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On pairwise Baire bitopological spaces

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Abstract. We introduce and investigate a notion of pairwise Baire bitopological space. We obtain the following characterization which is a generalization of the corresponding one in General Topology: (X, P, L) is pairwise Baire iff the intersection of every sequence of P-open (L-open) L-dense (P-dense) sets is P-dense (L-dense), and we find a variant of Baire's category theorem for a class of pairwise locally compact spaces. We use this bitopological concept to characterize a class of real normed lattices with the property that each sequence of open dense sets which are decreasing has a dense intersection.

1. Introduction

A bitopological space (X, P, L) is a set X with two topologies P and L on X. If G is an open set of P we say G is P-open; the same for close, dense, ... J. C. Kelly [8] initiated the systematic study of such spaces and several others authors have contributed to the development of the theory. In the mentioned paper, Kelly considered the study of the Baire concept in non symmetric structures and he proved that if (X, d) is a right K-sequentially complete quasi-pseudometric space then the conjugate (X, d^{-1}) is a Baire space. Moreover, this type of completeness, which has been recently studied in [13] and in [14], constitutes an appropriate tool to solve the problem of completeness of function spaces and hyperspaces in the setting of quasi-uniformities [10], [11], [12].

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But the study of the Baire property in a purely bitopological context was begun by T. Fukutake in [6] who defined the pairwise Baire concept for a bitopological space and in his Proposition 3.2 he gives the following characterization:

(X, P, L) is pairwise Baire if and only if the intersection of every sequence of P-open (respect. L-open) L-dense (respect. P-dense) sets is P-dense (respect. L-dense).

Then, such a concept in a pairwise Baire space (X, P, L) has a respective condition about the P-open L-dense (respect. L-open P-dense) sets: they have to be P-dense (respect. L-dense).

To overcome this inconvenient we will give another definition of pairwise Baire, closely to the given one by Fukutake, and then we will start a systematic study of a such spaces. First, it is a appropiate to recall that pairwise spaces like pairwise paracompactness [9] seem intractable. In Section 2 we will modify the definition of a P-L-Baire set in order to obtain in Section 3, with other results, the derived characterization:

(X, P, L) is pairwise Baire if and only if the intersection of every sequence of P-open (respect. L-open) L-dense (respect. P-dense) sets is L-dense (P-dense).

In Section 4 we will find a class of bitopological spaces for which a Baire category theorem is satisfied in the context of pairwise locally compact spaces [2]. In Section 5 we will be able to characterize a certain class of real normed lattices by mean of our pairwise Baire concept and so this bitopological concept is shown to be useful to describe a property of General Topology.

From now, $P\text{-}\operatorname{cl}(A)$ and $P\text{-}\operatorname{int}(A)$ will denote the P-closure and the P-interior of the subset A of X, respectively.

2. P-L-nowhere dense

The following definitions for a bitopological space (X, P, L) were given in [6].

Definition 2.1. A subset A of X is said to be P-L-nowhere dense in X if $L\text{-}\operatorname{int}(P\text{-}\operatorname{cl}(A)) = \emptyset$.

Definition 2.2. A subset A of X is said to be of P-L-first category in X if it is an at most countable union of P-L-nowhere dense sets in X. If A is not of P-L-first category we say that A is of P-L-second category.

We omit the proofs of the following propositions for a bitopological space (X, P, L).

Proposition 2.3. *Let A and B be two subsets of X. If A is P-L-nowhere dense in X and $B \subset A$ then B is P-L-nowhere dense in X.*

Proposition 2.4. *The subset A of X is P-L-nowhere dense in X if and only if $P\text{-}\operatorname{int}(X - A)$ is L-dense.*

Proposition 2.5. *Let $Y \in L$. If the subset A of Y is P-L-nowhere dense in X then A is P-L-nowhere dense in Y.*

Proposition 2.6. *Let Y be an L-dense subset of X. If the subset A of Y is P-L-nowhere dense in the subspace, then A is P-L-nowhere dense in X.*

The assumption for Y to be dense in X cannot be removed as the following counterexample shows.

