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Decreasing sequences of means appearing from non-decreasing functions

By LUCIO R. BERRONE (Rosario)

Abstract. Most of the classically studied means are functions of the form $\mathbb{R}^n \ni X \mapsto f^{-1}(A(f;X))$ where $A(\cdot, X)$ is a linear operator which may vary with X and f is a continuous and strictly monotonic function, the problem of comparison between different means being a very basic one of the theory. By using completely monotone functions in $(0, +\infty)$, a decreasing sequence of means is presently associated with a given non-decreasing function. Moreover, some interesting properties of completely monotone functions are made evident and an apparently new criterion to compare means is established.

1. Introduction

Let I denote an interval of $\mathbb{R}$. Along this paper we are concerned with continuous means; that is, functions $\mu : I^n \to \mathbb{R}$ which satisfy the two properties:

M1) μ is continuous;

M2) if $X = (x_1, x_2, \dots, x_n) \in I^n$, then $\min_{1 \le i \le n} x_i \le \mu(X) \le \max_{1 \le i \le n} x_i$.

Furthermore, a continuous mean μ is said to be *symmetric* if the property

M3) $\mu(\pi(X)) = \mu(X),\ X \in I^n$, for every permutation π of the coordinates,

holds. From property **M2)**, which is usually known as ([17] and also [2], [3]), we easily derive that if $x \in I$ and $X = (x, x, \dots, x)$, then $\mu(X) = x$.

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Several types of means are classically associated with a continuous and strictly monotonic function f defined on an interval $I \subseteq \mathbb{R}$. For example, when $X = (x_1, x_2, \dots, x_n) \in I^n$ and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a real non-negative n-tuple with $\sum_{i=1}^n \lambda_i = 1$, the *quasilinear f -mean* of X with weight Λ is defined by

$$M_n(X; \Lambda, f) = f^{-1}\left(\sum_{i=1}^{n} \lambda_i f(x_i)\right). \tag{1}$$

The quasilinear f-mean with weight $\Lambda = (1/n, \dots, 1/n)$ is symmetric and it is called *quasiarithmetic mean* and denoted by $M_n(X; f)$. We will frequently limit ourselves to consider means in two variables and then a simplified notation will be used. The subscript in $M_n(X; f)$ is omitted when $n = 2$, so that $M(x, y; f)$ stands for the quasiarithmetic f-mean in two variables; i.e., $M(x, y; f) = M_2((x, y); f)$.

Let $x, y \in I$, $x < y$. If we set $n = m + 1$ in (1) and we define $X_m = (x + (y - x)i/m)_{i=0}^m$ and $\Lambda_m = (\lambda_i)_{i=0}^m$ with $\lambda_0 = 0$, $\lambda_i = 1/m$, $i = 1, 2, \dots, m$; then, by letting $m \uparrow +\infty$ we obtain the *Lagrangian mean of x, y associated with f*:

$$L(x, y; f) = \lim_{m \uparrow +\infty} M_{m+1}(X_m; \Lambda_m, f) = f^{-1}\left(\frac{1}{y - x}\int_x^y f(\xi)\, d\xi\right). \tag{2}$$

The definition of $L(x, y; f)$ for every x, y is completed by symmetry ($L(x, y; f) = L(y, x; f)$, $x, y \in I$) and imposing $L(x, x; f) = x$, $x, y \in I$, which is compatible with the obvious property

$$\lim_{y \downarrow x} L(x, y; f) = x.$$

Diverse names have been given to the means defined by (2). In [4], G. Aumann called them *functional means* (funktionalmittel). Because of their connections with the mean value theorem of the Calculus, these means are called *mean-value means* in the book [8]. In this respect, note that if we consider the function $F(x) = \int^x f(\xi)\, d\xi$, $x \in I$, from (2) the following identity can be derived

$$F(y) - F(x) = (y - x)F'(L(x, y; f)),$$

which makes clear that $L(x, y; f)$ is the intermediate point in the Lagrange's mean value theorem when applied to the strictly convex (or strictly concave) and differentiable function F. LiKe in [7], we favor the more concise term ‘Lagrangian’ to denote this class of means.

Our next example of continuous mean arises from considering two continuous and strictly monotonic functions f and σ defined in I. Then, the function

$$C(x,y;f,\sigma) = \left| \begin{array}{ll} f^{-1}\left[\frac{1}{\sigma(y)-\sigma(x)}\int_x^y f(\xi)d\sigma(\xi)\right], & x,y\in I,\ x\neq y \\ x & x,y\in I,\ x=y \end{array} \right., \tag{3}$$

defines a continuous and symmetric mean on I which we call *Cauchy* (f,σ)-*mean*. It should be noted that if $id(x)\equiv x$, then $C(x,y;f,id)\equiv L(x,y;f)$. Moreover, $C(x,y;f,f)\equiv M(x,y;f)$. Indeed, it is not a difficult task to trace back the origin of these means up to the Cauchy mean-value theorem from Calculus. In fact, if $r,s\in\mathcal{C}^1(I)$ and $x,y\in I$, then we know that

$$\frac{r(y)-r(x)}{s(y)-s(x)} = \frac{r'(\theta)}{s'(\theta)}, \tag{4}$$

holds for certain $x<\theta<y$. Setting $f(\xi)=r'(\xi)/s'(\xi)$ and $\sigma(\xi)=s(\xi)$, $\xi\in I$, the identity (4) may be rewritten as

$$\frac{1}{\sigma(y)-\sigma(x)}\int_x^y f(\xi)d\sigma(\xi) = f(\theta),$$

and hence definition (3) is deduced after assuming f strictly monotonic.

