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On the prime power divisors of the iterates of the Euler-φ function

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Abstract. In this paper we prove that for each fixed $k \geq 1$,

$$\Delta(\varphi_k(n)) := \Omega(\varphi_k(n)) - \omega(\varphi_k(n)) = (1+o(1))\frac{1}{k}(\log\log x)^k(\log\log\log\log x)$$

holds for almost all n, and that

$$\lim_{x\to\infty}\frac{1}{x}\left\{n \leq x \;\Big|\; \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{\log\log x}(\log\log\log\log x)} < z\right\} = \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}dt,$$

where $s(x) = (\log\log x)(\log\log\log\log x) + c_1\log\log x + o(\log\log x)$.

1. Introduction

Let $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$ $(\varphi_0(n) = n,\ \varphi_1(n) = \varphi(n))$ be the k-fold iterate of the Euler totient function. Let P be the set of primes, and the letters p, q, π, Q with and without suffixes denote prime numbers. Φ is the Gaussian distribution function. $\omega(n)$ counts the number of distinct prime factors of n, $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. Let $\Delta(n) = \Omega(n) - \omega(n)$. As usual $p^r \parallel n$ means that $p^r \mid n$ but $p^{r+1} \nmid n$.

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For the variable x let $x_1 = \log x$, $x_r = \log x_{r-1}$ $(r = 2, 3, \dots)$.

In our recent paper [1] we proved that for each fixed k

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \;\middle|\; \frac{\omega(\varphi(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z) \tag{1.1}$$

$$\lim \frac{1}{\pi(x)} \# \left\{ p \le x \;\middle|\; \frac{\omega(\varphi(p-1)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z) \tag{1.2}$$

where $a_k = \frac{1}{(k+1)!}$, $b_k = \frac{1}{k!\sqrt{2k+1}}$.

We are interested in the distribution of $\Delta(\varphi_k(n))$.

Theorem 1. *For ech fixed $k \geqq 1$, for all but $o(x)$ integers $n \leqq x$,*

$$\Delta(\varphi_k(n)) = a_{k-1}(1 + o(1)) x_2^k x_4 \tag{1.3}$$

holds.

Remark 1. Theorem 1 implies that (1.1) remains valid if we change ω by Ω. For an arbitrary additive function f and an interval $I \subseteq [1, \infty)$ let

$$f(n \mid I) = \sum_{\substack{q^\beta \| n \\ q \in I}} f(q^\beta),$$

and let $f_z(n) = f(n \mid [0, z])$.

We shall prove

Theorem 2. *Let*

$$s(x) := \sum_{p \le x} \frac{\Omega_{x_2^2}(p-1)}{p} + \pi(x_2) + \sum_{x_2 \le q \le x_2^2} (1 - e^{-x_2/q}) - \omega x_2,$$

where

$$\omega = \int_1^\infty \frac{\exp(-\xi)}{\xi^2} d\xi.$$

Then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x \;\middle|\; \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{x_2} x_4} < z \right\} = \Phi(z).$$

Remark 2. One can prove that $s(x) = x_2 x_4 + c_1 x_2 + o(x_2)$.

2. Lemmata

Let

$$s(x,D,\ell) := \sum_{\substack{p\le x\\ p\equiv \ell \pmod D}} 1/p.$$

Lemma 1. *Uniformly in $1\le D\le x$ we have*

$$s(x,D,1)\le \frac{cx_2}{\varphi(D)}. \tag{2.1}$$

Proof. From sieve theorems we know that the number of primes $p\equiv 1 \pmod D$ in the interval $[aD,2aD]$ is less than $c_1\frac{aD}{\log a}\frac{1}{\varphi(D)}$. Applying this for $a=2,2^2,2^3,\dots$ one gets that

$$s(x,D,1)\le \sum_{\substack{p<2D\\ p\equiv 1 \pmod D}} \frac{1}{p} + \frac{c_1}{\varphi(D)}\sum_{j=1}^{j_0}\frac{1}{2^jD}\,\frac{2jD}{\log 2^j},$$

where j_0 is the largest integer for which $2^{j_0}D\le x$. The first sum is at most $\frac{1}{D+1}$, the second sum is less than $\frac{c_1}{\varphi(D)}\frac{1}{\log 2}\sum_{j=1}^{j_0}\frac{1}{j} < \frac{c_2}{\varphi(D)}\log j_0 < \frac{c_3}{\varphi(D)}x_2$. Hence (2.1) is immediate. □

