Publ. Math. Debrecen
55 / 1-2 (1999), 33–45

On the stability of the homogeneous equation

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Abstract. Let f be a function defined on a subset $U(\mathbb{R}_0 U \subset U)$ of the real linear space X with the values in the sequentially complete locally convex linear topological Hausdorff space Y. We will show that if there exist a bounded subset $V \subset Y$, a non-empty subset $A \subset \mathbb{R}_0$, $\delta : A \to [0, \infty)$, $K : U \to [0, \infty)$ such that for all $\alpha \in A$, $x \in U$ the condition $\alpha^{-1} f(\alpha x) - f(x) \in \delta(\alpha) K(x) V$ holds, then, under certain assumpitons on A and K, there exists a unique homogeneous mapping $F : U \to Y$ such that the difference $F(x) - f(x)$ is suitably bounded on U.

0. Introduction

Józef Tabor proved in [4] that every mapping from a real vector space X into a normed space Y satisfying

$$\|\alpha^{-1} f(\alpha x) - f(x)\| \le \varepsilon \quad \text{for } \alpha \in \mathbb{R} \setminus \{0\},\ x \in X, \tag{1}$$

where $\varepsilon \ge 0$ is given, is homogeneous. This result has next been generalized successively in a different directions.

Zygfryd Kominek and Janusz Matkowski investigated in [2] the condition

$$\alpha^{-1} f(\alpha x) - f(x) \in V, \quad \text{for } \alpha \in A,\ x \in S, \tag{2}$$

for the mapping f from a cone $S \subset X$ into a sequentially complete locally convex linear topological Hausdorff space Y over $\mathbb{R}$ and a subset

Mathematics Subject Classification: 39B72.
Key words and phrases: homogeneous function, linear topological space, Hausdorff space, locally convex space, sequentially complete space.

$A \subset (1, \infty)$. This result has been generalized in [3]. J. Schwaiger has examined the condition

$$f(\alpha x) - \phi(\alpha) f(x) \in V(\alpha), \quad \text{for } \alpha \in A,\ x \in X, \tag{3}$$

where

- G is a semigroup with unit acting on the non-empty set X;
- Y is a sequentially complete locally convex linear topological Hausdorff space Y over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$;
- $A \subset G$ generates G as a semigroup;
- $\phi : G \to \mathbb{K}$ is a function;
- $V : G \to \mathcal{B}(Y)$ is a mapping from G into the set $\mathcal{B}(Y)$ of bounded subsets of Y.

It is proved there that if functions f and ϕ satisfy (3) and $f(X)$ is unbounded, then ϕ is a multiplicative function and there is a function $F : X \to Y$ satisfying

$$F(\alpha x) = \phi(\alpha) F(x) \quad \text{for } \alpha \in G,\ x \in X,$$

(we say then that F is ϕ-homogeneous) and such that the difference $F - f$ is suitably bounded on X.

On the other hand, in [1] and [5] it has been investigated the inequality

$$\|f(\alpha x) - \alpha^v f(x)\| \leq g(\alpha, x), \quad \alpha \in \mathbb{R} \setminus \{0\},\ x \in X,$$

with a constant $v \in \mathbb{R} \setminus \{0\}$ and a function g mapping $\mathbb{R} \times X$ into $\mathbb{R}$.

In this paper we investigate the stability condition

$$\alpha^{-1} f(\alpha x) - f(x) \in \delta(\alpha) K(x) V, \quad \alpha \in A \subset \mathbb{R} \setminus \{0\},\ x \in U,$$

and on this way we will unite together two mentioned above generalizations.

1. Main result

Through this paper the letters $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_0$ stand for natural numbers, non-negative integers, reals, non-negative reals, and reals different from zero, respectively. From now on X stands for a real linear space and Y – for a sequentially complete locally convex linear topological Hausdorff space. The sequentially closure of V will be denoted by $\operatorname{seq\,cl} V$. By $\operatorname{conv} V$ we will denote the convex hull of V. The subset $V \subset Y$ is said to be bounded if for each neighbourhood W of zero there exists an $r \in \mathbb{R}_0$ such that $rV \subset W$. By $\langle A \rangle$ we denote a multiplicative group generated by the set $A \subset \mathbb{R}_0$.

