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On Poincaré stability

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Abstract. In this paper, the author presents a mathematical theory for Poincaré stability, and establishes some criteria for this kind of stability.

1. Introduction

Among many stabilities in dynamical systems the best-known and most investigated is undoubtedly the Lyapunov stability. But, as pointed out by E. A. JACKSON [1], Lyapunov stability is rather restrictive, for it implies that two trajectories close to each other must be for the same value of time. This kind of stability is sometimes referred to as an ischronous correspondence of the two solutions. In some cases even an anharmonic oscillator is unstable in this sense.

To relax this restriction and thereby include an important type of stability which would be physically more practical in may cases, Poincaré introduced the concept of Poincaré stability or orbital stability as follows [1], [2], [4]:

Poincaré stability. (a) Let Γ be the orbit defined by $u(t)$ for all t, and Γ' be any orbit defined by $v(t)$ for all t, Γ is said to be Poincaré stable or orbitally stable if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, if $|u(0) - v(\tau)| < \delta(\varepsilon)$ for some τ, then there exists a continuous monotonically increasing function $t' = t'(t)$ such that $|u(t) - v(t')| < \varepsilon$ for all $t > 0$.

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(b) The orbit Γ is said to be asymptotically Poincaré stable, if Γ is Poincaré stable and Γ' tends toward Γ as $t \to \infty$, or more precisely, there exists a strictly increasing function $t'(t)$, such that $|u(t) - v(t'(t))| \to 0$, as $t \to \infty$.

Examples. Consider the following system in polar-coordinates:

$$\dot{r} = 0, \qquad \dot{\theta} = r\theta \tag{1.1}$$

Take a solution $\phi(t)$ to (1.1), which is of the form

$$r(t) = r_0, \qquad \theta(t) = \theta_0 \exp(r_0 t)$$

with the initial condition (r_0, θ_0) satisfying $r_0 > 0$.

To see its stability, consider another solution $\bar{\phi} : \bar{r} = \bar{r}_0$, $\bar{\theta} = \bar{\theta}_0 \exp(\bar{r}_0 t)$ with initial condition $(\bar{r}_0, \bar{\theta}_0)$ satisfying $\bar{r}_0 < r_0$. It is easy to see from the expression

$$\begin{aligned} \theta(t) - \bar{\theta}(t) &= \theta_0 \exp(r_0 t) - \bar{\theta}_0 \exp(\bar{r}_0 t) \\ &= \exp(r_0 t)(\theta_0 - \bar{\theta}_0 \exp[(\bar{r}_0 - r_0)t]) \end{aligned}$$

that there exists a sequence $t_k \to \infty$ such that $\theta(t_k) - \bar{\theta}(t_k) = (2k+1)\pi$. Therefore,

$$\begin{aligned} \|\phi(t_k) - \bar{\phi}(t_k)\| &= \Big[(r(t_k)\cos\theta(t_k) - \bar{r}(t_k)\cos\bar{\theta}(t_k))^2 \\ &\qquad + (r(t_k)\sin\theta(t_k) - \bar{r}(t_k)\sin\bar{\theta}(t_k))^2\Big]^{1/2} \\ &= r(t_k) + \bar{r}(t_k) \\ &= r_0 + \bar{r}_0. \end{aligned}$$

If follows that $\phi(t)$ is not Lyapunov stable.

However, it is easy to see that $\phi(t)$ is Poincaré stable. In fact, for every $\varepsilon > 0$, in case of $|r_0 - \bar{r}_0| < \varepsilon$, $|\theta_0 - \bar{\theta}_0| < \varepsilon$, we can take the function $t' = r_0 t / \bar{r}_0 + (\ln\theta_0 - \ln\bar{\theta}_0)/\bar{r}_0$, such that

$$|r(t) - \bar{r}(t)| = |r_0 - \bar{r}_0| < \varepsilon, \qquad |\theta(t) - \bar{\theta}(t')| = 0.$$

Therefore $\|\phi(t_k) - \bar{\phi}(t_k)\| < \varepsilon$.

Similarly, it is easy to prove that in the following system

$$\dot{r} = r(1-r), \qquad \dot{\theta} = r\theta, \tag{1.2}$$

the orbit $r = 1$ is asymptotically Poincaré stable but not asymptotically Lyapunov stable.