Let $U_k(x;D) := \#\{n\le x,\ D\mid \varphi_k(n)\}$.

Lemma 2. *Fo every $k\ge 0$, $r\ge 0$ there exist numerical values $c(k,r)\le c(k,r+1)$ for which*

$$U_k(x;D)\le C(k,\Omega(D))\frac{xx_2^{k\Omega(D)}}{D}$$

whenever $e^e<x$, $1\le D\le x$.

Proof. The assertion is true for $k=0$. $C(0,r)=1$ is a suitable choice. Assume that it is true for $k-1$ instead of k. Let $D=p_1^{a_1}\dots p_r^{a_r}$, and the prime decomposition of $\varphi_{k-1}(n)$ let $\prod \pi^{\delta_\pi}$. Then

$$\varphi_k(n)=\prod \pi^{(\delta_\pi-1)}\prod(\pi-1)=E_1E_2.$$

Let $p_j^{e_j}\parallel E_1$, $b_j=\min(e_j,a_j)$, $D_1=\prod_{j=1}^r p_j^{b_j}$, $D_2=\prod_{j=1}^r p_j^{a_j-b_j}$.

Assume now that $D \mid \varphi_k(n)$. Then, either $D_2 = 1$, or $D_2 > 1$ and there is a product partition of D_2 as $\xi_1 \dots \xi_s$, $\xi_i > 1$ $(i = 1, \dots, s)$ for which $\xi_\ell \mid \pi_\ell - 1$, $\pi_\ell \mid \varphi_{k-1}(n)$ $(\ell = 1, \dots, s)$. It is clear that $s \le \sum_{j=1}^r (a_j - b_j)$. For a fixed choice of $\xi_1, \dots, \xi_s$ let $T = T_s$ run over those numbers $\pi_1, \dots, \pi_s \le x$ for which $\xi_\ell \mid \pi_\ell - 1$ $(\ell = 1, \dots, s)$. Then

$$U_k(x; D) \le \sum_{D_1 D_2 = D} \sum_s \sum_{\substack{\xi_1, \dots, \xi_s \\ \xi_1, \dots, \xi_s = D_2}} \sum_T U_{k-1}(x, D_1 T).$$

Since $\Omega(D_1 T) = \Omega(D_1) + s \le \Omega(D_1) + \Omega(D_2) = \Omega(D)$, from our hypothesis we obtain that

$$U_k(x; D) \le C(k-1, \Omega(D)) x x_2^{(k-1)\Omega(D)} \sum_{D_1 D_2 = D} \sum_s \sum_{\xi_1, \dots, \xi_s} \sum_T \frac{1}{D_1 T}.$$

From Lemma 1 we deduce that the sum $\sum \frac{1}{T}$ is less than $c^s x_2^s / \varphi(\xi_1) \cdots \varphi(\xi_s)$. Furthermore $\varphi(\xi_j) \ge \frac{1}{2}\xi_j$. Since the number of solutions of $D = D_1 D_2$ and that of the number of the product partition of D_2 is bounded by a function of $\Omega(D)$, and $s \le \Omega(D)$, we obtain that the assertion is true for k.