We start with the following

Lemma 1. *Let $V \subset Y$ and $0 \le \lambda \le 1$. Then $\lambda V \subset \operatorname{conv}(V \cup \{0\})$. Moreover, if V is symmetric with respect to 0, then $\lambda V \subset \operatorname{conv} V$.*

Proof. The above assertions follow from the following conditions

$$\lambda V \ni \lambda x = \lambda x + (1-\lambda)0 \in \operatorname{conv}(V \cup \{0\})$$

and

$$\lambda V \ni \lambda x = \lambda x + \frac{(1-\lambda)}{2}(-x) + \frac{(1-\lambda)}{2}(x) \in \operatorname{conv} V. \qquad \square$$

The simply conclusion from Lemma 1 is

Corollary 1. *Let $V \subset Y$ and $0 \le \alpha \le \beta$. Then $\alpha V \subset \beta \operatorname{conv}(V \cup \{0\})$. Moreover, if V is symmetric with respect to 0, then $\alpha V \subset \beta \operatorname{conv} V$.*

We have the following

Lemma 2. *Let $V \subset Y$ be a convex set. Then the set $\operatorname{seq\,cl} V$ is convex.*

Now we can formulate

Theorem 1. *Let $\emptyset \ne U \subset X$ be such that $\mathbb{R}_0 U \subset U$. Let a set $A \subset \mathbb{R}_0$ contain at least one element α such that $|\alpha| \ne 1$, and let $\delta : A \to \mathbb{R}_+$ be a mapping. Assume that the function $K : U \to \mathbb{R}_+$ satisfies*

$$K(\alpha x) \le |\alpha|^p K(x) \quad \text{for } \alpha \in \mathbb{R}_0,\ x \in U, \tag{4}$$

with some $p \in \mathbb{R} \setminus \{1\}$. Let $V \subset Y$ be a bounded set and let $f : U \to Y$ satisfy

$$\alpha^{-1} f(\alpha x) - f(x) \in \delta(\alpha)\, K(x)\, V \quad \text{for } \alpha \in A,\ x \in U. \tag{5}$$

Then there exists a unique function $F : U \to Y$ satisfying

$$F(\alpha x) = \alpha F(x) \quad \text{for all } \alpha \in \langle A \setminus \{-1\} \rangle,\ x \in U,$$

and such that

$$F(x) - f(x) \in c\, K(x)\ \operatorname{seq}\operatorname{cl}\operatorname{conv}(V \cup (-V)), \quad x \in U,$$

where

$$c := \min\left(\inf_{\alpha \in A_p} \frac{\delta(\alpha)}{1 - |\alpha|^{p-1}}, \inf_{\alpha \in A^p} \frac{\delta(\alpha)}{|\alpha|^{p-1} - 1} \right),$$

$$A_p := \{\alpha \in A : |\alpha|^{p-1} < 1\}, \quad A^p := \{\alpha \in A : |\alpha|^{p-1} > 1\},$$

(by $\inf \emptyset$ we mean $+\infty$).

Proof. Let us fix an $\alpha \in A$ such that $|\alpha| \neq 1$. Consider two cases
1) $|\alpha|^{p-1} < 1$,
2) $|\alpha|^{p-1} > 1$.