Although Poincaré stability has a long history, it has been rarely investigated mathematically "because it can be very difficult to establish" [1].

In this paper, the author investigates Poincaré stability mathematically and obtains some meaningful results.

In Section 2, the required preliminaries are given and in Section 3, several criteria on Poincaré stability are established.

2. Preliminaries

Firstly, let M [3] be a Riemannian manifold with a C^∞ Riemannian metric $\langle\cdot,\cdot\rangle$ for its tangent bundle TM, which gives positive definite inner product $\langle\cdot,\cdot\rangle_x$ on each tangent vector space TM_x, $x \in M$ and $\rho : M \times M \to \mathbb{R}$ is the distance function on M induced by $\langle\cdot,\cdot\rangle$. Secondly, let $V(x)$ be a vector field on M, $\phi(t,p) : \mathbb{R} \times M \to M$ a flow of $V(x)$ with initial point $p \in M$. For $\phi(t,p)$, $t \geq 0$, we define its neighborhood as follows:

Let $f : \mathbb{R}^+ \times D^{n-1} \to M$ ($\mathbb{R}^+ = \{t \mid t \geq 0\}$, D^{n-1} an $n-1$ dimensional unit disc) be an immersion such that $f(t, D^{n-1})$ is locally and $n-1$ dimensional geodesic disc of radius r perpendicular to $V(\phi(t,p))$ at $\phi(t,p)$, with r small enough. Then we define the image $f(\mathbb{R}^+, D^{n-1})$ as the neighborhood of $\phi(t,p)$ ($t \geq 0$). By the tubular manifold theorem, such a neighborhood does exist.

Remarks. Generally speaking f is not an embedding, but this is no disadvantage to the following discussions.

In the case that $M = \mathbb{R}^n$, the neighborhood of $\phi(t,p)$ is just written as $\phi(t,p) \times D_r^{n-1}$, where $t \geq 0$, D_r^{n-1} is an $n-1$ dimensional disc with radius r such that at $\phi(t,p)$, $\phi(t,p) \times D_r^{n-1}$ is the $n-1$ dimensional disc perpendicular to $V(\phi(t,p))$ with center $\phi(t,p)$ and radius r.

Finally, let N be the orthogonal complement bundle of the tangent bundle TM:

$$N = \bigcup_{x \in M} N_x,$$

where

$$N_x = \{(x, v) \in TM_x \mid \langle v, V(x) \rangle = 0\},$$

and θ be the projection of TM onto N, i.e.:

$$\theta(x, v) = (x, \pi_x(v)),$$

where

$$\pi_x(v) = \begin{cases} 0 & V(x) = 0, \\ v - \dfrac{\langle v, V(x) \rangle V(x)}{\langle V(x), V(x) \rangle} & \text{otherwise.} \end{cases}$$

For the flow $\phi(t, p)$ generated by $V(x)$, the Jacobi matrix of the one parameter diffeomorphism $\phi(t, p)$ at $p \in M$ is denoted by $D\phi(t, p)$, clearly $D\phi(t, p)$ is a one-parameter linear map from TM to TM.

3. Main results

In what follows we assume that the trajectory considered does not tend to a rest point, i.e., its ω-limit set contains no rest point. For convenience, we denote the neighborhood of $\phi(t, p)$ by $\phi(t, p) \times D_r^{n-1}$ instead of $f(\phi(t, p) \times D_r^{n-1})$.

Theorem 3.1. *Let $\phi(t, p)$ be a flow contained in a compact set of M, and suppose that there exists a $T > 0$ such that*

$$\|\pi(D\phi(T + t, p), w)\| < 1 \quad \forall t \geq 0,\ w \in N_{\phi(t,p)} \|w\| = 1 \tag{3.1}$$

then $\phi(t, p)$ is asymptotically Poincaré stable.

Proof. For the neighborhood $\phi(t, p) \times D_r^{n-1}$ of $\phi(t, p)$, let us first consider the following maps:

$$T_k : \phi(KT, p) \times D_r^{n-1} \to \phi((K + 1)T, p) \times D_r^{n-1} \tag{3.2}$$

which is induced by the one-parameter diffeomorphism $\phi(t, p) : \mathbb{R}^+ \times M \to M$, in the same fashion as a Poincaré map, and $T_{k+1} = T_1 \circ T_k$ clearly holds.

