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A Class of Diophantine Equations

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Abstract. A method is given to solve all equations of the type $F(2^a 3^b) = \pm 2^c 3^d$, where F is a polynomial with integral coefficients and at least two distinct roots.

A method is developed to solve the Diophantine equation

$$F(2^a 3^b) = \pm 2^c 3^d \tag{1}$$

where F is a polynomial with at least two distinct roots. Let S be a set of integers divisible only by primes from a given, finite set of prime numbers. Let $F \in \mathbf{Z}[X]$ have at least two distinct roots. In [4] [chapter 10] it is proved that if $n \in \mathbf{Z}$ and $F(n) \in S$ then $|n| \leq C(S, F)$ where $C(S, F)$ is an effectively computable constant. This result is proved by Baker's method and its p–adic analogue.

Equation (1) arises when one wishes to calculate reducible polynomials of discriminant only divisible by two and there. Effective bounds for algebraic integers with discriminants with given prime divisors was first given in [1]. In a later paper we shall consider over 1000 equations of the above type in one step so our method must not depend on the factorization of F in some number field but only on the sizes of the coefficients of F. The equations (1) arise when one wishes to consider the discriminant of a polynomial some of whose roots are rational. One decomposes the polynomial into irreducible factors. One can then factor the discriminant of the polynomial. One of the factors will then give an equation of the type considered here.

The method used in this paper relies on the application of Bakers method. This provides quite large bounds which are then reduced with

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the aid of the reduction techniques of De Weger, [6]. I must thank B.M.M. De Weger for initially giving me the idea for this method. We first bound all the variables in the equation. In what follows $c_1, c_2, \ldots$ and $K_1, \ldots$ will be explicitly given constants depending on the degree and coefficients of F.

Theorem 1. *Let $S = \{2^a 3^b : a, b \in \mathbf{N}\}$ and let*

$$F = f_n X^n + f_{n-1} X^{n-1} + \cdots + f_1 X + f_0 \tag{2}$$

be a polynomial with integer coefficients, of degree n and having at least two distinct roots. Assume (for simplicity) that $f_n \in S$, (for other f_n a small change in the following will suffice). Set $\delta = (n-1)/n$. Let $X = 2^N 3^M \in S$ be such that $\pm F(x) \in S$. Then either $x \leq c_9$ or x is determined by the solution of a system of the following form:

$$\Lambda = A \ln 2 + B \ln 3 \tag{3}$$

where A and B are integer variables dependent on N and M respectively and if $H = \max(|A|, |B|)$ then

$$|\Lambda| < c_6 2^{-(1-\delta)H}, \quad H < K_1 \tag{4}$$

Proof. Denote $e = \mathrm{ord}_2(f_0)$, $f = \mathrm{ord}_3(f_0)$. We need to introduce the following constants:

$$\begin{aligned}
c_1 &= (|f_{n-1}| + n - 1), & c_2 &\left(1 + \textstyle\sum_{i=0}^{n-1} |f_i|\right) / f_n, \\
c_3 &= \left(\max_{i=0,\ldots,n-2} (|f_i|^{1/(n-1-i)}\right), & c_4 &= 2^e 3^f, \\
c_5 &= c_2 + |f_0|/f_n, & c_6 &= c_1 (c_1 + 1)^{1-\delta}, \\
c_7 &= 1.237 \cdot 10^{15}, & c_8 &= 1.095, \\
c_9 &= \max(c_2, c_3, c_4, c_5).
\end{aligned}$$

The constants c_7 and c_8 arise from the application of Waldschmidt's theorem. In the case $F(x) = -y$ we have that x is less than the maximum root of $F = 0$. (As $f_n > 0$ and $x > 0$.) Therefore by elementary estimates we have $x \leq c_2$.