Counterexample 2.7. On the real line $\Re$ we consider the topologies P and L which have as proper P-open and L-open sets the intervals $]a, +\infty[$ and $]-\infty, a[$, $a \in \Re$, respectively. Let Y be the interval $[0, +\infty[$ with the topologies induced by P and L. If $A = \{0\}$ then A is P-L-nowhere dense in Y but $L\text{-}\operatorname{int}(P\text{-}\operatorname{cl}(A)) =]-\infty, 0[$ and so A is not P-L-nowhere dense in $\Re$.

Proposition 2.8.

(i) *A is of P-L-first category in X if and only if $A \subset \bigcup_{n=1}^{\infty} C_n$ with C_n P-closed and $L\text{-}\operatorname{int}(C_n) = \emptyset$, $n \in \mathbb{N}$.*

(ii) *If $A \subset B \subset X$ and B is of P-L-first category in X then A is of P-L-first category in X.*

(iii) *If B_n, $n = 1, 2, \dots$ are of P-L-first category in X then $\bigcup_{n=1}^{\infty} B_n$ is of P-L-first category in X.*

Proof. (i) Suppose $A = \bigcup_{n=1}^{\infty} F_n$ with $L\text{-}\operatorname{int}(P\text{-}\operatorname{cl}(F_n)) = \emptyset$, $n \in \mathbb{N}$, then, if we put $C_n = P\text{-}\operatorname{cl}(F_n)$, $n \in \mathbb{N}$, the desired conclusion follows.

Conversely, if $A \subset \bigcup_{n=1}^{\infty} C_n$ with C_n P-L-nowhere dense in X, $n \in \mathbb{N}$, then $A = \bigcup_{n=1}^{\infty} (C_n \cap A)$ and from Proposition 2.3, $C_n \cap A$ is P-L-nowhere dense in $X, n \in \mathbb{N}$.

(ii) and (iii) are consequences of (i).

3. Pairwise Baire spaces

We begin the section by giving a definition which is a slight modification of the one given in [6].

Definition 3.1. We say that X is P-L-Baire if each nonvoid P-open set is L-P-second category in X.

Definition 3.2. We say that X is pairwise Baire if X is P-L-Baire and L-P-Baire.

The above definition is generalization of Baire space. The following examples show that it is a good bitopoligical definition in the following sense: A pairwise Baire spaces need not be a pair of Baire spaces and vice-versa and besides we can find a pairwise Baire space (X, P, L) with $P \neq L$ which is pairwise Hausdorff [8].

Example 3.3. Consider the set of non-negative integers $\mathbb{N}$ with the cofinite topology P and the topology L whose has as proper open sets the sets $\{0, 1, \ldots, n\}$, $n \in \mathbb{N}$. Then $(\mathbb{N}, P, L)$ is pairwise Baire and $(\mathbb{N}, P)$ is not Baire.

Example 3.4. Let $\mathbb{Z}$ be the set of integer numbers. Let P be the discrete topology in $\mathbb{Z}$ and L the topology which base is given by the partition $\{\{-1, 0, 1\},\ \{-n, n\}, n \geq 2\}$. Then $(\mathbb{Z}, P)$ and $(\mathbb{Z}, L)$ are obviously Baire spaces but $(\mathbb{Z}, P, L)$ is not L-P-Baire; in fact, $G = \{-2, 2\}$ is L-open and $L\text{-}\operatorname{int}(P\text{-}\operatorname{cl}(\{-2\})) = L\text{-}\operatorname{int}(P\text{-}\operatorname{cl}(\{2\})) = \emptyset$ and G is P-L-first category in $(\mathbb{Z}, P, L)$.