Now assume that the function f possesses derivatives of all orders in the open interval $I=(a,b)$, $(-\infty\le a<b\le+\infty)$, and that $f^{(n)}$ does not change of sign in I for every $n\in\mathbb{N}_0$. The class of these functions will be denoted by $\mathcal{C}^\infty_{NV}(I)$ along this paper. Since the work of S. N. Bernstein [6], it is known that a function $f\in\mathcal{C}^\infty_{NV}(I)$ is actually analytic. For a member f of $\mathcal{C}^\infty_{NV}(I)$ we can define a class of means which generalizes the Lagrangian one. To this end, we fix $n\in\mathbb{N}$ and we consider $f\in\mathcal{C}^\infty_{NV}(I)$ such that $f^{(n+2)}$ does not vanish identically in I. Observe that the derivatives $f^{(k)}$, $0\le k\le n$, of such function f are all strictly increasing in I. In fact, by definition, $f^{(k+1)}$ and $f^{(k+2)}$ do not change of sign in I. Since $f^{(k+2)}$ does not vanishes identically in I, if $f^{(k+1)}(\xi)=0$ for a certain $\xi\in I$, then we should have $f^{(k+1)}(x)=0$ for $x\in(a,\xi]$ or $x\in[\xi,b)$ and therefore $f^{(k+1)}\equiv 0$ in I by analyticity, a contradiction. Then we set

$x, y \in I,\ x < y$, and we recall the Taylor's formula with remainder in Lagrange's form for the function f:

$$
\begin{aligned}
f(y) &= f(x) + f'(x)(y-x) + \dots \\
&\quad + \frac{f^{(k-1)}(x)}{(k-1)!}(y-x)^{k-1} + \frac{f^{(k)}(\theta_k)}{k!}(y-x)^k,
\end{aligned}
\tag{5}
$$

where $1 \le k \le n$ and $x < \theta_k < y$. The strict monotonicity of $f^{(k)}$ implies the intermediate value θ_k is unique, so defining a mean value between x and y which shall be denoted henceforth by $T^{(k)}(x, y; f)$. An explicit representation of $T^{(k)}(x, y; f)$ can be given with the help of the integral form of the remainder for the Taylor's formula. In fact, recalling that

$$
\begin{aligned}
f(y) &= f(x) + f'(x)(y-x) + \dots + \frac{f^{(k-1)}(x)}{(k-1)!}(y-x)^{k-1} \\
&\quad + \frac{1}{(k-1)!}\int_x^y f^{(k)}(\xi)(y-\xi)^{k-1}d\xi,
\end{aligned}
\tag{6}
$$

and taking into account (5) we see that

$$
\frac{f^{(k)}(\theta_k)}{k!}(y-x)^k = \frac{1}{(k-1)!}\int_x^y f^{(k)}(\xi)(y-\xi)^{k-1}d\xi,
$$

whence

$$
T^{(k)}(x, y; f) = \theta_k = \left(f^{(k)}\right)^{-1}\left[\int_x^y f^{(k)}(\xi)\frac{k(y-\xi)^{k-1}}{(y-x)^k}d\xi\right]. \tag{7}
$$

After a reiterated application of the L'Hospital rule to (7) we conclude that $\lim_{y\downarrow x} T^{(k)}(x, y; f) = x$, so that it make sense to define by continuity $T^{(k)}(x, x; f) = x$. Observe that for $k > 1$ and $x \neq y$ we generally have $T^{(k)}(y, x; f) \neq T^{(k)}(x, y; f)$. Finally, the identity $T^{(1)}(x, y; f) = L(x, y, f')$, $x, y > 0$, is obvious. We call *Lagrangian mean of order k associated with f* the continuous non-symmetric mean $T^{(k)}(x, y; f)$, $k \in \mathbb{N}$.

Let M and N be two means defined on I^n; M and N are said to be *comparable* if $M(X) \le N(X)$ or $M(X) \ge N(X)$ for every $X \in I$. In this paper we are principally interested in the problem of the comparison between the above means when calculated for the successive derivatives of a function f belonging to $\mathcal{C}^\infty_{NV}(I)$ and so, some results and terminology will now be exposed regarding to comparison of means and to the space $\mathcal{C}^\infty_{NV}(I)$. A necessary and sufficient criterion of comparison for quasilinear means is provided by the following result.

Theorem 1. *Let f and g be two continuous and strictly monotonic functions defined on I, g increasing (decreasing) then, for all n-tuples $X \in I^n$ and $\Lambda \geq 0$, we have*

$$M_n(X;\Lambda,f) \leq M_n(X;\Lambda,g), \tag{8}$$

if and only if $g \circ f^{-1}$ is convex (concave). If $g \circ f^{-1}$ is strictly convex, there is equality in (8) *if and only if all x_i with $\lambda_i > 0$ are equal, $i = 1, 2, \dots, n$. If g is decreasing (increasing) and $g \circ f^{-1}$ is convex (concave), the inequality* (8) *is reversed.*

This simple but important theorem is well known since the appearance of the treatise [10] at least (although the case of equality in (8) is not discussed there). The statement we have given above is essentially taken from [11] and [8].

Under the hypothesis in Theorem 1 we have
$M_n(X;\Lambda,f) = M_n(X;\Lambda,g)$ identically in n, X and Λ if and only if $g \circ f^{-1}$ is an affine function, that is, $f = \alpha g + \beta$ for some real constants α and β, $\alpha \neq 0$. This fact can be directly derived from Theorem 1 like in [11] (cf. [10] or [8] for a direct proof and references to the original sources). It is noteworthy that the condition $f = \alpha g + \beta$ for constants α and β, $\alpha \neq 0$, is also necessary in order that $M_2(X;\Lambda,f) = M_2(X;\Lambda,g)$, $X \in I^2$, for a fixed weight Λ. A concise statement of this result, whose proof can be found in [10], [2] and [3], is given below.