Clearly,

$$DT_k = \pi(D\phi(T, \phi(KT, p))) \text{ at } \phi(KT, p).$$

Because $\phi(t, p)$ is contained in a compact set, we have two statements: on the one hand, there exists $0 \le c < 1$ such that (see (3.1)):

$$\|DT_k\| = \|\pi(D\phi(T, \phi(KT, p)))\| \le c < 1, \qquad K = 1, 2, \dots$$

and on the other hand, there exists a sufficiently small $\eta > 0$ and a constant $\bar{c}$ with $c \le \bar{c} < 1$, such that on the neighborhood

$$\phi(t, p) \times D_\eta^{n-1} \subset \phi(t, p) \times D_r^{n-1}$$
$$t \ge 0,\ D_\eta^{n-1} = \{x \mid x \in D_r^{n-1},\ \rho(x, 0) \le \eta\}$$

of $\phi(t, p)$, the map (3.2) satisfies

$$\rho(T_k(\phi(KT, p), b), \phi((K+1)T, p)) < \bar{c}\rho(b, 0), \qquad b \in D_\eta^{n-1} \tag{3.3}$$

uniformly for $K \in \mathbb{Z}^+$, due to the fact that $\phi(t, p)$ is contained in a compact set. It follows that any trajectory $\psi(t)$ with initial point b in $\phi(0, p) \times D_\eta^{n-1}$ satisfies (t_K defined as below)

$$\rho(\psi(t_{K+1}), b), \phi((K+1)T, p) < \bar{c}\rho(\psi(t_K), \phi(KT, p))$$

at the points where $\psi(t)$ passes through the sections

$$\phi(KT, p) \times D_\eta^{n-1}, \qquad K = 0, 1, 2, \dots$$

Denote by t_K a value such that $\psi(t_K)$ is in the section $\phi(KT, p) \times D_\eta^{n-1}$, then it follows from the above argument that

$$\rho(\psi(t_K), b), \phi(KT, p) < \bar{c}^K \rho(\psi(t_0), \phi(0, p)),$$

therefore

$$\lim_{K \to \infty} \rho(\psi(t_K), b), \phi(KT, p) = 0. \tag{3.4}$$

Secondly let us consider the map

$$T_K^x : \phi(KT, x) \times D_\eta^{n-1} \to \phi((K+1)T, x) \times D_\eta^{n-1}$$

where

$$x \in \sigma = \{\phi(t,p) \mid 0 \leq t \leq T\}.$$

In the same manner, it is not difficult to see that T_K^x satisfies the inequality (3.3). Since $[0,T]$ is compact, σ is also compact, and therefore T_K^x satisfies (3.3) uniformly.

Now for any $t > 0$, there exist $x \in \sigma$ and $K \geq 0$, such that

$$\phi(t,p) = \phi(KT, x).$$

If follows from (3.3) that in the section $\phi(KT,x) \times D_\eta^{n-1}$, we have

$$\rho\big(\psi(t(t)), \phi(KT,x)\big) < \bar{c}^K M$$

where

$$M = \max_{t\in[0,T]} \|\psi(t,b) - \phi(t,p)\|, \qquad b \in \phi(0,p) \times D_\eta^{n-1}.$$

It can be seen from the fact that $K \to \infty$ as $t \to \infty$ that $\lim_{t\to\infty} \rho(\psi(\bar{t}(t)), \phi(t,p)) = 0$, and this completes the proof. □

Now we mainly discuss dynamical systems in Euclidean Space.

Theorem 3.2. *Let $\phi(t,p)$ be a solution of the system*

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n,\ F \in C^h(\mathbb{R}^n \cdot \mathbb{R}^n),\ h \geq 1 \tag{3.5}$$

and let $\phi(KT,p) \times D_r^{n-1}$, defined as in Section 2, be a neighborhood of $\phi(t,p)$. Suppose that

$$(y - \phi(t,p)) \cdot F(y) \leq 0 \tag{3.6}$$

where the dot denotes the inner product and

$$y \in \phi(t,p) \times D_r^{n-1}, \qquad t \geq 0,$$

then $\phi(t,p)$ is stable in Poincaré sense.