Example 3.5 (Sorgenfrey's bitopological line). Consider the real line $\Re$ with the topologies P and L (called right and left half-open interval topology respectively, or *RHO* and *LHO* topology) in which basic neighbourhoods of x are the stes $[x, z[$ for $z > x$ and $]z, x]$ for $z < x$ respectively. As a consequence of Proposition 4.12 we will see in Example 4.13 that Sorgenfrey's bitopological line $(\Re, P, L)$ is pairwise Baire. Moreover, since P and L are finer than the usual topology in $\Re$, the space $(\Re, P, L)$ is pairwise Hausdorff and $P \neq L$.

The main theorem is the following one:

Theorem 3.6. *The following are equivalent:*

(i) *X is pairwise Baire.*

(ii) *The intersection of each sequence of L-open (respect. P-open) P-dense (respect. L-dense) sets is P-dense (respect. L-dense).*

(iii) *For every countable family $\{F_n : n \in \mathbb{N}\}$ formed by L-closed (respect. P-closed) sets with $P\text{-}\operatorname{Int}(F_n) = \emptyset$ (respect. $L\text{-}\operatorname{int}(F_n) = \emptyset$) we have that $P\text{-}\operatorname{int}\left(\bigcup_{n=1}^{\infty} F_n\right) = \emptyset$ (respect. $L\text{-}\operatorname{int}\left(\bigcup_{n=1}^{\infty} F_n\right) = \emptyset$).*

(iv) *If M is L-P-first category (respect. of P-L-first category) in X then $X - M$ is P-dense (respect. L-dense).*

Proof. We will suppose in (i) that X is P-L-Baire and we will omit the dual proofs corresponding to the cases enclosed by brackets.

(i)→(ii)

Suppose X is P-L-Baire. Let $\{D_n : n \in \mathbb{N}\}$ be a countable family of L-open P-dense sets of X. Suppose $X \neq P\text{-}\operatorname{cl}\left(\bigcap_{n=1}^{\infty} D_n\right)$, then $G = X - P\text{-}\operatorname{cl}\left(\bigcap_{n=1}^{\infty} D_n\right)$ is a proper P-open set and it satisfies $G \subset X - \left(\bigcap_{n=1}^{\infty} D_n\right) = \bigcup_{n=1}^{\infty} (X - D_n)$.

Since $P\text{-}\operatorname{int}(L\text{-}\operatorname{cl}(X - D_n)) = P\text{-}\operatorname{int}(X - D_n) = X - P\text{-}\operatorname{cl}(D_n) = \emptyset$, for each $n \in \mathbb{N}$, by Proposition 2.8 (i), G is of L-P-first category in X, which is absurd.

(ii)→(iii).

Let $\{F_n : n\in\mathbb{N}\}$ be a countable family of L-closed sets with $P\text{-}\operatorname{Int}(F_n)=\emptyset$, $n \in \mathbb{N}$. Then $\{X - F_n : n \in \mathbb{N}\}$ is a family of L-open sets which are P-dense since

$$P\text{-}\operatorname{cl}(X - F_n) = X - P\text{-}\operatorname{int}(F_n) = X, \quad n \in \mathbb{N}.$$

Now

$$X = P\text{-}\operatorname{cl}\left(\bigcap_{n=1}^{\infty} (X - F_n)\right) = P\text{-}\operatorname{cl}\left(X - \bigcup_{n=1}^{\infty} F_n\right) = X - P\text{-}\operatorname{int}\left(\bigcup_{n=1}^{\infty} F_n\right)$$

and thus

$$P\text{-}\operatorname{int}\left(\bigcup_{n=1}^{\infty} F_n\right) = \emptyset.$$