In the first case, for all $m, n \in \mathbb{N}_0$ and $x \in U$, we have

$$\begin{aligned} &\alpha^{-(m+n)} f(\alpha^{m+n} x) - \alpha^{-m} f(\alpha^m x) \\ &\quad = \sum_{k=1}^{n} \alpha^{-(m+k-1)} [\alpha^{-1} f(\alpha\, \alpha^{m+k-1} x) - f(\alpha^{m+k-1} x)] \\ &\quad \in \sum_{k=1}^{n} \alpha^{-(m+k-1)}\, \delta(\alpha)\, K(\alpha^{m+k-1} x) V, \end{aligned}$$

but from (4), by Corollary (1), we get

$$\begin{aligned} &\sum_{k=1}^{n} \alpha^{-(m+k-1)}\, \delta(\alpha)\, K(\alpha^{m+k-1} x) V \\ &\quad \subset \sum_{k=1}^{n} |\alpha|^{-(m+k-1)}\, \delta(\alpha) |\alpha^{m+k-1}|^p K(x)\ \operatorname{conv}(V \cup (-V)) \\ &\quad = |\alpha|^{m(p-1)} \frac{\delta(\alpha)\, K(x)}{1 - |\alpha|^{p-1}} \left(1 - |\alpha|^{n(p-1)}\right) \operatorname{conv}(V \cup (-V)) \\ &\quad \subset |\alpha|^{m(p-1)} \frac{\delta(\alpha)\, K(x)}{1 - |\alpha|^{p-1}} \operatorname{conv}(V \cup (-V)). \end{aligned}$$

Then

$$\alpha^{-(m+n)} f(\alpha^{m+n}x) - \alpha^{-m} f(\alpha^m x) \in |\alpha|^{m(p-1)} \frac{\delta(\alpha)\, K(x)}{1-|\alpha|^{p-1}} \ \mathrm{conv}(V \cup (-V)), \tag{6}$$

and $(\alpha^{-n} f(\alpha^n x) : n \in \mathbb{N})$ is the Cauchy sequence for each $x \in U$. Thus we may define a function $F_\alpha : U \to Y$,

$$F_\alpha(x) := \lim_{n\to\infty} \alpha^{-n} f(\alpha^n x),$$

which satisfies

$$F_\alpha(\alpha x) = \lim_{n\to\infty} \alpha^{-n} f(\alpha^{n+1}x) = \alpha \lim_{n\to\infty} \alpha^{-(n+1)} f(\alpha^{n+1}x) = \alpha F_\alpha(x) \tag{7}$$

for every $x \in U$. From (6), with $m = 0$, we obtain

$$\alpha^{-n} f(\alpha^n x) - f(x) \in \frac{\delta(\alpha)\, K(x)}{1-|\alpha|^{p-1}} \mathrm{conv}(V \cup (-V)),$$

so

$$F_\alpha(x) - f(x) \in \frac{\delta(\alpha) K(x)}{1-|\alpha|^{p-1}} \ \mathrm{seq\,cl\,conv}(V \cup (-V)) \quad \text{for } x \in U. \tag{8}$$

Consider the second case. Replacing in (5) x by $\alpha^{-1}x$ we get

$$\alpha^{-1} f(x) - f(\alpha^{-1}x) \in \delta(\alpha)\, K(\alpha^{-1}x) V,$$

and so, from (4) and Corollary 1

$$\alpha f(\alpha^{-1}x) - f(x) \in \delta(\alpha)|\alpha|^{1-p} K(x)\, \mathrm{conv}(V \cup (-V)) \quad \text{for } \alpha \in A,\ x \in U. \tag{9}$$

Then, similarly as in the first case, one can show that

$$\alpha^{m+n} f(\alpha^{-(m+n)}x) - \alpha^m f(\alpha^{-m}x) \in |\alpha|^{m(1-p)} \frac{\delta(\alpha) K(x)}{|\alpha|^{p-1}-1} \ \mathrm{conv}(V \cup (-V)), \tag{10}$$

and $(\alpha^n f(\alpha^{-n}x) : n \in \mathbb{N})$ is the Cauchy sequence for every $x \in U$. The function $F_\alpha : U \to Y$ given by

$$F_\alpha(x) := \lim_{n\to\infty} \alpha^n f(\alpha^{-n}x)$$

is well defined and satisfies

$$F_\alpha(\alpha x) = \alpha F_\alpha(x) \quad \text{for every } x \in U. \tag{11}$$