Theorem 2. *Let f and g satisfy the assumptions of Theorem 1, and let Λ be equal to (λ_1, λ_2) with $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$; then*

$$M_2(X;\Lambda,f) = M_2(X;\Lambda,g),\ X \in I^2,$$

if and only if $f = \alpha g + \beta$ for some constants α and β, $\alpha \neq 0$.

In view of Lagrangian means are, by (2), limits of quasiarithmetic ones, we should hope that theorems similar to 1 and 2 hold for that means too. For instance, Theorem 2 is always true in at least one direction: denoting by $H(x,y;f)$ a mean belonging to any class so far considered, we obviously have $H(x,y;\alpha f + \beta) \equiv H(x,y;f)$ for every real constants $\alpha \neq 0$ and β. Lagrangian means also satisfy the converse statement as it is proved in Chapter 6 of [8] and, by using different techniques, in [7].

Theorem 3. *In order that* $L(x, y; f) = L(x, y; g),\ x, y \in I$ *it is necessary and sufficient that* $f = \alpha g + \beta$ *for some constants* α *and* β, $\alpha \neq 0$.

Now, a result is given furnishing sufficient conditions of comparison between Lagrangian means of arbitrary order.

Theorem 4. *Let be given* $n \in \mathbb{N}$ *and let* f *and* g *belong to* $\mathcal{C}^{\infty}_{NV}(I)$ *with* $g^{(n)}$ *strictly increasing (decreasing); then, the inequality*

$$T^{(n)}(x, y; f) \leq T^{(n)}(x, y; g), \tag{9}$$

holds when $g^{(n)} \circ \left(f^{(n)}\right)^{-1}$ *is convex (concave). If* $g^{(n)}$ *is strictly decreasing (increasing) and* $g^{(n)} \circ \left(f^{(n)}\right)^{-1}$ *is convex (concave), the inequality* (9) *is reversed.*

An analogous result is also valid for Cauchy means. This is by no way an accidental fact because the means so far considered admit the same general representation expressed by

$$\mu(x, y; f, \rho) = f^{-1}\left(\int_x^y f(\xi)d\rho(x, y; \xi)\right), \quad x, y \in I,\ x < y, \tag{10}$$

where $\{\rho(x, y; \cdot) : x, y \in I\}$ is a family of positive measures on I such that $\rho(x, y; \cdot)$ becomes a probability measure when restricted to $[x, y]$; i.e.,

$$\int_x^y d\rho(x, y; \xi) = 1. \tag{11}$$

Thus, the sufficiency of the conditions "g increasing (decreasing), $g \circ f^{-1}$ convex (concave) in I" in order that $\mu(x, y; f, \rho) \leq \mu(x, y; g, \rho)$ holds for $x, y \in I$, is a simple consequence of the Jensen's inequality. Extensive generalizations of this fact (see [5] and Chapter 4 of [18]) have been obtained which rest on the abstract version of the Jensen's inequality given by B. Jessen in [12].

Now we turn to briefly discuss a specific subclass of $\mathcal{C}^{\infty}_{NV}(I)$. A function $f \in \mathcal{C}^{\infty}_{NV}(I)$ is said to be *completely monotone in* I when the inequality $(-1)^n f^{(n)}(x) \geq 0$, $x \in I$, is satisfied for every $n \in \mathbb{N}_0$. Completely monotone functions in the semi-axis $(0, +\infty)$ have very nice properties; for instance, they are analytic in the half-plane $\mathrm{Re}(z) > 0$. Perhaps, the nicest property of these functions is the Hausdorff–Bernstein representation theorem that follows (cf. Chapter 4 of [21]).

Theorem 5. *A necessary and sufficient condition that the function f should be completely monotone in $(0,+\infty)$ is that*

$$f(x)=\int_0^{+\infty} e^{-xt}d\alpha(t), \tag{12}$$

where α is bounded and non-decreasing and the integral converges for $x\in(0,+\infty)$.

An interesting consequence of the Hausdorff–Bernstein theorem is stated in the following corollary.

Corollary 6. *If f is a completely monotone function in $(0,+\infty)$; then, the absolute value of the successive derivatives $f^{(n)}$ are all log-convex functions; i.e., $\ln\left|f^{(n)}\right|$ is a convex function on $(0,+\infty)$ for every $n\in\mathbb{N}_0$.*

The usual proof of this corollary utilizes the representation (12) to show that the inequalities $f^{(n)}(x)f^{(n+2)}(x)-\left(f^{(n+1)}(x)\right)^2\geq 0,\ x\in(0,+\infty)$ are satisfied for every $n\in\mathbb{N}_0$ (see pg. 167 of [21] or pg. 222 of [15]).

Our plan for the next two sections is the following. We first exhibit the relationships existing between the means corresponding to the successive derivatives of a completely monotone function in $(0,+\infty)$. Then, Theorem 5 is used to associate a sequence of comparable means with a non-decreasing function $\alpha:(0,+\infty)\to\mathbb{R}$. The techniques used to reach these results are insufficient to compare Lagrangian means of different orders associated with a given completely monotone function. Notwithstanding, a restricted comparison result is shown to hold for these means whose proof is finally extended to embrace a fairly general type of means. In Section 3 we show that simple relationships exist between different classes of means associated with a completely monotone function. The convergence of sequences of comparable means is also studied there. Since several means we are considering arise as intermediate points in elementary theorems from the Calculus, remarkable properties of completely monotone functions are brought out in this connection.