(iii)→(iv)
Let M_n be an L-P-nowhere dense set for each $n \in \mathbb{N}$ and suppose $M = \bigcup_{n=1}^{\infty} M_n$. Consider the family $\{C_n : n \in \mathbb{N}\}$ where $C_n = L\text{-}\operatorname{cl}(M_n)$, for each $n \in \mathbb{N}$; by the hypothesis we have

$$P\text{-}\operatorname{int}\left(\bigcup_{n=1}^{\infty} C_n\right) = \emptyset,$$

hence

$$X = X - P\text{-}\operatorname{int}\left(\bigcup_{n=1}^{\infty} C_n\right) = P\text{-}\operatorname{cl}\left(X - \bigcup_{n=1}^{\infty} C_n\right) \subset P\text{-}\operatorname{cl}(X - M).$$

(iv)→(i)
Let G be a nonvoid P-open set. If G is L-P-first category in X then $X - G$ will be L-dense and thus

$$X = P\text{-}\operatorname{cl}(X - G) = X - P\text{-}\operatorname{int}(G) = X - G$$

and therefore $G = \emptyset$ which is a contradiction. □

Proposition 3.7. *Let (X, P, L) be P-L-Baire. If Y is a P-open P-dense subset of X then Y is P-L-Baire.*

Proof. Let G be a P-open set in Y. Then G is of L-P-second category in X and by Proposition 2.6, G is of L-P-second category in Y. □

Proposition 3.8. *If be bitopological space (X, P, L) contains a P-open, P-dense, P-L-Baire subset Y, then X is P-L-Baire.*

Proof. Suppose G is of P-open and of L-P-first category in X. Then $G \cap Y$ is a non-void P-open set of X and by (ii) of Proposition 2.8, $G \cap Y$ is of L-P-first category in X. Since Y is P-L-Baire, $G \cap Y$ is of L-P-second category in Y and of by Proposition 2.5, $G \cap Y$ is of L-P-second category in X which is a contradiction. □

Proposition 3.9. *Let X be P-L-Baire. If the subset A of X is of L-P-first category in X then $X - A$ is P-L-Baire.*

Proof. Suppose A is of L-P-first category in X. By (iv) of Theorem 3.6, $X - A$ is P-dense. Now, let B be an L-P-first category subset of

$X - A$. By Proposition 2.6 B is of L-P-first category in X and then $A \cup B$ is of L-P-first category in X. By (iv) of Theorem 3.6, $X - (A \cup B) = (X - A) - B$ is P-dense in X and therefore $(X - A) - B$ is P-dense in $X - A$; the conclusion follows from (iv) of Theorem 3.6. □

We omit the proof of the following proposition which characterizes the L-P-first category subsets of a P-L-Baire space.

Proposition 3.10. *Let Y be a subset of the P-L-Baire space X. Y is of L-P-first category in X if and only if $X - Y$ contains a P-dense $L - G_\delta$ subset.*

Corollary 3.11. *If X is P-L-Baire and A is an $L - G_\delta$ P-dense subset of X, then A is P-L-Baire.*

Proof. By the last proposition $X - A$ is of L-P-first category and from Proposition 3.9, $X - (X - A) = A$ is P-L-Baire. □

The following theorem shows a local property which is a generalization of the corresponding one in Baire spaces.

Theorem 3.12. *If each point of the bitopological space (X, P, L) has a P-L-Baire (respect. L-P-Baire) P-open P-neighbourhood (respect. L-open L-neighbourhood) then X is pairwise Baire.*

Proof. Let $\{D_n : n \in \mathbb{N}\}$ be a family of L-open P-dense sets of X. Let $x \in X$ and suppose x has a P-neighbourhood U which is P-L-Baire. Consider the family $\{D_n \cap U : n \subset \mathbb{N}\}$ where $D_n \cap U$ is a nonvoid L-open subset in the space U, for each $n \in \mathbb{N}$. Also $D_n \cap U$ is P-dense in U for each $n \in \mathbb{N}$, and since U is P-L-Baire

$$U \subset P\text{-}\,\mathrm{cl}\left(\bigcap_{n=1}^{\infty}(D_n \cap U)\right)$$

