R. If R is α-reduced, then $R[X;\alpha,\delta]$ is reduced. In this case, $\alpha(e) = e$ and $\delta(e) = 0$ for any idempotent $e \in R$. Conversely, if R is a strongly regular ring and if $R[X;\alpha,\delta]$ is reduced, then R is α-reduced.*

Proof. Let R be a α-reduced ring. Suppose, on the contrary, that $R[X;\alpha,\delta]$ is not reduced. Then there exists a nonzero element $f \in R[X;\alpha,\delta]$ such that $f^2 = 0$. Since R is reduced, $f \notin R$. Let $f = \sum_{i=0}^{m} a_i X^i$ with $a_0, \dots, a_m (\neq 0) \in R$. Since $f^2 = 0$, we have $a_m \alpha^m(a_m) = 0$. By Lemma 1(1) we obtain $a_m^2 = 0$, and hence $a_m = 0$, a contradiction. Therefore $R[X;\alpha,\delta]$ is reduced. Note that every idempotent in a reduced ring is central (see [5, Lemma I.12.2, p. 40]). Let e be any idempotent in R. Then e is central in $R[X;\alpha,\delta]$, and hence $eX = Xe = \alpha(e)X + \delta(e)$. This implies $\alpha(e) = e$ and $\delta(e) = 0$. Next assume that R is a strongly regular ring. If R is not α-reduced, then there exists a nonzero element $r \in R$ such that $r\alpha(r) = 0$. Since R is strongly regular, there exists an element $x \in R$ such that $r^2x = r$ and $rx = xr$. If we set $e = rx$, then e is a nonzero central idempotent of R and $e\alpha(e) = xr\alpha(r)\alpha(x) = 0$. Let $f = eXe - eX$. Then $f^2 = 0$, but $f = e(\alpha(e)X + \delta(e))e - eX = e\delta(e) - eX \neq 0$. Therefore $R[X;\alpha,\delta]$ is not reduced.

The following theorem generalizes [1, Theorem 3] and also improves [4, Theorem 4.5].

Theorem 6. *Let R be a strongly regular ring, let α be an automorphism of R, and let δ be an α-derivation of R. Then the following statements are equivalent:*

(1) *R is strongly invariant in $R[X;\alpha,\delta]$.*

(2) *$\Psi(R)=R$ for any automorphism Ψ of $R[X;\alpha,\delta]$.*

(3) *R is α-reduced.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Suppose that R is not α-reduced. Then, by the same way as in the proof of Lemma 5, we can find a nonzero central idempotent e of R such that $e\alpha(e)=0$ and $f=eXe-eX=e\delta(e)-eX\neq 0$. Let Ψ denote the automorphism of $R[X;\alpha,\delta]$ defined by $\Psi(a)=(1+f)a(1-f)$ for all $a\in R[X;\alpha,\delta]$. Then $\Psi(e)=(1+f)e(1-f)=e-e\delta(e)+eX\notin R$, and hence $\Psi(R)\not\subseteq R$.

(3) $\Rightarrow$ (1). Since R is α-reduced, $R[X;\alpha,\delta]$ is reduced by Lemma 5. Let S be a ring with an automorphism β and a β-derivation ∂ and assume that $\Psi: R[X;\alpha,\delta]\to S[Y;\beta,\partial]$ is an isomorphism. Since $S[Y;\beta,\partial]$ is reduced, [3, Theorem 3.15] implies that the set of all idempotents in $S[Y;\beta,\partial]$ is contained in S. Let P be any prime ideal of R. Since R is strongly regular, for each $a\in P$, Ra is generated by a central idempotent (cf. [5, Proposition I.12.3, p. 40]). Since $P=\sum_{a\in P}Ra$, there exists a set $\{e_i \mid i\in I\}$ of central idempotents such that $P=\sum_{i\in I}Re_i$. By Lemma 5, $\alpha(e_i)=e_i$ and $\delta(e_i)=0$ for each $i\in I$. Hence P is stable under α and δ. Similarly $\sum_{i\in I}S\Psi(e_i)$ is stable under β and ∂. Since $\Psi(P(R[X;\alpha,\delta]))=\Psi(\sum_{i\in I}e_iR[X;\alpha,\delta])=\sum_{i\in I}\Psi(e_i)S[Y,\beta,\partial]$, Ψ induces an isomorphism $\bar{\Psi}:(R/P)[X;\bar{\alpha},\bar{\delta}]\to(S/\sum_{i\in I}S\Psi(e_i))[Y;\bar{\beta},\bar{\partial}]$, where $\bar{\alpha}$, $\bar{\delta}$, $\bar{\beta}$ and $\bar{\partial}$ are the maps induced by α, δ, β and ∂ respectively. Since R is strongly regular, R/P is a division ring. By Corollary 4 we obtain $\bar{\Psi}(R/P)=S/\sum_{i\in I}S\Psi(e_i)$, that is, $S=\Psi(R)+\sum_{i\in I}S\Psi(e_i)$. Hence we have $\Psi(R)\subseteq S$. We need to prove $\Psi(R)=S$. Suppose, on the contrary, that $\Psi(R)\subsetneqq S$. Then $\Psi^{-1}(S)\subsetneqq R$. Hence there is an element $f(X)=a_nX^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0\in\Psi^{-1}(S)$ with $n>0$ and some $a_0,\ldots,a_n(\neq 0)\in R$. Recall that the prime radical $N(R)$ of R is the intersection of all prime ideals of R. Since $N(R)$ is a nil ideal by [5, Proposition XV.1.2, p. 283] and since R is reduced, $N(R)=0$, that is, the intersection of all prime ideals of R is zero. Since $a_n\neq 0$, there exists a prime ideal Q of R such that $a_n\notin Q$. By a similar way as above, we can easily see that Ψ induces an

isomorphism $\tilde{\Psi} : (R/Q)[X; \tilde{\alpha}, \tilde{\delta}] \to (S/\Psi(P)S)[Y; \tilde{\beta}, \tilde{\partial}]$, where $\tilde{\alpha}$, $\tilde{\delta}$, $\tilde{\beta}$ and $\tilde{\partial}$ are the maps induced by α, δ, β and ∂, respectively. Since R is strongly regular, R/Q is a division ring, so that $\tilde{\Psi}(R/Q) = S/\Psi(Q)S$ by Corollary 4. Therefore we have $S = \Psi(R) + \Psi(Q)S$. Hence $\Psi(f(X)) \in S = \Psi(R) + \Psi(Q)S \subseteq \Psi(R + Q(R[X; \alpha, \delta]))$. This implies $f(X) \in R + Q(R[X; \alpha, \delta])$, and hence $a_n \in Q$, a contradiction. Consequently we obtain $\Psi(R) = S$.

References

[1] E. P. Armendariz, H. K. Koo and J. K. Park, Isomorphic Ore extensions, *Comm. in Algebra* **15** (1987), 2633–2652.

[2] K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, *Cambridge University Press*, 1989.

[3] A. A. M.Kamal, Idempotents in polynomial rings, *Acta Math. Hung.* **59** (1992), 355–363.

[4] A. A. M. Kamal, Strongly Ore invariant rings, *Tamkang J. Math.* **26** (1995), 277–282.

[5] B. Stenström, Rings of Quotients, *Springer-Verlag*, *Berlin, Heidelberg, New York*, 1975.

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