Proof. Contradiction argument. For any $\varepsilon > 0$, let us take a neighborhood $\phi(t,p) \times D_\eta^{n-1}$ of $\phi(t,p)$ satisfying

(1) $\phi(t,p) \times D_\eta^{n-1} \subset \phi(t,p) \times D_r^{n-1}, \qquad t \geq 0,$
(2) $\eta < \varepsilon,$

where

$$D_\eta^{n-1} = \{x \mid x \in D_r^{n-1}, |x| \le \eta < \min(r, \varepsilon)\}.$$

Consider any trajectory $\psi(t, q)$ with initial value in $\phi(0, p) \times D_\eta^{n-1}$. If there exists a $t'(t) > 0$, such that

$$\psi(t', q) \in \phi(t, p) \times D_\eta^{n-1},$$

and

$$\|\psi(t', q) - \phi(t, p)\| > \eta,$$

then it follows from the fact that $\|\psi(0, q) - \phi(0, p)\| \le \eta$ that there must exist $0 \le \bar{t} < t$ such that

$$\frac{d}{ds} \|\psi(s, \psi(t'(\bar{t}), q)) - \phi(s, \phi(\bar{t}, p))\| \Big|_{s=0} > 0,$$

which is equivalent to

$$\frac{d}{ds} \left[\psi(s, \psi(t'(\bar{t}), q)) - \phi(s, \phi(\bar{t}, p))\right]^2 \Big|_{s=0} > 0,$$

but

$$\begin{aligned} &\frac{d}{ds} \left[\psi(s, \psi(t'(\bar{t}), q)) - \phi(s, \phi(\bar{t}, p))\right]^2 \Big|_{s=0} \\ &\quad = 2\left[\psi(t'(\bar{t}), q) - \phi(\bar{t}, p)\right] \cdot \left[F(\psi(t'(\bar{t}), q)) - F(\phi(\bar{t}, p))\right] \\ &\quad = 2\left[\psi(t'(\bar{t}), q) - \phi(\bar{t}, p)\right] \cdot F(\psi(t'(\bar{t}), q)) \le 0, \end{aligned}$$

(note that $\psi(t'(\bar{t}), q) - \phi(\bar{t}, p)$ is perpendicular to $F(\psi(t'(t), q))$ leading to a contradiction).

It is easy to obtain the following results:

Corollary 3.3. *Let $\phi(t, p)$ be a trajectory as in Theorem 3.2, and N be a neighborhood of $\phi(t, p)$. If for every point $x \in N$, the following hold*

$$(x - \phi(t, p)) \cdot F(\phi(t, p)) = 0 \tag{3.7}$$

$$(x - \phi(t, p)) \cdot F(x) \le 0 \tag{3.8}$$

then $\phi(t, p)$ is stable in Poincaré sense.

Corollary 3.4. *Denote by $DF(x)$ the Jacobi matrix of $F(x)$, if*

$$(V - \phi(t,p)) \cdot DF(\phi(t,p))V < 0, \qquad V \in \mathbb{R}^n$$

where V satisfies

$$(V - \phi(t,p)) \cdot F(\phi(t,p)) = 0,$$

then $\phi(t,p)$ is stable in Poincaré sense.

For the asymptotic Poincaré stability we need the following lemma:

Lemma 3.5. *Let $\phi(t,p)$ be a trajectory of the system (3.5) in Theorem 3.2. Suppose that there exists a compact neighborhood $N = \phi(t,p) \times D_r^{n-1}$ of $\phi(t,p)$, such that every trajectory $\psi(t,q)$ with initial point $\psi(t_0,q) \in \phi(t_0,p) \times D_r^{n-1}$ is contained in N. Then the function defined by $\bar{t}(t) =$ the value of time at which $\psi(\bar{t}(t),q) \in \phi(t,p) \times D_r^{n-1}$, is differentiable and satisfies $(d/dt)\bar{t}(t) \geq \delta$, where δ is a positive constant.*