and therefore

$$x \in U \subset P\text{-}\,\mathrm{cl}\left(\bigcap_{n=1}^{\infty}(D_n \cap U)\right) \subset P\text{-}\,\mathrm{cl}\left(\bigcap_{n=1}^{\infty} D_n\right)$$

i.e.,

$$P\text{-}\,\mathrm{cl}\left(\bigcap_{n=1}^{\infty} D_n\right) = X.$$

Thus, X is P-L-Baire. With a similar argument X is L-P-Baire. □

4. A Baire category theorem for bitopological spaces

We are going to find a Baire category theorem in terms of bitopological spaces. To begin with, we need some definitions and results.

Definition 4.1. A topological space is said to be quasi-regular [7] if each nonvoid open set contains the closure of a nonvoid open set.

If X is a regular space then it is quasi-regular. What follows is a generalization for a bitopological space (X, P, L) given here for the first time.

Definition 4.2. X is P-L-quasi-regular if each nonvoid P-open set contains the L-closure of a nonvoid P-open set.

Definition 4.3. X is pairwise quasi-regular if it is P-L-quasi-regular and L-P-quasi-regular.

The proof of the following proposition is trivial.

Proposition 4.4. *If X is P-L-reguar* [8] *then it is P-L-quasi-regular.*

The converse is not true as we see in the following.

Example 4.5. Consider the bitopological space $(\mathbb{N}, P, L)$ of Example 3.3. It is easy to observe that P and L are not quasi-regular, $(\mathbb{N}, P, L)$ is pairwise quasi-regular and $\mathbb{N}$ is not P-L-regular.

The following definitions are given in a bitopological space (X, P, L).

Definition 4.6 [2]. A cover U of X is called pairwise open if $U \subset P \cup L$ and both $U \cap P$ and $P \cap L$ contain a nonvoid set.

Definition 4.7 [4]. X is called pairwise compact if every pairwise open cover of X has a finite subcover.

Deinition 4.8 [2]. X is called pairwise locally compact if for each $x \in X$ there is a P-open (respect. L-open) neighbourhood V of x such that $L\text{-}\operatorname{cl}(V)$ (respect. $P\text{-}\operatorname{cl}(V)$) is pairwise compact.

If X is pairwise compact it is pairwise locally compact.

It is well-known [7] that if X is locally compact and quasi-regular then X is Baire. In the corresponding terms of bitopological spaces this is not true as the following counterexample shows.

Counterexample 4.9. In the set $\mathbb{N}$ of the nonnegative integers we consider the cofinite topology P and the discrete topology L. We see that $(\mathbb{N}, P, L)$ is pairwise compact:

Let $U = \{U_i : i \in I\}$ be a pairwise open cover of X; then there is $k \in I$ such that $U_k \in P$ and then $X - U_k$ is a finite subset of $\mathbb{N}$; therefore, it's obvious that $\mathbb{N}$ is covered by a finite subcover of U which contains U_k.

Also, it is easy to see that $(\mathbb{N}, P, L)$ is pairwise regular. Finally, we prove that $\mathbb{N}$ is not P-L-Baire: Consider the family $\{G_i : i \in \mathbb{N}\}$ with $G_i = \{i,\ i+1,\ i+2, \dots\}$. Each G_i is P-dense L-open but $\bigcap_{i \in \mathbb{N}} G_i$ is not dense since it is empty.

So, we are looking for a class of pairwise locally compact quasi-regular spaces for which we will prove they are pairwise Baire, but we have also in mind Theorem 3.5 of T. Fukutake:

Theorem 4.10 [6]**.** *Let (X, P, L) be a bitopological space and $P \leq L$. If L is metarizable and complete, then X is a P-L-Baire space.*

In order to extend a similar result and to obtain as conclusion that X is pairwise Baire, we need a weaker condition than $P \leq L$, since this condition with its dual condition $L \leq P$ gives the conclusion $P = L$ and so the theorem would not be a bitopological result, but a known topological one. Therefore, we give the following definition which is new:

Definition 4.11. The bitopological space (X, P, L) is pairwise fine if each nonvoid P-open (respect. L-open) set has a nonvoid L-open (respect. P-open) subset.