The proof is complete. □

As a consequence we have

Lemma 3. *Let $K_x \to \infty$ arbitrarily slowly, $\omega_0 = K_x$, $\omega_j = x_2^{2j}$ $(j = 1, 2, \dots)$. Let $k \ge 0$ be fixed. Then the number M of that integers $n \le x$ for which there exists at least one prime $p \ge \omega_k$ such that $p^2 \mid \varphi_k(n)$ is $o(x)$.*

Proof. It is enough to observe that

$$M \le \sum_{p \ge \omega_k} U_k(x, p^2), \qquad \sum_{p \ge \omega_k} \frac{1}{p^2} \ll \frac{1}{\omega_k x_3}, \qquad \square$$

Lemma 4. *For $e^e \le y \le x$ we have*

$$\sum_{p \le x} (\Omega_y(p-1) - \log\log y)^2 \ll \frac{x}{\log x} \log\log y,$$

and

$$\sum_{p \le x} \frac{1}{p} \left| \Omega_y(p-1) - \log\log y \right| \ll x_2 (\log\log y).$$

Proof. The first, Turán–Kubilius type inequality can be proved by squaring out, applying the Siegel–Walfisz theorem. The second inequality is an easy consequence of the first one. □

3. Proof of Theorem 1

Let the prime decomposition of $\varphi_{k-1}(n)$ be $\prod \pi^{\delta_\pi}$, $y = x_2^{2k}$. From Lemma 3 we obtain that $\Delta_y(\varphi_k(n)) = \Delta(\varphi_k(n))$ for all but $o(x)$ integers $n \le x$. Furthermore,

$$\Omega_y(\varphi_k(n)) = \sum_{\substack{\pi \le y \\ \pi \mid \varphi_{k-1}(n)}} (\delta_\pi - 1) + \sum_{\pi \mid \varphi_{k-1}(n)} \Omega_y(\pi - 1).$$

Let $\eta(n)$ denote the second sum. The first sum is $\Delta_y(\varphi_{k-1}(n))$. Thus we have

$$\Delta_y(\varphi_k(n)) = \Delta_y(\varphi_{k-1}(n)) + \eta(n) - \omega_y(\varphi_k(n)).$$

Since

$$\sum_{n \le x} \omega_y(\varphi_k(n)) = \sum_{p \le y} U_{k-1}(x; p),$$

from Lemma 2 and from the obvious inequality $U_{k-1}(x;p) \le x$ we obtain that the right hand side is less than

$$x\pi(x_2^k) + C(k-1,1)xx_2^k \sum_{x_2^k \le p \le y} 1/p \ll xx_2^k.$$

Consequently

$$\frac{1}{x} \#\{n \le x \mid \omega_y(\varphi_k(n)) \ge x_2^k x_5\} \to 0 \qquad (x \to \infty).$$

Let $\widetilde{\eta}(n) = (\log\log y)\omega(\varphi_{k-1}(n))$. Then

$$|\eta(n) - \widetilde{\eta}(n)| \le \sum_{\pi \mid \varphi_{k-1}(n)} |\Omega_y(\pi - 1) - \log\log y|,$$

and so by Lemma 2, and Lemma 4

$$\sum_{n\le x}|\eta(n)-\widetilde{\eta}(n)| \le \sum_{\pi\le x}|\Omega_y(\pi-1)-\log\log y|U_{k-1}(x;\pi)$$

$$\ll xx_2^{k-1}\sum_{\pi\le x}|\Omega_y(\pi-1)-\log\log y|\frac{1}{\pi} \ll xx_2^k(\log\log y)^{1/2},$$

Thus

$$\frac{1}{x}\#\left\{n\le x \;\middle|\; |\eta(n)-\widetilde{\eta}(n)| > x_2^k x_4^{3/4}\right\}\to 0 \qquad (x\to\infty).$$

Consequently,

$$\Delta(\varphi_k(n)) = \Delta_y(\varphi_{k-1}(n)) + (\log\log y)\omega(\varphi_{k-1}(n)) + O\left(x_2^k x_4^{3/4}\right)$$

for all but $o(x)$ integers $n\le x$.