2. Means associated with completely monotone functions

For a function f belonging to $\mathcal{C}^{\infty}_{NV}(I)$ whose successive derivatives do not vanish identically in I, the function f itself and their derivatives are all strictly monotonic and continuous functions and therefore, it make sense to consider means associated with $f^{(n)}$ for every $n \in \mathbb{N}_0$. In this regard, let us introduce some useful notation. For the sake of simplicity, we restrict ourselves to consider means in two variables. Let σ be a continuous and strictly monotonic function and let f belong to $\mathcal{C}^{\infty}_{NV}(I)$ such that $f^{(n)}$ does not vanishes identically, $n \in \mathbb{N}_0$. Given $n \in \mathbb{N}_0$, we define

$$\begin{aligned} M^{(n)}(x, y; f) &= M(x, y; f^{(n)}), \\ L^{(n)}(x, y; f) &= L(x, y; f^{(n)}), \\ C^{(n)}(x, y; f, \sigma) &= C(x, y; f^{(n)}, \sigma). \end{aligned} \tag{13}$$

Our first result shows that these sequences of means are decreasing in n when f is a completely monotone function in $(0, +\infty)$.

Theorem 7. *Let f be a completely monotone function in $(0, +\infty)$ which is not a constant. Then, the inequalities*

$$\begin{aligned} M^{(n+1)}(x, y; f) &\leq M^{(n)}(x, y; f), \\ L^{(n+1)}(x, y; f) &\leq L^{(n)}(x, y; f), \\ C^{(n+1)}(x, y; f, \sigma) &\leq C^{(n)}(x, y; f, \sigma), \quad x, y > 0, \end{aligned} \tag{14}$$

hold for every $n \in \mathbb{N}_0$.

Roughly speaking, the second inequality in (14) expresses that, if f is a completely monotone function in $(0, +\infty)$, then the intermediate value of the mean value theorem applied to $f^{(n)}$ moves to the left as n increases.

Proof. Let us prove the first inequality of (14); i.e.,

$$\begin{aligned} &\left(f^{(n+1)}\right)^{-1}\left(\frac{f^{(n+1)}(x) + f^{(n+1)}(y)}{2}\right) \\ &\qquad \leq \left(f^{(n)}\right)^{-1}\left(\frac{f^{(n)}(x) + f^{(n)}(y)}{2}\right), \qquad x, y > 0. \end{aligned} \tag{15}$$

Suppose that n is even. Since f does not reduce to a constant, we have $f^{(n+2)} > 0,\ x > 0$ (cf. pgs. 167–168 of [21]); i.e., $f^{(n+1)}$ is strictly increasing and then (15) is equivalent to

$$\begin{aligned}\tag{16} &\frac{f^{(n+1)}(x)+f^{(n+1)}(y)}{2}\\ &\leq \left(f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}\right)\left(\frac{f^{(n)}(x)+f^{(n)}(y)}{2}\right), \qquad x,y>0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\tag{17} &\left(f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}\right)''\left(f^{(n)}(x)\right)\\ &=\frac{f^{(n+3)}(x)f^{(n+1)}(x)-\left(f^{(n+2)}(x)\right)^2}{\left(f^{(n+1)}(x)\right)^3}, \qquad x>0,\end{aligned}$$

and

$$\tag{18} f^{(n+3)}(x)f^{(n+1)}(x)-\left(f^{(n+2)}(x)\right)^2\geq 0, \qquad x>0.$$

The identity (17) is immediate and the inequality (18) holds by Corollary 6 from the introduction. Since $f^{(n+1)}<0$, $x>0$, from (17) and (18) we obtain

$$\left(f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}\right)''\leq 0,$$

Thus, the function $f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}$ is concave and therefore inequality (16) holds. The case in which n is odd can be similarly treated. This case also follows by realizing that $-f^{(n)}$ is completely monotone when n is odd and that a general mean defined by (10) and (11) remains unchanged when the function f is replaced by $\alpha f+\beta$, $(\alpha\neq 0)$ (see the observation immediately below Theorem 2 from the introduction). The proof of the remaining inequalities in (14) is similar and it will not be given here. $\square$

Let us illustrate Theorem 7 with some examples and remarks. First consider the completely monotone function $f(x)=x^{-\gamma}$, $x>0$, $(\gamma>0)$. We have $f^{(n)}(x)=C(n,\gamma)x^{-(\gamma+n)}$ for a certain constant $C(n,\gamma)$ and so

$$M^{(n)}(x,y;f)=\frac{xy}{\left(\frac{x^{\gamma+n}+y^{\gamma+n}}{2}\right)^{1/(\gamma+n)}}=\frac{xy}{\mathcal{M}_{\gamma+n}(x,y)},\ x,y>0,$$

where $\mathcal{M}_r(x,y) = ((x^r+y^r)/2)^{1/r}$ is the *power mean of order* r. Theorem 7 provides $M^{(n+1)}(x,y;f) \le M^{(n)}(x,y;f)$, which implies $\mathcal{M}_{\gamma+n}(x,y) \le \mathcal{M}_{\gamma+n+1}(x,y)$, a particular case of a well known property of the power mean of order r ([10], [8], [18]). By choosing $\sigma(x) = x^\alpha$, $x>0$, $(\alpha \ne \gamma+n)$, we obtain

$$C^{(n)}(x,y;f,\sigma) = \left(\frac{\alpha-\gamma-n}{\alpha}\frac{y^\alpha - x^\alpha}{y^{\alpha-\gamma-n} - x^{\alpha-\gamma-n}}\right)^{\frac{1}{\gamma+n}} = \mathcal{E}(\alpha-\gamma-n,\alpha),$$

where $\mathcal{E}(r,s) = [r(x^s-y^s)/(s(x^r-y^r))]^{1/(s-r)}$ denotes the extended mean introduced by K. B. Stolarsky in [19] and studied by E. B. Leach and M. C. Sholander in [13] and [14]. Thus, from Theorem 7 we deduce $\mathcal{E}(\alpha-\gamma-n-1,\alpha) \le \mathcal{E}(\alpha-\gamma-n,\alpha)$, a particular instance of the monotonicity of extended means (cf. [13]).