From (10), with $m = 0$, we get

$$\alpha^n f(\alpha^{-n}x) - f(x) \in \frac{\delta(\alpha)\, K(x)}{|\alpha|^{p-1} - 1} \operatorname{conv}(V \cup (-V)),$$

so

$$F_\alpha(x) - f(x) \in \frac{\delta(\alpha)\, K(x)}{|\alpha|^{p-1} - 1} \operatorname{seq\,cl\,conv}(V \cup (-V)) \quad \text{for } x \in U. \tag{12}$$

Let us notice then from (7) and (11) it follows

$$F_\alpha(\alpha^n x) = \alpha^n F_\alpha(x) \quad \text{for } \alpha \in A \setminus \{-1, 1\},\ x \in U,\ n \in \mathbb{N}, \tag{13}$$

or equivalently, since $\mathbb{R}_0 U \subset U$,

$$F_\alpha(\alpha^{-n} x) = \alpha^{-n} F_\alpha(x) \quad \text{for } \alpha \in A \setminus \{-1, 1\},\ x \in U,\ n \in \mathbb{N}. \tag{14}$$

We prove that $F_\alpha = F_\beta$ for $\alpha, \beta \in A \setminus \{-1, 1\}$. In this purpose we consider three cases:

1) $|\alpha|^{p-1} < 1$ and $|\beta|^{p-1} < 1$;
2) $|\alpha|^{p-1} < 1$ and $|\beta|^{p-1} > 1$;
3) $|\alpha|^{p-1} > 1$ and $|\beta|^{p-1} > 1$;

1) For an arbitrary $x \in U$, from (8), by Corollary 1 and Lemma 2 we get

$$\begin{aligned}\beta^{-n} F_\alpha(\beta^n x) - \beta^{-n} f(\beta^n x) &\in \frac{\delta(\alpha)\, K(\beta^n x)}{|\beta|^n (1 - |\alpha|^{p-1})} \operatorname{seq\,cl\,conv}(V \cup (-V)) \\ &\subset |\beta|^{n(p-1)} \frac{\delta(\alpha)\, K(x)}{1 - |\alpha|^{p-1}} \operatorname{seq\,cl\,conv}(V \cup (-V)).\end{aligned}$$

Hence

$$\lim_{n\to\infty} \beta^{-n} F_\alpha(\beta^n x) = F_\beta(x) \quad \text{for each } x \in U. \tag{15}$$

By (8) and (13) we get

$$\begin{aligned}
\alpha^{-n} F_\beta(\alpha^n x) - \beta^{-n} F_\alpha(\beta^n x) &= (\alpha\beta)^{-n}[\beta^n F_\beta(\alpha^n x) - \alpha^n F_\alpha(\beta^n x)] \\
&= (\alpha\beta)^{-n}[F_\beta(\alpha^n\beta^n x) - f(\alpha^n\beta^n x) + f(\alpha^n\beta^n x) - F_\alpha(\alpha^n\beta^n x)] \\
&\in |\alpha\beta|^{n(p-1)} \left(\frac{\delta(\beta)}{1-|\beta|^{p-1}} + \frac{\delta(\alpha)}{1-|\alpha|^{p-1}}\right) K(x) \operatorname{seq}\operatorname{cl}\operatorname{conv}(V \cup (-V)),
\end{aligned}$$

so, from (15)

$$F_\alpha(x) - F_\beta(x) = \lim_{n\to\infty} \big(\alpha^{-n} F_\beta(\alpha^n x) - \beta^{-n} F_\alpha(\beta^n x)\big) = 0 \text{ for every } x \in U.$$