In the case $F(x) = y$ we let $x = 2^N 3^M$, $y = 2^p 3^q$, $f = 2^r 3^t$. We obviously have

$$\min(N, p) \leq \mathrm{ord}_2(f_0), \quad \min(M, q) \leq \mathrm{ord}_3(f_0).$$

Assume $x > c_3$ then $|f_i|x^i < x^{n-1}$ for $i \le n-2$. We now obtain,

$$|f_n x^n - y| = |f_{n-1}x^{n-1} + \cdots + f_0| \le |f_{n-1}|x^{n-1} + \cdots + |f_0| \le c_1 x^{n-1}$$

Let $X = x^n$ and $\chi = f_n X = 2^{nN+r}3^{nM+t} = 2^\sigma 3^\tau$. We then get

$$|\chi - y| \le c_1 \chi^\delta$$

We now have four cases to consider.

Case 1: $\underline{\min(N,p) = N,\ \min(M,q) = M.}$

We then obtain $x \le c_4$.

Case 2: $\underline{\min(N,p) = p,\ \min(M,q) = q.}$

We have $y \le |f_0|$ so $x \le c_5$.

Case 3: $\underline{\min(N,p) = N,\ \min(M,q) = q.}$

We have $0 \le |3^{\tau-q} - 2^{p-\sigma}| \le c_1 3^{(\tau-q)\delta}$.

Case 4: $\underline{\min(N,p) = p,\ \min(M,q) = M.}$

We have $0 \le |2^{\sigma-p} - 2^{q-\tau}| \le c_1 2^{(\sigma-p)\delta}$.

In our two remaining cases, 3 and 4, we have a situation given by the following in which a and b are variables in $\mathbf{N}$.

$$0 < |p_1^a - p_2^b| < c_1 p_1^{\delta a}\ , \quad a, b \in \mathbf{N},\ \{p_1, p_2\} = \{2, 3\}$$

Let $H = \max(a, b)$ and note that $\max(p_1^a, p_2^b) \ge 2^H$. We now have two cases to consider.

i) $\underline{p_1^a > p_2^b}$.

Set $\Lambda_1 = \ln(p_1^a/p_2^b) > 0$. We obtain

$$0 \le \Lambda_1 < (p_1^a/p_2^b) - 1 < c_1 p_1^{a\delta}/p_2^b < c_1 2^{-(1-\delta)H}.$$

ii) $\underline{p_2^b > p_1^a}$.

Set $\Lambda_2 = \ln(p_2^b/p_1^a) > 0$. Now if $p_1^a < p_2^b/(c_1+1)$ then $p_1^a < p_1^{\delta a}$ which is impossible, since $0 < \delta < 1$. Therefore we obtain

$$0 \le \Lambda_2 < c_1 p_1^{a(\delta-1)} \le c_1 p_2^{-b(1-\delta)}(c_1+1)^{1-\delta} < c_6 2^{-(1-\delta)H}.$$

Hence in both of these cases we obtain for

$$\Lambda = \pm\Lambda_1 = \mp\Lambda_2 = A\ln 2 + B\ln 3,$$

where $A = \pm a$ and $B = \mp b$, the inequality;

$$0 \le |\Lambda| < c_6 2^{-(1-\delta)H}.$$

In Case 3 we have $N \le e$ and $nM + t \le |B| \le H$, whereas in Case 4 we have $M \le f$ and $nN + r \le |A| \le H$. Thus N and M are bounded by H.

We now apply Waldschmidt's Theorem to the linear form Λ, see [5] or [6] [Lemma 2.1]. We obtain $|\Lambda| > \exp(-c_7(\ln H + c_8))$. We note in passing that one could use better estimates as we have only two logarithms, see [3], but the method does not depend critically on these constants. In fact, the constant c_6 is the one which it is most important to keep small.

This leads us to deduce that $H < K_1$, (see [6] [Lemma 2.3]) where K_1 is given by:

$$K_1 = 2\frac{(\ln c_6 + c_7 c_8 + c_7 \ln(c_7/((1-\delta)\ln 2)))}{(1-\delta)\ln 2} \tag{5}$$

Q.E.D.