The sequences of means given by (14) may well be finite. An extreme example of this situation is furnished by $f(x) = e^{-x}$, $x>0$, for which all terms in (14) are coincident. More generally, two members coincide in any inequality (14) when $f^{(n+1)} \equiv \alpha f^{(n)} + \beta$ for any $n \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \ne 0$. As a simple inductive reasoning can convince us, this differential equation possesses completely monotone solutions only if $\alpha < 0$ and they are given by

$$f(x) = C_1 e^{\alpha x} + C_2, \quad x > 0, \tag{19}$$

with $C_1, C_2 \in \mathbb{R}$ ($C_1 \ne 0$ if we search for non-constant completely monotone solutions). Thus, aside from the trivial case (19) in which f is of exponential type, (14) give rise to an infinite sequence of inequalities.

Theorem 7 may be slightly improved as follows. Let f a non constant function belonging to $\mathcal{C}^\infty_{NV}(0,+\infty)$ such that $-f'$ is completely monotone in $(0,+\infty)$. From Theorem 7 we obtain

$$\begin{aligned} M^{(n+1)}(x,y;f) &\le M^{(n)}(x,y;f), \\ L^{(n+1)}(x,y;f) &\le L^{(n)}(x,y;f), \\ C^{(n+1)}(x,y;f,\sigma) &\le C^{(n)}(x,y;f,\sigma), \quad x,y>0, \end{aligned} \tag{20}$$

for every $n \in \mathbb{N}$. Now, taking into account that $f' < 0$ and

$$\left(f' \circ f^{-1}\right)''(f(x)) = \frac{f'''(x)f'(x) - (f''(x))^2}{(f'(x))^3} \le 0, \quad x > 0,$$

we realize that the same argument employed in the proof of the theorem shows that inequalities (20) are valid for $n = 0$ too. Take, for instance, $f(x) = -\ln x,\ x > 0$. Although f changes of sign in $(0, +\infty)$, we see that $-f'(x) = 1/x$ is completely monotone and so $L^{(n+1)}(x, y; f) \leq L^{(n)}(x, y; f)$, $x, y > 0$, holds for $n \in \mathbb{N}_0$ or, after some calculations,

$$\begin{aligned} e^{-1}\left(x^x y^{-y}\right)^{1/(x-y)} &\geq \frac{x-y}{\ln x - \ln y} \geq \sqrt{xy} \geq \cdots \\ &\geq \left[\frac{x^{n-1}y^{n-1}(x-y)}{(n-1)(x^{n-1}-y^{n-1})}\right]^{1/n} \geq \cdots . \end{aligned} \tag{21}$$

Recalling the definition of the *generalized logarithmic means* of Stolarsky ([19], [20], [8]) $\mathcal{S}_p = \left[(x^p - y^p)/(p(x-y))\right]^{1/(p-1)}$, $(p \neq 0, 1)$, $\mathcal{S}_1 = \lim_{p\to 1}\mathcal{S}_p = e^{-1}\left(x^x y^{-y}\right)^{1/(x-y)} = \mathcal{I}(x, y)$ (the *identric mean*), $\mathcal{S}_0 = \lim_{p\to 0}\mathcal{S}_p = (x-y)/(\ln x - \ln y)$ (the *logarithmic mean*), the inequalities (21) can be written as

$$\mathcal{S}_1 \geq \mathcal{S}_0 \geq \mathcal{S}_{-1} \geq \cdots \geq \mathcal{S}_{1-n} \geq \cdots . \tag{22}$$

The second inequality in this series; namely $\mathcal{L}(x, y) \geq \mathcal{G}(x, y)$ (as usual, $\mathcal{G}(x, y) = \mathcal{S}_{-1}(x, y) = \sqrt{xy}$ denotes the *geometric mean*) is due to Carlson ([9]). A particular case of the monotonicity in p of $\mathcal{S}_p$ is expressed by inequalities (22) if globally considered (see [19], [20] and pg. 347 of [8]).

The title of the paper originates in the following result.

Theorem 8. *Let $\alpha : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing function which make convergent the integral $\int_0^{+\infty} e^{-xt}d\alpha(t)$ for every $x > 0$. If the function $f(x) = \int_0^{+\infty} e^{-xt}d\alpha(t)$, $x > 0$, does not reduces to a constant, then, the inequalities* (14) *hold for f.*

Proof. It is a simple consequence of Theorems 7 and 5. □

From we said in the introduction we realize that a result like Theorem 7 could be established for the sequence of means $\{\mu(x, y; f^{(n)}, \rho)\}$ where μ is defined by (10) and (11) and f is completely monotone in $(0, +\infty)$. In spite of the generality of such result, it would be inadequate to the purpose of comparing Lagrangian means $T^{(n)}(x, y; f)$ corresponding to different orders n: the measure $d\rho_n(x, y; \xi) = n\,(y-\xi)^{n-1}d\xi/(y-x)^n$ varies with n. However, our next result shows that the sequence $\{T^{(n)}(x, y; f)\}$ is decreasing for a completely monotone function f whenever $x \leq y$.

Theorem 9. *Let f be a completely monotone function in $(0,+\infty)$ which is not a constant. Then, the inequality*

$$T^{(n+1)}(x,y;f) \leq T^{(n)}(x,y;f), \quad y \geq x > 0, \tag{23}$$

holds for every $n \in \mathbb{N}$.

In other words, Theorem 9 says that the intermediate value appearing in the remainder of the Taylor's expansion up to order n of f decreases as n increases whenever f is completely monotone in $(0,+\infty)$.