2) Let $|\alpha|^{p-1} < 1$ and $|\beta|^{p-1} > 1$. For an arbitrary $x \in U$, as above, from (8), we get

$$\begin{aligned}
\beta^n F_\alpha(\beta^{-n} x) - \beta^n f(\beta^{-n} x) &\in |\beta|^n \frac{\delta(\alpha)\, K(\beta^{-n} x)}{1-|\alpha|^{p-1}} \operatorname{seq}\operatorname{cl}\operatorname{conv}(V \cup (-V)) \\
&\subset |\beta|^{n(p-1)} \frac{\delta(\alpha)\, K(x)}{1-|\alpha|^{p-1}} \operatorname{seq}\operatorname{cl}\operatorname{conv}(V \cup (-V)),
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \beta^n F_\alpha(\beta^{-n} x) = F_\beta(x) \quad \text{for } x \in U. \tag{16}$$

On the other hand, by (12), we obtain

$$\begin{aligned}
\alpha^{-n} F_\beta(\alpha^n x) - \alpha^{-n} f(\alpha^n x) &\in \frac{\delta(\beta)\, K(\alpha^n x)}{|\alpha|^n(|\beta|^{p-1} - 1)} \operatorname{seq}\operatorname{cl}\operatorname{conv}(V \cup (-V)) \\
&\subset |\alpha|^{n(p-1)} \frac{\delta(\beta)\, K(x)}{|\beta|^{p-1} - 1} \operatorname{seq}\operatorname{cl}\operatorname{conv}(V \cup (-V)),
\end{aligned}$$

so

$$\lim_{n\to\infty} \alpha^{-n} F_\beta(\alpha^n x) = F_\alpha(x) \quad \text{for each } x \in U. \tag{17}$$

Thus by (8), (12), (13) and (14) we have

$$\begin{aligned}\alpha^{-n}F_\beta(\alpha^n x)-\beta^n F_\alpha(\beta^{-n}x) &= \left(\frac{\beta}{\alpha}\right)^n[\beta^{-n}F_\beta(\alpha^n x)-\alpha^n F_\alpha(\beta^{-n}x)]\\ &= \left(\frac{\beta}{\alpha}\right)^n[F_\beta(\alpha^n\beta^{-n}x)-f(\alpha^n\beta^{-n}x)+f(\alpha^n\beta^{-n}x)-F_\alpha(\alpha^n\beta^{-n}x)]\\ &\in \left|\frac{\beta}{\alpha}\right|^{n(1-p)}\left(\frac{\delta(\beta)}{|\beta|^{p-1}-1}+\frac{\delta(\alpha)}{1-|\alpha|^{p-1}}\right)K(x)\ \mathrm{seq\,cl\,conv}(V\cup(-V)),\end{aligned}$$

and consequently, by (16) and (17)

$$F_\alpha(x)-F_\beta(x)=\lim_{n\to\infty}\bigl(\alpha^{-n}F_\beta(\alpha^n x)-\beta^n F_\alpha(\beta^{-n}x)\bigr)=0\quad\text{for } x\in U.$$

3) Let $|\alpha|^{p-1}>1$ and $|\beta|^{p-1}>1$. For an arbitrary $x\in U$, as above, from (12) we obtain

$$\begin{aligned}\beta^n F_\alpha(\beta^{-n}x)-\beta^n f(\beta^{-n}x) &\in |\beta|^n\frac{\delta(\alpha)\,K(\beta^{-n}x)}{|\alpha|^{p-1}-1}\ \mathrm{seq\,cl\,conv}(V\cup(-V))\\ &\subset |\beta|^{n(1-p)}\frac{\delta(\alpha)\,K(x)}{|\alpha|^{p-1}-1}\ \mathrm{seq\,cl\,conv}(V\cup(-V)),\end{aligned}$$

so

$$\lim_{n\to\infty}\beta^n F_\alpha(\beta^{-n}x)=F_\beta(x)\quad\text{for each } x\in U. \tag{18}$$

