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Moufang loops of order $2m$

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Abstract. Let L be a Moufang loop of order $2m$, $(2,m)=1$. Then F. LEONG and P. E. TEH have proven in [3] that there exists a normal subloop M of order m in L. Furthermore, they have shown that if $|M|=p^2$, then L is a group. We extend this result with M which is an abelian group of order $p_1^2\cdots p_n^2$ where $p_1,\dots,p_n$ are distinct primes and also $M=C_p\times C_{p^n}$.

1. Definitions

1. A loop $\langle L,\cdot\rangle$ is a Moufang loop if $(wx\cdot w)y=w(x\cdot wy)$ $\forall w,x,y\in L$.

2. $(x,y,z)=(x\cdot yz)^{-1}(xy\cdot z)$ for $x,y,z\in L$. L_a is the subloop generated by all (x,y,z) in L. Clearly, L is a group if and only if $L_a=\{1\}$.

3. Define

$gR(x,y)=(gx\cdot y)(xy)^{-1}$

$gL(x,y)=(yx)^{-1}(y\cdot xg)$

$gT(x)=x^{-1}\cdot gx$

$I(L)=\langle R(x,y),L(x,y),T(x)\mid x,y\in L\rangle$ is called the inner mapping group of L.

4. Let K be a subloop of L and π a set of primes

(i) K is a normal subloop of L, if $K\theta=K$ for all $\theta\in I(L)$ where $K\theta=\{k_i\theta\mid k_i\in K\}$.

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(ii) K is a π-loop if the order of every element of K is a π-number. (A positive integer n is a π-number if every prime divisor of n lies in π.)

(iii) K is a Hall π-subloop of the finite loop L if $|K|$ is the largest π-number that divides $|L|$.

(iv) K is a Sylow p-subloop of L if K is a Hall π-subloop of L and $\pi = \{p\}$ (p is a prime).

2. Known results with Moufang loops

Let L be a Moufang loop.

$\mathbf{R}_1$ L is diassociative, that is $\langle x, y\rangle$ is a group for any x, y in L. [1, p. 115, Lemma 3.1].

$\mathbf{R}_2$ If $(x, y, z) = 1$ for some $x, y, z \in L$, then $\langle x, y, z\rangle$ is a group. [1, p. 117, Moufang's Theorem].

$\mathbf{R}_3$ If $x \in L$ and $\theta \in I(L)$, then $(x^n)\theta = (x\theta)^n$ for any integer n. [1, p. 120, (4.1)].

$\mathbf{R}_4$ If H is a subloop of L, u is an element of L and d is the smallest positive integer such that $u^d \in H$, then $|\langle H, u\rangle| \geq d|H|$. [2, p. 5, Lemma 0].

$\mathbf{R}_5$ Suppose $|L| = 2m$, $(2, m) = 1$. Then there exists a normal subloop M of order m such that $L = C_2 \rtimes M$, i.e., $L = C_2M$ and $C_2 \cap M = \{1\}$. [3, p. 411, Lemma 1].

$\mathbf{R}_6$ Suppose $|L| = 2p^2$, $(2, p) = 1$. Then L is a group. [3, p. 411, Lemma 2].

$\mathbf{R}_7$ Let M be a normal subloop of the finite loop L. If H is a Hall subloop of M such that H is normal in M, then H is normal in L. [4, p. 879, Lemma 1].

3. Moufang loops of order $2m$

Theorem 1. *Let L be a Moufang loop of order $2m$, $(2, m) = 1$ and $m = p_1^2 \cdots p_n^2$ where $p_1, \ldots, p_n$ are distinct primes. Suppose there exists* *a normal subloop M of order m which is an abelian group. Then L is a group.*

Proof. By $\mathbf{R}_5$, we can write $L = C_2 \rtimes M$ for some cyclic group C_2. We proceed by induction on n. If $n = 1$, then by $\mathbf{R}_6$, L is a group. So we can assume that the result holds for $1 \leq n \leq k$. Now let $n = k+1$ and suppose P_i is a Sylow p_i-subloop of M for $i = 1, 2$. Then $P_i \lhd M$ as M is abelian. Since P_1 is a Hall subloop of M and $P_1 \lhd M$, $P_1 \lhd L$ by $\mathbf{R}_7$. Similarly, $P_2 \lhd L$.

Now L/P_i is a group by induction. So $L_a \subset P_i$ for both $i = 1$ and 2. Then $L_a \subset P_1 \cap P_2 = \{1\}$. Thus L is a group.

Remark. In [3], it was proven that all Moufang loops of order $2m$ are groups provided the three conditions below are satisfied

(i) $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_i \leq 2$, p_i are distinct odd primes,

(ii) $p_i \neq 1 \bmod p_j$ for all i and j,

(iii) $p_i^2 \neq 1 \bmod p_j$ if $\alpha_i = 2$, for all i and j.

Essentially our proof in Theorem 1 is identical to the proof of the above theorem. The authors similarly prove that the Moufang loop could be written as $C_2 \rtimes M$ where $|M| = m$. Although in our situation, we assume that M is an abelian group, the authors here prove that M is an abelian group (when the three conditions are satisfied).