Proof. The inequality

$$\begin{aligned} &\left(f^{(n+1)}\right)^{-1}\left[\int_x^y f^{(n+1)}(\xi)\frac{(n+1)(y-\xi)^n}{(y-x)^{n+1}}d\xi\right] \\ &\leq \left(f^{(n)}\right)^{-1}\left[\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi\right], \quad x,y>0, \end{aligned}$$

must be proved for every $n \in \mathbb{N}$. First assume that n is even. By reasoning like in the proof of Theorem 7, we see that the inequality (24) is equivalent to the following one:

$$\begin{aligned} &\int_x^y f^{(n+1)}(\xi)\frac{(n+1)(y-\xi)^n}{(y-x)^{n+1}}d\xi \\ &\leq \left(f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}\right)\left[\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi\right], \quad y \geq x > 0. \end{aligned} \tag{25}$$

Since $f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}$ is concave (see the proof of Theorem 7), from Jensen's inequality we obtain

$$\begin{aligned} &\left(f^{(n+1)}\circ\left(f^{(n)}\right)^{-1}\right)\left[\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi\right] \\ &\geq \int_x^y f^{(n+1)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi, \quad y \geq x > 0. \end{aligned} \tag{26}$$

The case $y = x$ being trivial, in order to establish (25) it will be sufficient to show that

$$\begin{aligned} &\int_x^y f^{(n+1)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi \\ \geq &\int_x^y f^{(n+1)}(\xi)\frac{(n+1)(y-\xi)^n}{(y-x)^{n+1}}d\xi, \quad y > x > 0. \end{aligned} \tag{27}$$

In fact, if $y > x$ we have

$$\begin{aligned} &\int_x^y f^{(n+1)}(\xi)\left[\frac{n(y-\xi)^{n-1}}{(y-x)^n} - \frac{(n+1)(y-\xi)^n}{(y-x)^{n+1}}\right]d\xi \\ &= \int_x^y f^{(n+1)}(\xi)d\left[\frac{(y-\xi)^{n+1}}{(y-x)^{n+1}} - \frac{(y-\xi)^n}{(y-x)^n}\right] \\ &= \int_x^y \left[\frac{(y-\xi)^{n+1}}{(y-x)^{n+1}} - \frac{(y-\xi)^n}{(y-x)^n}\right]df^{(n+1)}(\xi) \\ &= \int_x^y \left[\frac{(y-\xi)^n}{(y-x)^n} - \frac{(y-\xi)^{n+1}}{(y-x)^{n+1}}\right]f^{(n+2)}(\xi)d\xi, \end{aligned} \tag{28}$$

and for $x \leq \xi \leq y$,

$$\frac{(y-\xi)^n}{(y-x)^n} - \frac{(y-\xi)^{n+1}}{(y-x)^{n+1}} = \frac{(y-\xi)^n(\xi-x)}{(y-x)^{n+1}} \geq 0. \tag{29}$$

Since n has been assumed even, $f^{(n+2)} \geq 0$ and so, from (28) and (29) the inequality (27) is derived. The proof follows similar lines for the case in which n is odd. □

It should be observed that the inequality (27) reverses when $y < x$. Indeed, for $y < x$ we have

$$\int_x^y f^{(n+1)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi = \int_y^x f^{(n+1)}(\xi)\frac{n(\xi-y)^{n-1}}{(x-y)^n}d\xi,$$

and

$$\begin{aligned} &\int_y^x f^{(n+1)}(\xi)\left[\frac{n(\xi-y)^{n-1}}{(x-y)^n} - \frac{(n+1)(\xi-y)^n}{(x-y)^{n+1}}\right]d\xi \\ &= -\int_y^x \frac{(\xi-y)^n(x-\xi)}{(x-y)^{n+1}}f^{(n+2)}(\xi)d\xi \leq 0. \end{aligned}$$

Thus, the validity of an inequality like (23) for $x > y$ can not be generally expected. To exemplify Theorem 9 we choose the function $f(x) = e^{-x}$, $x > 0$. Then, we obtain

$$T^{(n)}(x, y; f) = -\ln\left[\int_x^y e^{-\xi} \frac{n(y-\xi)^{n-1}}{(y-x)^n} d\xi\right]. \tag{30}$$

By using the confluent hypergeometric function $M(a, b, z)$ (see pg. 505 of [1]), the equation (30) can be rewritten as follows

$$T^{(n)}(x, y; e^{-x}) = x - \ln M(1, n+1, x-y),$$

and therefore, the inequality (23) from Theorem 9 becomes

$$x - \ln M(1, n+2, x-y) \leq x - \ln M(1, n+1, x-y), \quad y \geq x > 0,$$

or, taking into account that $x \mapsto \ln x$ is increasing and defining $z = x - y \leq 0$,

$$M(1, n+2, z) \geq M(1, n+1, z), \quad z \leq 0.$$

Then, we see that the inequality (23) reduces to one involving the confluent hypergeometric function M when $f(x) \equiv e^{-x}$.

The last result of this section has to do with a generalization of the proof of Theorem 9 to means expressed by

$$\nu(x, y; f, \rho) = f^{-1}\left(\int_x^y f(\xi)\rho(x, y; \xi) d\xi\right), \quad x, y \in I, \tag{31}$$

where f is a continuous and strictly monotonic function and $\{\rho(x, y; \cdot) : x, y \in I\}$ is a family of measurable non-negative weights which depend continuously on $(x, y) \in I^2$ and verifying the following conditions:

GM1) $\int_x^y \rho(x, y; \xi)\, d\xi = 1$, $x, y \in I$, $x < y$.

GM2) $\rho(y, x; \cdot) = -\rho(x, y; \cdot)$, $x, y \in I$.