Now, in a pairwise fine space (X, P, L) we notice that D is P-dense if and only if it is L-dense and then, for such spaces, it is true that X is pairwise Baire in Fukutake's sense if and only if X is pairwise Baire in our sense. So, the following proposition is a generalization of Theorem 3.5 [6] for pairwise Baire spaces.

Proposition 4.12. *Let (X, P, L) be pairwise fine. X is pairwise Baire if and only if (X, P) and (X, L) are Baire.*

Proof. It is straightforward. □

Example 4.13. Since each interval $[a, b[$ $(a < b)$ of the real line $\Re$ contains an interval $]c, d]$ $(c < d)$ and vice-versa, it is obvious that the Sorgenfrey bitopological line (Example 3.5) is pairwise fine and in consequence it is paire Baire since $\Re$ with the *RHO* or *LHO* topology is Baire.

Lemma 4.14. *Let (X, P, L) be pairwise fine.*

(i) *is pairwise quasi-regular if and only if (X, P) and (X, L) are quasi-regular.*

(ii) *If X pairwise locally compact then (X, P) and (X, L) are locally compact.*

Proof. (i) Suppose X is pairwise quasi-regular. We only will see (X, P) is quasi-regular.

Let G be a nonvoid P-open set. Since X is pairwise fine there is a nonvoid L-open set H such that $H \subset G$; by the pairwise quasi-regularity there is a nonvoid L-open M with $M \subset P\text{-}\operatorname{cl}(M) \subset H$. Also, there is a nonvoid P-open set N with $N \subset M$ that satisfies $N \subset P\text{-}\operatorname{cl}(N) \subset P\text{-}\operatorname{cl}(M) \subset G$.

Conversely, we will only see X is P-L-quasi-regular. Suppose $G \neq \emptyset$ is P-open. Since X is pairwise fine there is a nonvoid L-open set H such that $H \subset G$ and since (X, L) is quasi-regular there is a nonvoid L-open set M with $L\text{-}\operatorname{cl}(M) \subset H$. Now, there is a nonvoid P-open set N such that $N \subset M$ that satisfies $N \subset L\text{-}\operatorname{cl}(N) \subset L\text{-}\operatorname{cl}(M) \subset G$.

(ii) Suppose X pairwise locally compact. We will show (X, P) is locally compact.

Let $x \in X$. Since X is pairwise locally compact there is a P-open neighbourhood G of x such that $M = L\text{-}\operatorname{cl}(G)$ is pairwise compact. We will prove that M is P-compact.

Let U be a P-open cover of M. If $V \in U$ $(V \neq \emptyset)$, then since X is pairwise fine there is a nonvoid L-open set H such that $H \subset V$. Then $U \cup \{H\}$ is a pairwise open cover of M and it admits a finite subcover $U' \subset U \cup \{H\}$. If $H \notin U'$ then $U \cup \{H\}$ is a finite P-open subcover of U; in the other case $U' \cup \{V\}$ is a finite P-open subcover of U. So, M is P-compact. □

Theorem 4.15. *Let (X, P, L) be pairwise fine. If X is pairwise quasi-regular and pairwise locally compact then X is pairwise Baire.*

Proof. By the previous lemma (X, P) and (X, L) are quasi-regular and locally compact and then they are Baire spaces. The conclusion follows from Proposition 4.12. □

5. Pairwise Baire spaces and normed lattices

If T and $\leq$ are a topology and an order, respectively, on X then $(X, T, \leq)$ is said to be a topological ordered space provided the graph $\mathbb{G}$ of the order $\leq$ is closed. The quasi-uniformity $\mathbb{U}$ in X is said to determine $(X, T, \leq)$ whenever $T = T(\mathbb{U}^*)$ and $\mathbb{G} = \bigcap_{V \in \mathbb{U}} V^{-1}$ where $\mathbb{U}^*$ is the uniformity in X which has $\mathbb{U} \cup \mathbb{U}^{-1}$ as a subbase ($\mathbb{U}^{-1}$ is the conjugate quasi-uniformity of $\mathbb{U}$). These concepts are found in FLETCHER and LINDGREN's book [5].