Since $\omega(\varphi_{k-1}(n)) = a_{k-1}x_2^k + O\left(x_2^{k-1/2}x_5\right)$ (see (1.1)) for almost all n, and by induction on k we may assume that $\Delta_y(\varphi_{k-1}(n)) = O\left(x_2^{k-1}x_4\right)$, therefore

$$\Delta(\varphi_k(n)) = a_{k-1}x_2^k x_4 + O\left(x_2^k x_4^{3/4}\right)$$

for almost all n. The proof is complete.

4. Further lemmata

Let $V=\exp(\exp(\sqrt{x_2}))$, $Y=\exp(x_1e^{-\sqrt{x_2}})$, $u=\frac{x_1}{\log Y}$, $\beta=\log\frac{\log Y}{\log V} = x_2-2\sqrt{x_2}$, $J_1=\left[\frac{x_2}{x_3},x_2\right]$, $J_2=\left[x_2,x_2^2\right]$, $J=J_1\cup J_2$, $L=[V,Y]$.

Let

$$\psi_0(n) := \prod_{\substack{p<V\\ p\mid n}}(p-1),\quad \psi_1(n) := \prod_{\substack{p\mid n\\ p\in L}}(p-1),\quad \psi_2(n) := \prod_{\substack{p>Y\\ p\mid n}}(p-1),$$

$\psi(n) := \psi_0(n)\psi_1(n)\psi_2(n)$, $T(n)=\prod_{p\mid n}p^{\alpha_p(n)-1}$ where $\alpha_p(n)$ is defined as the exponent for which $p^{\alpha_p(n)}\parallel n$.

Let

$$\tau(k) := \prod_{\substack{\pi\in L\\ \pi\equiv 1 \pmod k}}(1-1/\pi).$$

From the prime number theorem for arithmetical progressions one can get easily that

$$\tau(k) = \exp(-\beta/\varphi(k)) \left(1 + O\left(\frac{1}{\varphi(k)\log V}\right)\right), \tag{4.1}$$

uniformly as $k \in [\sqrt{x_2}, x_2^4]$, say.

Assume from now on that $q, q_1, q_2 \in J$.

Let

$$\varrho(q_1, q_2) := \frac{\tau(q_1)\tau(q_2)}{\tau(q_1 q_2)}.$$

Since $\tau(q_1 q_2) = 1 + O\left(\frac{x_2}{q_1 q_2}\right) = 1 + O\left(\frac{x_3^2}{x_2}\right)$, therefore

$$\varrho(q_1, q_2) = \tau(q_1)\tau(q_2)\left(1 + O\left(\frac{x_2}{q_1 q_2}\right)\right). \tag{4.2}$$

Let

$$\mathcal{A}(q) := \sum_{\substack{n \le x \\ \psi_1(n) \not\equiv 0 \pmod{q}}} 1, \tag{4.3}$$

$$\mathcal{B}(q_1, q_2) := \sum_{\substack{n \le x \\ \psi_1(n) \equiv 0 \pmod{q_1} \\ \psi_1(n) \not\equiv 0 \pmod{q_2}}} 1, \tag{4.4}$$

$$\mathcal{C}(q_1, q_2) := \sum_{\substack{n \le x \\ \psi_1(n) \not\equiv 0 \pmod{q_j} \\ j=1,2}} 1. \tag{4.5}$$

As an immediate consequence of Theorem 2.5 in Halberstam–Richert [7], we obtain that

$$\mathcal{A}(q) = x\tau(q)(1 + O(e^{-u})), \tag{4.6}$$

$$\mathcal{C}(q_1, q_2) = x\tau(q_1)\tau(q_2)\left(1 + O\left(\frac{x_2}{q_1 q_2}\right)\right). \tag{4.7}$$

Since $\mathcal{B}(q_1, q_2) + \mathcal{C}(q_1, q_2) = \mathcal{A}(q_2)$, therefore

$$\mathcal{B}(q_1, q_2) = x\tau(q_2)(1 - \tau(q_1)) + O\left(\frac{x x_2}{q_1 q_2}\right). \tag{4.8}$$

