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On the equivalence of the rectangular and the rhombic functional equations

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Abstract. We construct a bijection of the set of continuous solutions of a generalized rectangular functional equation onto the set of continuous solutions of a corresponding generalized rhombic functional equation.

1. Introduction

The *rectangular functional equation*

$$f(x_1+y_1,x_2+y_2)+f(x_1+y_1,x_2-y_2)+f(x_1-y_1,x_2+y_2) + f(x_1-y_1,x_2-y_2)=4f(x_1,x_2) \tag{1.1}$$

for all $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$ expresses that the value of the function $f:\mathbb{R}^2\to\mathbb{C}$ at the centre of any rectangle, the sides of which are parallel to the coordinate axes, equals its mean value over the four vertices. The related functional equation

$$f(x_1+y_1,x_2)+f(x_1-y_1,x_2) +f(x_1,x_2+y_2)+f(x_1,x_2-y_2)=4f(x_1,x_2) \tag{1.2}$$

is for obvious geometric reasons called the *rhombic functional equation.* Aczél, Haruki, McKiernan and Sakovič [1] observed that (1.1) and (1.2) have the same set of solutions.

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Chung, Ebanks, Ng, Sahoo and Zeng [2] generalized the domain $\mathbb{R}^2$ of (1.1) and (1.2) to a product $G = G_1 \times \cdots \times G_n$ of groups. Apart from normalizing factors their equations are

$$\frac{1}{2^n} \sum_{\sigma_1,\dots,\sigma_n=\pm 1} f(x_1 y_1^{\sigma_1}, \dots, x_n y_n^{\sigma_n}) = f(x)h_1(y) + h_2(y), \tag{1.3}$$

and

$$\sum_{i=1}^{n} \frac{1}{2} \sum_{\sigma_i=\pm 1} f(x_1, \dots, x_{i-1}, x_i y_i^{\sigma_i}, x_{i+1}, \dots, x_n) = f(x)p(y) + q(y), \tag{1.4}$$

where $x = (x_1, \dots, x_n) \in G$ and $y = (y_1, \dots, y_n) \in G$. They proved that the same set of f's occurs in the solutions of (1.3) and (1.4).

In this paper we generalize the left hand sides still further, keeping the right hand sides of (1.3) and (1.4) unchanged, so that we get a further generalized rectangular functional equation (2.1) and a generalized rhombic functional equation (2.2). We construct a bijection of the set of solutions of the generalized rectangular functional equation onto the set of solutions of the generalized rhombic functional equation. The bijection takes not just f, but also the other functions h_1, h_2 and p, q of the solutions into account. The result reveals that it is not necessary that $G_1, \dots, G_n$ be groups.

The way we proceed is inspired by [2; Sections 4 and 5]. It should be pointed out that we work with complex valued solutions, while the solutions of [2] may take their values in any quadratically closed field of characteristic different from 2.

2. The bijection

Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be topological spaces and put $X = X_1 \times \cdots \times X_n$ and $Y = Y_1 \times \cdots \times Y_n$. We let $K_1, \dots, K_n$ be compact spaces equipped with complex Borel measures $\mu_1, \dots, \mu_n$ such that $\mu_1(K_1) = \cdots = \mu_n(K_n) = 1$. We assume that the given continuous mappings $\phi_i : X_i \times K_i \times Y_i \to X_i$ for $i = 1, \dots, n$, satisfy the following condition: For each $i = 1, \dots, n$ there exists an $e_i \in Y_i$ such that $\phi_i(x_i, k_i, e_i) = x_i$ for all

$x_i \in X_i$ and $k_i \in K_i$. If there are more than one possibility for the e_i's we fix one for each $i = 1, \dots, n$.

Writing $x = (x_1, \dots, x_n) \in X$ and $y = (y_1, \dots, y_n) \in Y$ we formulate the following version of the generalized rectangular functional equation

$$\tag{2.1} \begin{aligned} \int\limits_{K_1 \times \dots \times K_n} & f(\phi_1(x_1, k_1, y_1), \dots, \phi_n(x_n, k_n, y_n)) d\mu_1(k_1) \dots d\mu_n(k_n) \\ &= f(x)h_1(y) + h_2(y), \quad x \in X,\ y \in Y, \end{aligned}$$

and of the generalized rhombic functional equation

$$\tag{2.2} \begin{aligned} \sum_{i=1}^{n} \int\limits_{K_i} & F(x_1, \dots, \phi_i(x_i, k_i, y_i), \dots, x_n) d\mu_i(k_i) \\ &= F(x)p(y) + q(y), \quad x \in X,\ y \in Y. \end{aligned}$$

The functions to be determined are $f \in C(X)$ and $h_1, h_2 \in C(Y)$, resp. $F \in C(X)$ and $p, q \in C(Y)$. We use the standard notation of $C(X)$ for the set of complex valued continuous functions on a topological space X.