Then, by (14)

$$\begin{aligned}\alpha^n F_\beta(\alpha^{-n}x)-\beta^n F_\alpha(\beta^{-n}x) &= (\alpha\beta)^n[\beta^{-n}F_\beta(\alpha^{-n}x)-\alpha^{-n}F_\alpha(\beta^{-n}x)]\\ &= (\alpha\beta)^n[F_\beta(\alpha^{-n}\beta^{-n}x)-f(\alpha^{-n}\beta^{-n}x)+f(\alpha^{-n}\beta^{-n}x)-F_\alpha(\alpha^{-n}\beta^{-n}x)]\\ &\in |\alpha\beta|^{n(1-p)}\left(\frac{\delta(\beta)}{|\beta|^{p-1}-1}+\frac{\delta(\alpha)}{|\alpha|^{p-1}-1}\right)K(x)\ \mathrm{seq\,cl\,conv}(V\cup(-V)),\end{aligned}$$

and from (18)

$$F_\alpha(x)-F_\beta(x)=\lim_{n\to\infty}\bigl(\alpha^n F_\beta(\alpha^{-n}x)-\beta^n F_\alpha(\beta^{-n}x)\bigr)=0\quad\text{for each } x\in U.$$

Thus we may define

$$F:=F_\beta,\quad \beta\in A\setminus\{-1,1\},$$

which satisfies

$$F(\alpha x) = \alpha F(x) \quad \text{for all } \alpha \in A \setminus \{-1\},\ x \in U.$$

One can show, by induction, that

$$F(\lambda_1 \cdots \lambda_n \mu_1^{-1} \cdots \mu_m^{-1} x) = \lambda_1 \cdots \lambda_n \mu_1^{-1} \cdots \mu_m^{-1} F(x)$$

for all $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m \in A \setminus \{-1\}$ and each $x \in U$, so

$$F(\alpha x) = \alpha F(x) \quad \text{for } \alpha \in \langle A \setminus \{-1\} \rangle,\ x \in U. \tag{19}$$

From (8) and (12) we have

$$F(x) - f(x) \in c\, K(x) \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)) \quad \text{for every } x \in U,$$

where

$$c := \min\left(\inf_{\alpha \in A_p} \frac{\delta(\alpha)}{1 - |\alpha|^{p-1}}, \inf_{\alpha \in A^p} \frac{\delta(\alpha)}{|\alpha|^{p-1} - 1} \right).$$

To prove the uniquenes of F let us suppose that F and F' satisfy (19) and there exist non-negative real constants c and c' such that

$$\begin{aligned} F(x) - f(x) &\in c\, K(x) \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)), \quad && x \in U, \\ F'(x) - f(x) &\in c'\, K(x) \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)), && x \in U. \end{aligned}$$

Consider two cases:
1) there exists an $\alpha \in A$ such that $|\alpha|^{p-1} < 1$. Then, for an arbitrary $x \in U$,

$$\begin{aligned} F(x) - F'(x) &= \alpha^{-n} F(\alpha^n x) - \alpha^{-n} f(\alpha^n x) + \alpha^{-n} f(\alpha^n x) - \alpha^{-n} F'(\alpha^n x) \\ &\in \frac{c + c'}{|\alpha|^n} K(\alpha^n x) \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)) \\ &\subset |\alpha|^{n(p-1)} (c + c')\, K(x) \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)) \end{aligned}$$

for every $n \in \mathbb{N}$, so $F(x) - F'(x) = 0$ for each $x \in U$;

2) there exists an $\alpha \in A$ such that $|\alpha|^{p-1} > 1$. Then, for an arbitrary $x \in U$,

$$\begin{aligned} F(x) - F'(x) &= \alpha^n F(\alpha^{-n}x) - \alpha^n f(\alpha^{-n}x) + \alpha^n f(\alpha^{-n}x) - \alpha^n F'(\alpha^{-n}x) \\ &\in |\alpha|^n (c + c')\, K(\alpha^{-n}x) \operatorname{seq\,cl\,conv}(V \cup (-V)) \\ &\subset |\alpha|^{n(1-p)} (c + c')\, K(x) \operatorname{seq\,cl\,conv}(V \cup (-V)) \end{aligned}$$

for every $n \in \mathbb{N}$, so $F(x) - F'(x) = 0$ for each $x \in U$, which completes the proof. □