Lemma 5. *Let*

$$K_1 := \sum_{n\le x} \left[\omega(\psi_1(n)|J_2) - A_x\right]^2, \tag{4.9}$$

where

$$A_x = \sum_{q\in J_2} (1-\tau(q)). \tag{4.10}$$

Then

$$K_1 \ll x x_2. \tag{4.11}$$

Proof. Let $S(x) := \sum_{n\le x} \omega(\psi_1(n)|J_2)$; $F(x) := \sum_{n\le x} \omega^2(\psi_1(n)|J_2)$. Then

$$S(x) = \sum_{q\in J_2} (x - \mathcal{A}(q)) + O\left(x_2^2\right) = xA_x + O\Bigg(xe^{-u} \sum_{q\in J_2} \tau(q)\Bigg)$$
$$= xA_x + O(x),$$

since $\tau(q) \le 1$ and $\sum \tau(q) \le e^u$.

Furthermore, $F(x) = S(x) + \sum_1$, where

$$\textstyle\sum_1 = \displaystyle\sum_{\substack{q_1\ne q_2\\ q_1,q_2\in J_2}} \sum_{\substack{\psi_1(n)\equiv O \pmod{q_j}\\ j=1,2}} 1. \tag{4.12}$$

The inner sum of the right hand side of (4.12) equals to

$$[x] - \mathcal{B}(q_1,q_2) - \mathcal{B}(q_2,q_1) - \mathcal{C}(q_1,q_2).$$

Thus

$$\textstyle\sum_1 = [x] \displaystyle\sum_{q_1\ne q_2} 1 - x \sum_{q\ne q_2} \tau(q_2)(1-\tau(q_2)) + \tau(q_1)(1-\tau(q_2))$$
$$- x \sum_{q_1\ne q_2} \tau(q_1)\tau(q_2) + O(xx_2) = x \sum_{q_1\ne q_2} (1-\tau(q_1))(1-\tau(q_2)) + O(xx_2)$$
$$= xA_x^2 + O(xx_2) - x \sum_{q\in J_2} (1-\tau(q))^2.$$

Since

$$K_1 = F(x) - 2A_x S(x) + A_x^2[x],$$

we have

$$K_1 = xA_x^2 + O(xx_2) + xA_x + O(x) - 2A_x(xA_x + O(x)) + O(xA_x) - x\sum_{q\in J_2}(1-\tau(q))^2 \ll xA_x,$$

and finally, from

$$A_x \ll \sum \beta/q_1 \ll x_2,$$

we obtain that

$$K_1 \ll xx_2.$$

This completes the proof of Lemma 5.

Let

$$\mathcal{B}_x = \sum_{q\in J_1}\tau(q). \tag{4.13}$$

From (4.1) we obtain that

$$\mathcal{B}_x = \sum_{q\in J_1} e^{-\beta/q-1}\left(1+O\left(\frac{1}{q(\log V)^c}\right)\right) = \sum_{q\in J_1} e^{-\beta/q}\left(1+O\left(\frac{\beta}{q^2}\right)+O\left(\frac{1}{q(\log V)^c}\right)\right).$$

To estimate the main term

$$T := \sum_{q\in J_1} e^{-\beta/q},$$

we shall use the prime number theorem in the form

$$\Delta(z) := \pi(z) - \ell iz \ll ze^{-c_1\sqrt{x_3}} \quad \text{for } z\in J_1.$$

Thus

$$T = \int_{J_1} e^{-\beta/\eta}\frac{d\eta}{\log\eta} + \int_{J_1} e^{-\beta/\eta} d\Delta(\eta) = T_1 + T_2.$$

By partial intergation we have

$$T_2 = O\left(x_2 e^{-c_1\sqrt{x_3}}\right) + \int |\Delta(\eta)|\,|(e^{\beta/\eta})'|d\eta.$$

The integral on the right hand side is bounded by

$$\ll e^{-c_1\sqrt{x_3}} \int \eta \left(e^{-\beta/\eta}\right)' d\eta = O\left(x_2 e^{-c_1\sqrt{x_3}}\right) + e^{-c_1\sqrt{x_3}} \int_{J_1} e^{-\beta/\eta} d\eta.$$