As it can be easily seen, $\nu(x, y; f, \rho)$ is a symmetric and continuous mean in I. Although closely related to Jensen's inequality and to inequalities proved many years ago by J. F. Steffensen (cf. pgs. 114–115 of [16]), the following result seems do not have been previously registered in the literature.

Theorem 10. *Let f and g be continuous and strictly monotonic functions in I, g increasing (decreasing) and consider two continuous families of weights, ρ and σ, satisfying the conditions* GM1) *and* GM2). *Then, the inequality*

$$\nu(x,y;f,\rho) \le \nu(x,y;g,\sigma), \quad x,y \in I, \tag{32}$$

holds provided that $g \circ f^{-1}$ is convex (concave) and the weights satisfy

$$\int_x^\eta \rho(x,y;\xi)\,d\xi \ge \int_x^\eta \sigma(x,y;\xi)\,d\xi, \quad x,y,\eta \in I, \quad x<\eta<y. \tag{33}$$

Proof. Assume g strictly increasing; then, it will be enough to prove the inequality

$$\begin{gathered}(g\circ f^{-1})\left(\int_x^y f(\xi)\rho(x,y;\xi)\,d\xi\right) \le \int_x^y g(\xi)\sigma(x,y;\xi)\,d\xi,\\ x,y\in I, \quad x<y.\end{gathered} \tag{34}$$

Since $g \circ f^{-1}$ is convex, from Jensen's inequality we obtain

$$\begin{gathered}(g\circ f^{-1})\left(\int_x^y f(\xi)\rho(x,y;\xi)\,d\xi\right) \le \int_x^y g(\xi)\rho(x,y;\xi)\,d\xi,\\ dx,y,z\in I, \quad x<y.\end{gathered} \tag{35}$$

Now we show that

$$\int_x^y g(\xi)\rho(x,y;\xi)\,d\xi \le \int_x^y g(\xi)\sigma(x,y;\xi)\,d\xi \tag{36}$$

holds for $x,y \in I$, $x<y$. In fact, we have

$$\begin{aligned}\int_x^y & g(\xi)\sigma(x,y;\xi)\,d\xi - \int_x^y g(\xi)\rho(x,y;\xi)\,d\xi\\ &= \int_x^y g(\eta)\,d\left[\int_x^\eta \sigma(x,y;\xi)\,d\xi - \int_x^\eta \rho(x,y;\xi)\,d\xi\right]\\ &= \int_x^y \left[\int_x^\eta \rho(x,y;\xi)\,d\xi - \int_x^\eta \sigma(x,y;\xi)\,d\xi\right]\,dg(\eta).\end{aligned}$$

The external integral of the last member of these equalities is understood to be a Riemann–Stieltjes integral. In view of condition (33) we realize that this integral is non-negative, so proving (36). The case in which g is decreasing can be analogously treated. $\square$

Extensions of Theorem 10 to general means like (10) are not a difficult matter. Such extensions could include means arising from measures supported on a finite set of point, so embracing existing results of comparison for quasilinear means with different weights (see pg. 265 of [8] and the references given there).

We conclude this section by specializing the condition (33) from Theorem 10 to the case of Cauchy means (3). To this end, we consider the Cauchy means $C(x,y;f,\sigma_1)$ and $C(x,y;g,\sigma_2)$. Indeed, from (31) and (3) we see that

$$\rho(x,y;\xi)\,d\xi = \frac{1}{\sigma(y)-\sigma(x)}\,d\sigma(\xi),$$

whence condition (33) becomes

$$\frac{1}{\sigma_1(y)-\sigma_1(x)}\int_x^\eta d\sigma_1(\xi) \geq \frac{1}{\sigma_2(y)-\sigma_2(x)}\int_x^\eta d\sigma_2(\xi),$$
$$x,y,\eta\in I,\quad x<\eta<y,$$

or

$$\frac{\sigma_1(\eta)-\sigma_1(x)}{\sigma_1(y)-\sigma_1(x)} \geq \frac{\sigma_2(\eta)-\sigma_2(x)}{\sigma_2(y)-\sigma_2(x)},\qquad x,y,\eta\in I,\ x<\eta<y.$$

3. Other results

In this final section we establish some facts related to the comparison between the different classes of means $M^{(n)}$, $L^{(n)}$ and $T^{(n)}$ introduced in Section 2. Properties of convergence of the sequences $\{M^{(n)}\}$, $\{L^{(n)}\}$ and $\{T^{(n)}\}$ are also discussed. To begin with, we prove the following theorem.

Theorem 11. *Let f be a completely monotone function in $(0,+\infty)$ which is not a constant; then, the inequalities*

$$M^{(n)}(x,y;f) \leq L^{(n)}(x,y;f) \leq \frac{x+y}{2},\quad x,y>0, \tag{37}$$

holds for every $n \in \mathbb{N}_0$. *Furthermore, the inequality*

$$T^{(n)}(x,y;f) \le L^{(n)}(x,y;f), \quad y > x > 0, \tag{38}$$

holds for every $n \in \mathbb{N}$.

PROOF. The Jensen–Hadamard inequalities

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(\xi)\,d\xi \le \frac{f(x)+f(y)}{2}, \tag{39}$$

are true for $x, y \in I$ provided that f is convex in I ([10], [8]) and obviously they are reversed when f is concave. Now, let f be completely monotone in $(0,+\infty)$. If n is even then $f^{(n)}$ is decreasing and convex and so, from (39) we obtain, for $x, y > 0$,

$$\begin{aligned}\frac{x+y}{2} &\ge \left(f^{(n)}\right)^{-1}\left(\frac{1}{y-x}\int_x^y f^{(n)}(\xi)\,d\xi\right)\\ &\ge \left(f^{(n)}\right)^{-1}\left(\frac{f^{(n)}(x)+f^{(n)}(y)}{2}\right).\end{aligned} \tag{40}$$

Moreover, $f^{(n)}$ is increasing and concave when n is odd; therefore the inequalities (40) also hold in this case, then proving (37).