Proof. Given a time value t and an increment Δt, the length of the trajectory $\phi(t,p)$ between $\phi(t,p)$ and $\phi(t+\Delta t,p)$ is

$$\Delta s = \|F(\phi(t,p))\|\Delta t,$$

and the length of the trajectory $\psi(\bar{t},q)$ between $\psi(\bar{t}(t),q)$ and $\psi(\bar{t}+\Delta\bar{t},q) = \psi(\bar{t}(t+\Delta t),q)$ is

$$\Delta\bar{s} = \|F(\psi(\bar{t}(t),p))\|\Delta\bar{t}.$$

Now, on the compact tubular region $\phi(t',p) \times D_r^{n-1}$, $t' \in [t, t+\Delta t]$, we see in the light of the tubular manifold theorem that there exists a diffeomorphism $f : [0,1] \times D^{n-1} \to \phi(t',p) \times D_r^{n-1}$ such that

$$DfU = F(x),$$

where U is the vector field defined on $[0,1] \times D^{n-1}$ which has the form $U = (1,0,\ldots,0)$.

It follows that $\bar{t}(t)$ is differentiable and there exist positive constants δ' and δ'' such that

$$\frac{\Delta\bar{s}}{\Delta s} \geq \delta', \qquad \frac{\|F(\phi(t,p))\|}{\|F(\psi(\bar{t}(t),q))\|} \geq \delta''$$

hold uniformly on $\phi(t', p) \times D_r^{n-1}$ because of compactness.

Therefore

$$\frac{d\bar{t}}{dt} = \frac{d\bar{s}}{ds} \cdot \frac{\|F(\phi(t,p))\|}{\|F(\psi(\bar{t},q))\|} \geq \delta'\delta''.$$

Due to the compactness of N, we can find a constant δ, such that on N, $d\bar{t}/dt \geq \delta$.

Theorem 3.6. *Let $\phi(t,p)$ be a trajectory of the system* (3.5). *If there exists a compact neighborhood $N = \phi(t,p) \times D_r^{n-1}$ of $\phi(t,p)$, such that for every $y \in N$,*

$$(y - \phi(t,p)) \cdot F(y) \leq c(y - \phi(t,p))^2, \qquad c < 0,$$

then $\phi(t,p)$ is asymptotically stable in Poincaré sense.

Proof. Let $\psi(t,q)$ be a trajectory with initial value $\psi(t_0,q) \in \phi(t_0,p) \times D_r^{n-1}$.

From Theorem 3.2 and the inequality (3.8) we see that $\psi(t,q) \in N$. By Lemma 3.5 there exists a function $\bar{t}(t)$ with $d\bar{t}/dt \geq \delta$ ($\delta > 0$), such that $\psi(\bar{t}(t),q) \in \phi(t,p) \times D_r^{n-1}$.

Now the function $V(t) = \|\psi(\bar{t}(t),q) - \phi(t,p)\|^2$ satisfies

$$\begin{aligned}\frac{dV(t)}{dt} &= (\psi(\bar{t}(t),q) - \phi(t,p)) \cdot (\dot{\psi}(\bar{t}(t),q)\bar{t}'(t) - \phi(t,p)) \\ &= (\psi(\bar{t}(t),q) - \phi(t,p)) \cdot (F(\psi(\bar{t}(t),q))\bar{t}'(t) - F(\phi(t,p))) \\ &= (\psi(\bar{t}(t),q) - \phi(t,p)) \cdot F(\psi(\bar{t}(t),q))\bar{t}'(t) - 0 \\ &\leq c\delta(\psi(\bar{t}(t),q) - \phi(t,p))^2 \\ &= c\delta V(t),\end{aligned}$$

which implies that $V(t) \leq c_0 \exp(c\delta t)$.

Hence $\lim_{t\to\infty} V(t) = 0$, or equivalently, $\phi(t,p)$ is asymptotically stable in Poincaré sense.

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References

[1] E. A. JACKSON, Perspectives of nonlinear dynamics Vol. 1, *Cambridge University Press*, *Cambridge*, 1990.

[2] XIAOXIN LIAO, Theory of stability, *Central China Normal University Press*, *Wuhan*, 1988.

[3] M. SPIVAK, Differential geometry. Vol 1, 2nd edn, *Publish or Perish Inc.*, *Wilmington*, 1979.

[4] V. SZEBEHELY, Review of concepts of stability, *Celestial Mech.* **34** (1984), 49–64.

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