Since $\frac{1}{\log \eta} = \frac{1}{x_3} + O\left(\frac{x_4}{x_3}\right)$ in $\eta \in J_1$, therefore

$$T_1 = \frac{1}{x_3}\left(1 + O\left(\frac{x_4}{x_3}\right)\right) \int_{J_1} e^{-\beta/\eta} d\eta.$$

Introducing the new variable $\xi = \beta/\eta$ we have

$$\int_{J_1} e^{-\beta/\eta} d\eta = \beta \int_{\beta/x_2}^{\beta x_3/x_2} \frac{\exp(-\xi)}{\xi^2} d\xi. \tag{4.14}$$

Let

$$\omega := \int_1^\infty \frac{\exp(-\xi)}{\xi^2} d\xi. \tag{4.15}$$

The integral on the right hand side of (4.14) equals to

$$\omega + O\left(\frac{1}{\sqrt{x_2}}\right) + O\left(\frac{\exp\left(-\frac{\beta x_3}{x_2}\right)}{x_3^2}\right) = \omega + O\left(\frac{1}{\sqrt{x_2}}\right),$$

consequently

$$T_1 = \beta\omega + O(\sqrt{x_2}),$$

and

$$T_2 = O\left(x_2 e^{-x_1\sqrt{x_3}}\right).$$

One can prove similarly that

$$\sum_{q \in J_1} e^{-\beta/q} \left(O\left(\frac{\beta}{q^2}\right) + O\left(\frac{1}{q(\log V)^c}\right)\right) = O(1).$$

Consequently

$$\mathcal{B}_x = \omega\beta + O(\sqrt{x_2}) = \omega x_2 + O(\sqrt{x_2}). \tag{4.16}$$

Let $h(n)$ be the number of those primes q in J_1 for which $q \nmid \psi_1(n)$.

Lemma 6. *Let*

$$K_2 := \sum_{n\le x}(h(n)-\mathcal{B}_x)^2, \tag{4.17}$$

Then

$$K_2 = O(xx_2). \tag{4.18}$$

Proof. From (4.6), (4.7) we infer that

$$U(x) := \sum_{n\le x} h(n) = \sum_{q\in J_1}\mathcal{A}(q) = x(1+O(e^{-u}))\mathcal{B}_x. \tag{4.19}$$

$$V(x) := \sum_{n\le x} h^2(n) = U(x) + \sum\nolimits_2, \tag{4.20}$$

where

$$\begin{aligned}\sum\nolimits_2 &= \sum_{n\le x} h(n)(h(n)-1) = \sum_{q_1,q_2\in J_1,\, q_1\neq q_2} \mathcal{C}(q_1,q_2)\\ &= x(1+O(e^{-u}))\sum_{q_1\neq q_2,\, q_1,q_2\in J_1}\varrho(q_1,q_2) \qquad (4.21)\\ &= x(1+O(e^{-u}))\sum_{q_1,q_2\in J_1,\, q_1\neq q_2}\tau(q_1)\tau(q_2)\left(1+O\left(\frac{\beta}{q_1q_2}\right)\right).\end{aligned}$$

Consequently

$$\sum\nolimits_2 = x\left(\mathcal{B}_x^2 - \sum_{q\in J_1}\tau^2(q) + O(e^{-u}\mathcal{B}_x^2) + O\left(x_2\left(\sum_{q\in J_1}\frac{\tau(q)}{q}\right)^2\right)\right). \tag{4.22}$$

Since

$$K_2 = U(x) + \sum\nolimits_2 - 2\mathcal{B}_x U(x) + [x]\mathcal{B}_x^2,$$

therefore, by (4.19), (4.20), (4.22) we get that

$$K_2 = O(x\mathcal{B}_x) + O(e^{-u}x\mathcal{B}_x^2) + O\left(x\left(\sum_{q\in J_1}\tau^2(q) + x_2\left(\sum_{q\in J_1}\frac{\tau(q)}{q}\right)^2\right)\right).$$

From (4.16), $\mathcal{B}_x \ll x_2$, furthermore

$$\sum_{q\in J_1}\tau^2(q) \ll \frac{x_2}{x_3}, \qquad \sum_{q\in J_1}\frac{\tau(q)}{q} \ll \sum_{q\in J_1}\frac{1}{q} = o(1),$$

thus $K_2 \ll xx_2$.