In order to prove (38), let us assume $n \in \mathbb{N}$. We observe that, for $0 < x < y$,

$$\begin{aligned}\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi - \frac{1}{y-x}\int_x^y f^{(n)}(\xi)d\xi\\ = \int_x^y f^{(n)}(\xi)d\left[\frac{y-\xi}{y-x} - \frac{(y-\xi)^n}{(y-x)^n}\right]\\ = \int_x^y \left[\frac{(y-\xi)^n}{(y-x)^n} - \frac{y-\xi}{y-x}\right] f^{(n+1)}(\xi)d\xi,\end{aligned}$$

hence, the inequality

$$\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi \ge \frac{1}{y-x}\int_x^y f^{(n)}(\xi)\,d\xi, \quad y > x > 0,$$

is true for n even while the opposite inequality holds when n is odd. In the first case, $f^{(n)}$ is decreasing and so

$$\begin{aligned}&\left(f^{(n)}\right)^{-1}\left(\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi\right)\\ &\leq \left(f^{(n)}\right)^{-1}\left(\frac{1}{y-x}\int_x^y f^{(n)}(\xi)d\xi\right),\quad y>x>0.\end{aligned} \tag{41}$$

Since $f^{(n)}$ is increasing when n is odd, the inequality (41) is also valid in this case. This concludes the proof. $\square$

Note that inequality (38) expresses that, for a completely monotone function f, the Lagrangian intermediate value of $f^{(n)}$ keeps on the right of the intermediate value appearing in the Taylor expansion of f up to order n. Furthermore, inequalities (37) and (38) imply that the intermediate values in the interval $[x,y]$ corresponding to a completely monotone function in $(0,+\infty)$ always belong to the left half of $[x,y]$.

As we seen in the previous section, the sequences $\{M^{(n)}(x,y;f)\}$, $\{L^{(n)}(x,y;f)\}$ and $\{T^{(n)}(x,y;f)\}$ (for $x<y$) are non-increasing in n provided that f is completely monotone in $(0,+\infty)$. Therefore, there exist the limits of these sequences when $n\uparrow+\infty$ and they are means when considered as functions of (x,y). Particularly, our next result shows that $\lim_{n\uparrow+\infty}T^{(n)}(x,y;f)=\min\{x,y\}$.

Theorem 12. *Let f be a completely monotone function in $(0,+\infty)$ which is not a constant. Then,*

$$\lim_{n\uparrow+\infty} T^{(n)}(x,y;f)=x, \tag{42}$$

for $0<x\leq y$.

Proof. Since

$$\left(\left(f^{(n)}\right)^{-1}\right)''\left(f^{(n)}(x)\right)=-\frac{f^{(n+2)}(x)}{\left[f^{(n+1)}(x)\right]^3},\quad x>0,$$

we see that for a non-constant completely monotone function f is $\left(\left(f^{(n)}\right)^{-1}\right)''\geq 0$, $n\in\mathbb{N}_0$; i.e., $\left(f^{(n)}\right)^{-1}$ is convex for every n. Thus we can apply the Jensen's inequality to deduce, for $x < y$,

$$T^{(n)}(x,y;f) = \left(f^{(n)}\right)^{-1}\left(\int_x^y f^{(n)}(\xi)\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi\right)$$
$$\leq \int_x^y \xi\frac{n(y-\xi)^{n-1}}{(y-x)^n}d\xi = x + \frac{y-x}{n+1}.$$

But this inequality becomes an equality when $x = y$ so that, for $0 < x \leq y$ we can write

$$x \leq T^{(n)}(x,y;f) \leq x + \frac{y-x}{n+1},$$

and hence (42) follows after taking limits when $n \uparrow +\infty$. □

The convergence of $T^{(n)}(x,y;f)$ to $\min\{x,y\}$ when $n \uparrow +\infty$ as stated in Theorem 12 is no longer valid for other sequences of means such as $M^{(n)}$ or $L^{(n)}$. In fact, if f is of exponential type, say $f(x) = e^{-\alpha x}$, $x > 0$, $(\alpha > 0)$, then $M^{(n)}(x,y;f) = -\frac{1}{\alpha}\ln\left((e^{-\alpha x} + e^{-\alpha y})/2\right)$, $x, y > 0$, for every $n \in \mathbb{N}_0$. Note that $\lim_{a\uparrow+\infty}\left[-\frac{1}{\alpha}\ln\left((e^{-\alpha x} + e^{-\alpha y})/2\right)\right] = \min\{x,y\}$ and $\lim_{a\downarrow 0}\left[-\frac{1}{\alpha}\ln\left((e^{-\alpha x} + e^{-\alpha y})/2\right)\right] = (x+y)/2$ and then, by choosing an appropriate α, every point in the interval $(\min\{x,y\}, (x+y)/2)$ may be reached (in a trivial way) by the limit $\lim_{n\uparrow+\infty} M^{(n)}(x,y;f)$. On the other hand, there exist situations in which $\lim_{n\uparrow+\infty} L^{(n)}(x,y;f) = \min\{x,y\}$ holds. Consider, for instance, $f(x) = x^{-1}$, $x > 0$. In this case we have

$$L^{(n)}(x,y;f) = \mathcal{S}_{-n}(x,y) = \left[\frac{x^n y^n (y-x)}{n(y^n - x^n)}\right]^{1/(n+1)} \to \min\{x,y\},\ n \uparrow +\infty.$$

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LUCIO R. BERRONE
INSTITUTO DE MATEMÁTICA "BEPPO LEVI"
AV. PELLEGRINI 250
2000 – ROSARIO
ARGENTINA

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