Remark 1. If $0 \in U$, then, as it is easilly seen, from (19) we obtain $F(0) = 0$, so F satisfies

$$F(\alpha x) = \alpha F(x) \quad \text{for all } \alpha \in \langle A \setminus \{-1\} \rangle \cup \{0\},\ x \in U.$$

Then in the case $\langle A \setminus \{-1\} \rangle = \mathbb{R}_0$, the function F is homogeneous. If moreover $c = 0$, then the function f is homogeneous.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x + 2 & \text{for } x \in \langle 2 \rangle \\ x - 2 & \text{for } x \in -\langle 2 \rangle \\ 0 & \text{for } x \in \mathbb{R} \setminus \langle \{-2, 2\} \rangle. \end{cases}$$

One can check that f satisfies (5) with $A = \{-2, 2\}$, $\delta \equiv 1$, $K \equiv 1$, $p = 0$, $V = [-1, 1]$. The function

$$F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \begin{cases} x & \text{for } x \in \langle \{-2, 2\} \rangle \\ 0 & \text{for } x \in \mathbb{R} \setminus \langle \{-2, 2\} \rangle. \end{cases}$$

satisfies

$$F(\alpha x) = \alpha F(x) \quad \text{for } \alpha \in \langle \{-2, 2\} \rangle \cup \{0\},\ x \in \mathbb{R},$$

and moreover

$$F(x) - f(x) = \begin{cases} -2 & \text{for } x \in \langle 2 \rangle \\ 2 & \text{for } x \in -\langle 2 \rangle \\ 0 & \text{for } x \in \mathbb{R} \setminus \langle \{-2, 2\} \rangle. \end{cases}$$

On the other hand

$$A_0 = \{-2, 2\}, \quad A^0 = \emptyset,$$

so

$$c = \frac{1}{1 - 2^{-1}} = 2.$$

Thus

$$F(x) - f(x) \in [-2, 2] = 2 \cdot [-1, 1],$$

which shows that the estimation obtained in the above theorem is the best one.

The following result may be proved in much the same way as Theorem 1.

Theorem 2. *Let $S \subset X$ be a cone ($tS \subset S$ for $t > 0$) and let $B \subset (0, \infty)$ contain at least one element $\alpha \neq 1$. Let $\delta : A \to \mathbb{R}_+$ be a mapping. Assume that the function $K : S \to \mathbb{R}_+$ satisfies*

$$K(\alpha x) \leq \alpha^p K(x) \quad \text{for } \alpha \in (0, \infty),\ x \in S,$$

with a certain $p \in \mathbb{R} \setminus \{1\}$. Let $V \subset Y$ be a bounded set and let $f : S \to Y$ satisfy

$$\alpha^{-1} f(\alpha x) - f(x) \in \delta(\alpha)\, K(x)\, V \quad \text{for } \alpha \in B,\ x \in S.$$

Then there exists a unique function $F : S \to Y$ satisfying

$$F(\alpha x) = \alpha F(x) \quad \text{for all } \alpha \in \langle B \rangle,\ x \in S,$$

and such that

$$F(x) - f(x) \in c\, K(x)\, \operatorname{seq} \operatorname{cl} \operatorname{conv}(V \cup (-V)), \quad \text{for } x \in S,$$

where

$$c := \min\left(\inf_{\alpha \in B_p} \frac{\delta(\alpha)}{1 - \alpha^{p-1}}, \inf_{\alpha \in B^p} \frac{\delta(\alpha)}{\alpha^{p-1} - 1} \right),$$

with the sets B_p and B^p are defined as these ones for the set A.