We say that a non-negative real valued function q defined on a real linear space E is a quasi-norm if it satisfies for $x, y \in E$ and $t \geq 0$:

$$q(x) = q(-x) = 0 \rightarrow x = 0,$$

$$q(tx) = tq(x),$$

$$q(x+y) \leq q(x) + q(y).$$

Given a "quasi-normed space" (E, q) the function $d(x, y) = q(y - x)$ is a quasi-pseudometric on E which induces a quasi-uniform structure $\mathbb{U}_q$ on X. In Corollary 3.2 of [3] we showed the following result:

Every real linear normed lattice $(E, \| \|, \leq)$ is determined in L. Nachbin's sense by the quasi-uniformity $\mathbb{U}_q$ deduced from the quasi-norm $q(x) = \|x^+\|$ where $x^+ = \sup\{x, 0\}$.

We have the following:

Lemma 5.1. *A non-empty set is q-open if and only if it is open in the norm and decreasing.*

As a consequence, the quasi-normed space (E, q) deduced from the real linear normed lattice $(E, \| \|), \leq)$ is never Baire. We gave in [3] the following definition.

Definition 5.2. The real linear normed lattice $(E, \| \|, \leq)$ is quasi-Baire if each sequence of open and decreasing sets (i.e., of q-open sets) which are dense (in norm) has a dense intersection.

Before the main theorem we need the following lemma from [1] that we reproduce:

Lemma 5.3. *If G is q-dense (q^{-1}-dense) and increasing (decreasing) then G is dense.*

PROOF. Let $x \in E$ and $\varepsilon > 0$. Since G is q-dense there exists $y \in V_\varepsilon(x) \cap G$. We have

$$-y = -x - (y - x) \geq -x - (y - x)^+$$

hence

$$y \leq x^+(y - x)^+.$$

If we call $z = x + (y - x)^+$ then, since G is increasing, $z \in G$.

Finally

$$\|z - x\| = \|(y - x)^+\| = q(y - x) < \varepsilon$$

and G is dense.

The proof when G is q^{-1}-dense and decreasing is similar. □

Theorem 5.4. *The normed lattice E is quasi-Baire if and only if (E, q, q^{-1}) is pairwise Baire.*

PROOF. Suppose E is quasi-Baire and let $\{G_n\}$ be a sequence of q-open and q^{-1}-dense sets. By Lemma 5.1 G_n is decreasing ($n \in \mathbb{N}$) and then $\bigcap_{n=1}^{\infty} G_n$ is dense and since the topology deduced from the norm is finer than the one deduced from q^{-1}, $\bigcap_{n=1}^{\infty} G_n$ is q^{-1}-dense.

Analogously if $\{G_n\}$ is a sequence of q^{-1}-open and q-dense sets then $\bigcap_{n=1}^{\infty} G_n$ is q-dense.

For the converse, suppose (E, q, q^{-1}) is pairwise Baire and let $\{G_n\}$ be a sequence of decreasing open dense sets. Then G_n is q^{-1}-dense ($n \in \mathbb{N}$) and by Lemma 5.1 and the hypothesis, $\bigcap_{n=1}^{\infty} G_n$ is q^{-1}-dense. Now, since $\bigcap_{n=1}^{\infty} G_n$ is a decreasing set, by the previous lemma $\bigcap_{n=1}^{\infty} G_n$ is dense. □

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