The proof of the lemma is finished. □

Let H be the number of primes in J_1. Then, by Lemma 6

$$\sum_{n\le x}(\omega(\psi_1(n)|J_1) - (H - \mathcal{B}_x))^2 \ll xx_2,$$

consequently

$$\frac{1}{x}\#\left\{n \le x,\ |\omega(\psi_1(n)|J_1) - (H - \mathcal{B}_x)| > x_5\sqrt{x_2}\right\} \ll \frac{1}{x_5^2}. \tag{4.23}$$

Similarly, from Lemma 5,

$$\frac{1}{x}\#\left\{n \le x,\ |\omega(\psi_1(n)|J_2) - A_x| > x_5\sqrt{x_2}\right\} \ll \frac{1}{x_5}. \tag{4.24}$$

5. The distribution of $\Omega_{x_2^2}(\psi(n))$

Let

$$A(x) = \sum_{p\le x}\frac{\Omega_{x_2^2}(p-1)}{p}, \qquad \mathcal{B}^2(x) = \sum_{p\le x}\frac{\Omega_{x_2^2}^2(p-1)}{p}. \tag{5.1}$$

We observe that

$$f(n) := \Omega_{x_2^2}(\psi(n)) = \sum_{p|n}\Omega_{x_2^2}(p-1) = \sum_{p|n} f(p)$$

is a strongly additive function depending on the parameter x. To prove that

$$\lim_{x\to\infty}\frac{1}{x}\#\Big\{n\le x,\ \Omega_{x_2^2}(\psi(n))-A(x)\le z\mathcal{B}(x)\Big\}=\Phi(z), \tag{5.2}$$

we can apply Theorem 12.15 in Elliott [8], and the Berry–Esseen theorem (see Lemma 1.48 in [8]). The conditions of these theorems are satisfied. Indeed, by using the prime number theorem for arithmetical progressions and Lemma 1, we can deduce easily that

$$A(x)=x_2x_4+O(x_2),\qquad \mathcal{B}^2(x)=(1+o(1))x_2x_4^2, \tag{5.3}$$

and that

$$\sum_{p\le x}\frac{\Omega^3_{x_2^2}(p-1)}{p}\ll x_2x_4^3. \tag{5.4}$$

(5.3), (5.4) imply that

$$\sum_{\substack{x^\epsilon<p\le x\\ |f(p)|>\lambda\mathcal{B}(x)}}\frac{1}{p}\to 0$$

for arbitrary positive constants ϵ and λ, thus the condition of Theorem 12.15 holds. Since the left hand side of (5.4) is $o(\mathcal{B}^3(x))$, therefore the remainder term in the Berry–Esseen theorem is $o(1)$, thus (5.2) holds.

6. Proof of Theorem 2

It is clear that for $\ell=1,2$:

$$\begin{aligned}&\omega(\psi_1(n)|J_\ell)\le\omega(\psi(n)|J_\ell)\\ &\le\omega(\psi_1(n)|J_\ell)+\omega(\psi_0(n))|J_\ell)+\omega(T(n)|J_\ell).\end{aligned} \tag{6.1}$$

Let

$$\textstyle\sum_0=\sum_{n\le x}\omega(\psi_0(n)|J),\quad \sum_1=\sum_{n\le x}\omega(T(n)|J),\quad \sum_2=\sum_{n\le x}\omega(\psi_2(n)|J).$$