Lemma. *Let L be a Moufang loop of order $2p^{n+1}$, $(2,p) = 1$ such that $L = C_2 \rtimes M$ where $M = C_p \times C_{p^n}$. Write $C_2 = \langle w \rangle$, $C_p = \langle x \rangle$ and $C_{p^n} = \langle y \rangle$. Suppose L is not a group. Then*

(i) *$u \in L$ and $|u| = p^n \Rightarrow |\langle w, u \rangle| = 2p^n$ and $wuw = u^{-1}$.*

(ii) *$wxw = x^{-1}$.*

Proof. (i) Write $H = \langle w, u \rangle$. Then by $\mathbf{R}_1$, H is a group. Clearly, $2p^n \;\Big|\; |H|$. Suppose $|H| > 2p^n$. If $M \subset H$, then $|M| = p^{n+1} \;\Big|\; |H|$ and $|H| = 2p^{n+1} = |L|$. Hence L would be a group. This is a contradiction. So $M \not\subset H$ and there exists $v \in M - H$. So $|v| = p^\alpha$ for some $\alpha \geq 1$. Let d be the smallest positive integer such that $v^d \in H$. If $d < p$, then $(d, p^\alpha) = 1$, and there exist integers k and h such that $kd + hp^\alpha = 1$. Since $v^d, v^{p^\alpha} \in H$, $v = v^{kd+hp^\alpha} = (v^d)(v^{p^\alpha})^h \in H$. This contradicts $v \in M - H$. So $d \geq p$. Thus by $\mathbf{R}_4$, $|\langle H, y \rangle| \geq |H| > 2p^{n+1} = |L|$. This is a contradiction.

Thus $|H| = 2p^n$. Then $wuw = u$ or u^{-1}. Suppose $wuw = u$. Then $|wu| = 2p^n$. Thus $L = \langle wu, x\rangle$ is a group by $\mathbf{R}_1$. This is a contradiction. Therefore $wuw = u^{-1}$.

(ii) Suppose $wxwx \neq 1$. Since $M \lhd L$, $wxw \in M$. Thus $(wxw)x = x(wxw)$ as M is abelian. Also $[(wxw)x]^p = (wxw)^p x^p = wx^p w = 1$. Thus $|wxwx| = p$. If $wxwx \in \langle y\rangle$, then $wxwx = y^{cp^{n-1}}$, $1 \leq c \leq p-1$. Now $w(wxwx)w = w(y^{cp^{n-1}})w$. So $xwxw = (wyw)^{cp^{n-1}}$. Then $y^{cp^{n-1}} = (y^{-1})^{cp^{n-1}}$ by (i). Thus $y^{2cp^{n-1}} = 1$.

So $p^n \mid 2cp^{n-1}$, i.e., $p \mid 2c$. This is a contradiction. Therefore $wxwx \notin \langle y\rangle$ and $\langle w(wxwx)\rangle \cap \langle y\rangle = \{1\}$. Clearly $w[wxwx] = [wxwx]w$. So $\langle w, wxwx\rangle = \langle xwx\rangle = C_{2p}$. We know that $|\langle xwx, y\rangle| \geq |\langle xwx\rangle|\,|\langle y\rangle| = 2p \cdot p^n = |L|$. So $L = \langle xwx, y\rangle$ is a group by $\mathbf{R}_1$. This is a contradiction. Therefore $wxwx = 1$, i.e., $wxw = x^{-1}$.

Theorem 2. *Let L be a Moufang loop of order $2p^{n+1}$ where p is an odd prime. Suppose the normal subloop of order p^{n+1}, $M = C_p \times C_{p^n}$. Then L is a group.*

Proof. Assume L is not a group. Write $C_2 = \langle w\rangle$, $C_p = \langle x\rangle$ and $C_{p^n} = \langle y\rangle$. Now

$$
\begin{aligned}
w(x \cdot wy) &= (wx \cdot w)y && \text{by the Moufang identity}\\
&= x^{-1}y && \text{by } \mathbf{R}_1 \text{ and Lemma (ii).}
\end{aligned}
$$

Thus $$x(wy) = w(x^{-1}y) \qquad \text{as } w^{-1} = w.$$

Since $$\langle x^{-1}, y\rangle = C_p \times C_{p^n}, \qquad |x^{-1}y| = p^n.$$

Then
$$
\begin{aligned}
x(y^{-1}w) &= (x^{-1}y)^{-1}w && \text{by Lemma (i)}\\
&= (y^{-1}x)w\\
&= (xy^{-1})w && \text{as } M \text{ is abelian.}
\end{aligned}
$$

So $$(x, y^{-1}, w) = 1.$$

By $\mathbf{R}_2$, $$\langle x, y^{-1}, w\rangle = \langle x, y, w\rangle = L \qquad \text{is a group.}$$

Open question

Let L be a Moufang loop such that $L = C_2 \rtimes M$ where M is an abelian group. Then is L also a group?

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