In practice, the number K_1 will be too large to allow enumeration of the solutions so we will need a method to reduce the bound. As Λ is a linear form in two variables the continued fraction algorithm could be used to solve this problem in diophantine approximation. However, we shall use the L^3 algorithm, introduced in [2].

For any $C > 0$ let Γ denote the lattice in $\mathbf{Z}^2$ generated by the columns of the matrix:

$$\Omega = \begin{pmatrix} 1 & 0 \\ [C\ln 2] & [C\ln 3] \end{pmatrix},$$

where $[x]$ denotes the nearest integer to x. We can apply the L^3 algorithm and obtain a reduced basis $\{\underline{b}_1, \underline{b}_2\}$ in the sence of [2].

Theorem 2. *Let A and B be integers such that if $\Lambda = A\ln 2 + B\ln 3$ and $H = \max(|A|,|B|)$ then*

$$|\Lambda| < K_3 2^{-K_2 H}, \quad H < K_1$$

Choose $C > K_1^2$ and set

$$K_4 = \max\left(\frac{\|\underline{b}_1\|^2}{\|\underline{b}_1^*\|^2}, \frac{\|\underline{b}_1\|^2}{\|\underline{b}_2^*\|^2}\right)$$

If $\|\underline{b}_1\|^2 > 5K_4K_1^2$ then

$$H < \frac{1}{K_2\ln 2}\left(\ln CK_3 - \ln\left(\sqrt{\|\underline{b}_1\|^2/K_4 - K_1^2} - 2K_1\right)\right)$$

Proof. Set $\underline{x} = \Omega\binom{A}{B} = \binom{A}{\lambda} \in \Gamma$ where $\lambda = A[C\ln 2] + B[C\ln 3]$. We then obtain

$$|\lambda - C\Lambda| \leq 2K_1$$
$$|\lambda| < CK_3 2^{-K_2 H} + 2K_1.$$

Note also $A^2 + \lambda^2 \geq \|\underline{b_1}\|^2/K_4$ as $\{\underline{b_1}, \underline{b_2}\}$ is an L^3 basis. Hence $|\lambda| > \sqrt{\|\underline{b_1}\|^2/K_4 - K_1^2}$. Note the root is real by assumption. Combining the two inequalities for $|\lambda|$ gives the bound for H which is real by our assumption.

Q.E.D.

Example. Let $F(x) = 4x^4 - 3x^3 + 7x^2 - 3x + 8$. We apply Theorem 1 to obtain

$$\delta = 0.75,\ c_6 = 9.76,\ c_9 = 8,\ K_1 = 5.37 \cdot 10^{17}.$$

We reduce this bound for H by repeated application of Theorem 2.
i) With $C = 10^{40}$ we get $\|\underline{b}_1\| = 0.11 \cdot 10^{21}$ and hence we find $H < 279$.
ii) With $C = 10^6$ we get $\|\underline{b}_1\| = 1132$ and hence we find $H < 58$.
iii) With $C = 10^5$ we get $\|\underline{b}_1\| = 192$ and hence we find $H < 55$.
iv) Finally again with $C = 10^5$ we find $H < 54$.

With this reduced bound we have that either $x \le c_9 = 8$ or we are in Cases 3 and 4 of Theorem 1. In the latter cases a study of the proof reveals that; Either $N \le 3$ and $4M + 2 \le 54$ i.e. $M \le 13$, or $M = 0$ and $4N + 0 \le 54$ i.e. $N \le 13$ where $x = 2^N 3^M$. Now with these bounds it is an easy matter to show that there are no solutions to the equation $F(x) = \pm y$ with $x, y \in S$.

As can be seen, the computation for reducing the bounds is quite straightforward, requiring about 30 seconds CPU time for bounds on H of the size 10^{40}. However, the most computer time is spent searching for a solution below the final bound. As was remarked earlier it is the constant c_6 which is the most critical: the smaller that this is then the smaller will be the bounds and the easier the equation will be to solve.

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