First we estimate $\sum_1$. We have

$$\sum\nolimits_1 \ll \sum_{q\in J} \frac{x}{q^2} \ll x.$$

Similarly,

$$\sum\nolimits_2 \ll \sum_{q\in J} \sum_{\substack{n\le x \\ \psi_2(n)\equiv O \pmod q}} 1 \ll \sum_{q\in J} \sum_{\substack{p\equiv 1 \pmod q \\ Y<p\le x}} x/p$$

$$\ll x \log \frac{\log x}{\log Y} \sum_{q\in J} 1/q \ll x\sqrt{x_2}.$$

Finally

$$\sum\nolimits_0 \ll \sum_{q\in J} \sum_{\substack{n\le x \\ \psi_0(n)\equiv O \pmod q}} 1 \ll x \sum_{q\in J} \sum_{\substack{q\mid p-1 \\ p\le V}} 1/p$$

$$\ll x\sqrt{x_2} \sum_{q\in J} \frac{1}{q} = O(x\sqrt{x_2}\,).$$

We deduced that

$$\sum\nolimits_0 + \sum\nolimits_1 + \sum\nolimits_2 = O(x\sqrt{x_2}\,).$$

Consequently

$$\text{(6.2)} \qquad \frac{1}{x}\#\Big\{ n\le x \;\Big|\; \big|\omega(\psi(n)|J) - \omega(\psi_1(n)|J)\big| > x_5\sqrt{x_2} \Big\} \to 0.$$

We can observe that

$$\frac{1}{x}\#\Big\{ n\le x \;\Big|\; |\Delta(\varphi(n) - \Delta(\psi(n))| \ge K_x \Big\} \to 0,$$

if K_x is an arbitrary function which tends to infinity. Let $K_x = x_5$. Therefore it is enough to prove the theorem for $\psi(n)$ instead of $\varphi(n)$. From Lemma 3 we obtain that the asymptotic density of the integers $n \le x$ for which $p^2 \mid \psi(n)$ for some $p > x_2^2$ is zero. Thus $\Delta(\psi(n)) = \Delta_{x_2^2}(\psi(n))$ for all but $o(x)$ integers up to x. By sieve theorems one can deduce that the number of the integers $n \le x$ for which there is a prime $q \le x_2/x_3$ such that $q \nmid \psi(n)$ is less than

$$cx \sum_{q<x_2/x_3} \prod_{\substack{\pi\equiv 1 \pmod q \\ \pi<x}} \left(1-\frac{1}{\pi}\right) \le cx \sum_{q<x_2/x_3} \exp\left(-\frac{x_2}{q-1}\right) = o(x).$$

Thus, removing no more than $o(x)$ integers $n \le x$, for the others

$$\omega_{x_2/x_3}(\psi(n)) = \pi(x_2/x_3). \tag{6.3}$$

Hence, and from (5.2) we obtain that

$$\omega_{x_2^2}(\psi(n)) = \pi(x_2/x_3) + \omega(\psi_1(n)|J) + O(x_5\sqrt{x_2}\,)$$

for all but $o(x_2)$ integers $n \le x$. By using (4.23) and (4.24) we get that

$$\omega_{x_2^2}(\psi(n)) = \pi(x_2) + A_x - \mathcal{B}_x + O(x_5\sqrt{x_2}\,) \tag{6.4}$$

for all but $o(x)$ integers $n \le x$.

By (5.2) we have that

$$\frac{1}{x}\#\left\{n\le x \;\middle|\; \frac{\Delta(\varphi(n))-A(x)-\pi(x_2)-A_x+\mathcal{B}_x}{\sqrt{x_2x_4}} < z\right\} \to \Phi(z) \quad \text{as } x\to\infty.$$

From (4.1), (4.10) we can deduce easily that

$$A_x = \sum_{x_2\le q\le x_2^2} \left(1-\exp\left(-\frac{x_2}{q}\right)\right) + O(\sqrt{x_2}\,).$$

Then, by (4.16) our theorem follows.

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