Remark 2. Taking in Theorem 2

$$\delta = 0,\ K = 1,\ p = 0,\ B \subset (1, \infty),\ \operatorname{int} B \neq \emptyset, \quad (\text{then } \langle B \rangle = (0, \infty))$$

we obtain Theorem 1 from [2], that is Theorem 2 generalizes Theorem 1 from [2].

2. Corollaries

The condition (15) can be considered, in a special case, for a normed space Y. For this purpose we rewrite it in a different from. We take

$$V = \{y \in Y : \|y\| \leq 1\}.$$

Then (5) can be writen as

$$\|\alpha^{-1} f(\alpha x) - f(x)\| \leq \delta(\alpha)\, K(x) \quad \text{for } \alpha \in A,\ x \in U. \tag{20}$$

From Theorem 1 and Remark 1 we obtain

Theorem 3. *Let X, U, A, δ and K be as in Theorem 1. Let Y be a Banach space, $\langle A \setminus \{-1\}\rangle = \mathbb{R}_0$ and $0 \in U$. If a function $f : U \to Y$ satisfies* (20), *then there exists a unique homogeneous function $F : U \to Y$ such that*

$$\|F(x) - f(x)\| \leq c\, K(x) \quad \textit{for } x \in U,$$

where c is defined as in Theorem 1. If moreover $c = 0$, then the function f is homogeneous.

Jacek Tabor and Józef Tabor have examined in [5] the condition

$$\|f(\alpha x) - \alpha f(x)\| \leq |\alpha|^q K(x) \quad \text{for } \alpha \in \mathbb{K},\ x \in X, \tag{21}$$

for a vector space X and a normed space Y over a real or complex field $\mathbb{K}$, with a function $K : X \to \mathbb{R}_+$ satisfying (4) with a constant p, where $p, q \in \mathbb{R}_+$, $p \neq q$. They have proved that every function $f : X \to Y$ satisfying (21) is homogeneous

We rewrite (21) in the following form

$$\|\alpha^{-1} f(\alpha x) - f(x)\| \leq |\alpha|^{q-1} K(x). \tag{22}$$

From Theorem 3 (the details are left to the reader) we obtain

Corollary 2. *Let X, Y, K be as in Theorem 3. Let $p, q \in \mathbb{R}$ and let $A \subset \mathbb{R}_0$ be such that $\langle A \setminus \{-1\}\rangle = \mathbb{R}_0$. If there exists a sequence $(\alpha_n : n \in \mathbb{N}) \subset A$ such that $\alpha_n \to 0$ whenever*

$$p > 1 \quad \text{and } q > 1 \quad \text{or } p < 1 \quad \text{and } q > p,$$

or, if there exists a sequence $(\beta_n : n \in \mathbb{N}) \subset A$ *such that* $|\beta_n| \to \infty$ *whenever*

$$p < 1 \quad \text{and } q < 1 \quad \text{or } p > 1 \quad \text{and } q < p,$$

then every mapping $f : X \to Y$ *satisfying the condition* (22) *for* $\alpha \in A$, $x \in X$ *is homogeneous.*

Moreover, one can deduce from above Corollary the following

Corollary 3. *Every function* $f : X \to Y$ *satisfying*

$$\|\alpha^{-1} f(\alpha x) - f(x)\| \leq \varepsilon \quad \text{for } \alpha \in (-\gamma, 0),\ x \in X, \tag{23}$$

with some positive real constant γ, *is homogeneous.*

Remark 3. As far as it is for the author known, in the case $p = 1$, the problem of the stability of the homogeneous equation is open.

Remark 4. As one can see, the result of this paper is incomparable with those ones given by J. Schwaiger in [3]. On the one hand, the mentioned results are proved in more general setting, on the other, the bound in considered there conditions (beside the result being a generalization of the one from [5]) does not depend on the variable x from the set X.

Acknowlegement. The author wishes to express his gratitude to Professor Józef Tabor for his helpful suggestions during the preparation of this paper.

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(Received November 18, 1996; revised August 10, 1998)