Examples 2.1. 1. In [2] we have for $i = 1, \dots, n$ that $X_i = Y_i = G_i$, $K_i = \{\pm 1\}$, $\mu_i(\{1\}) = \mu_i(\{-1\}) = \frac{1}{2}$, $\phi_i(x_i, k_i, y_i) = x_i y_i^{k_i}$ and e_i is the neutral element of the group G_i. If G_i is abelian then a more general instance is given by $\phi_i(x_i, +1, y_i) = x_i + y_i$, $\phi_i(x_i, -1, y_i) = x_i + \sigma_i y_i$ where $\sigma_i : G_i \to G_i$ is a continuous homomorphism such that $\sigma_i \circ \sigma_i$ equals the identity map (see [3]).

2. We get a more general example than the one in [2] as follows: For $i = 1, \dots, n$ we let K_i be a compact topological transformation group of a topological group G_i such that the map $x_i \to k_i \cdot x_i$ of G_i into G_i is either an automorphism or an anti-automorphism for each fixed $k_i \in K_i$. Furthermore we let dk_i denote the normalized Haar measure on K_i. Then

$$\int\limits_{K_1} \cdots \int\limits_{K_n} f(x_1(k_1 \cdot y_1), \dots, x_n(k_n \cdot y_n)) dk_n \dots dk_1 = f(x)h_1(y) + h_2(y)$$

where $x = (x_1, \dots, x_n) \in G_1 \times \dots \times G_n$ and $y = (y_1, \dots, y_n) \in G_1 \times \dots \times G_n$, is a generalized rectangular functional equation that satisfies our requirements. We get Example 1 by taking $K_i = \mathbb{Z}_2$. Another choice of K_i can be $K_i = \mathbb{Z}_N$ acting on $X_i = \mathbb{C}$ as follows: $\mathbb{Z}_N = \{\omega^n \mid n = 0, 1, \dots, N-1\}$ where $\omega = \exp(2\pi i/N)$, and $\omega^n \cdot z = \omega^n z$ for $z \in \mathbb{C}$.

Theorem 2.2. *Let the assumptions be as above. For any continuous solution (f, h_1, h_2) of the generalized rectangular functional equation* (2.1) *we define*

$$\Phi(f, h_1, h_2) := \begin{cases} (f, p, q) & \text{if } f \text{ is not a constant} \\ (f, nh_1, nh_2) & \text{if } f \text{ is a constant} \end{cases}$$

where

$$p(y) := \sum_{i=1}^{n} h_1[y_i] \quad \text{and} \quad q(y) = \sum_{i=1}^{n} h_2[y_i] \quad \text{for } y \in Y$$

with the notation $[y_i] := (e_1, \dots, e_{i-1}, y_i, e_{i+1}, \dots, e_n) \in Y$ when $y_i \in Y_i$.

Then Φ is a bijection of the set of continuous solutions of the generalized rectangular functional equation (2.1) *onto the set of continuous solutions of the generalized rhombic functional equation* (2.2)*.*

The inverse map Ψ is given by

$$\Psi(f, p, q) := \begin{cases} (f, h_1, h_2) & \text{if } f \text{ is not a constant} \\ (f, p/n, q/n) & \text{if } f \text{ is a constant} \end{cases}$$

where

$$h_1(y) := \prod_{i=1}^{n} \{p[y_i] - (n-1)\},$$
$$h_2(y) := \sum_{k=1}^{n} q[y_k] \prod_{i=1}^{k-1} \{p[y_i] - (n-1)\}$$

for $y = (y_1, \dots, y_n) \in Y$.

Proof. The statements are easy to check if f is constant, so we may in the proof restrict our attention to solutions for which f is not constant. For brevity we write $dk_i = d\mu_i(k_i)$ for $i = 1, \dots, n$, $dk = dk_1 \dots dk_n$ and $K = K_1 \times \cdots \times K_n$ in the proof. Let (f, h_1, h_2) be a continuous solution of the generalized rectangular functional equation (2.1). Putting $y = [y_i]$

into it and using the conditions on the mappings $\phi_1, \dots, \phi_n$ we find that

$$
\begin{aligned}
&f(x)h_1[y_i] + h_2[y_i] \\
&\quad = \int_K f(\phi_1(x_1, k_1, e_1), \dots, \phi_i(x_i, k_i, y_i), \dots, \phi_n(x_n, k_n, e_n))dk \\
&\quad = \int_K f(x_1, \dots, \phi_i(x_i, k_i, y_i), \dots, x_n)dk \qquad (2.3)\\
&\quad = \int_{K_i} f(x_1, \dots, \phi_i(x_i, k_i, y_i) \dots, x_n)dk_i.
\end{aligned}
$$

Summing this over $i = 1, \dots, n$ we find that

$$
\sum_{i=1}^{n} \int_{K_i} f(x_1, \dots, \phi_i(x_i, k_i, y_i), \dots, x_n)dk_i = f(x) \sum_{i=1}^{n} h_1[y_i] + \sum_{i=1}^{n} h_2[y_i],
$$

which shows that (f, p, q) is a continuous solution of (2.2) with $p(y) := \sum_{i=1}^n h_1[y_i]$ and $q(y) = \sum_{i=1}^n h_2[y_i]$.

We will next show that Φ is injective. Let (f, h_1, h_2) and (f', h'_1, h'_2) be two solutions such that $\Phi(f, h_1, h_2) = \Phi(f', h'_1, h'_2)$. Then obviously $f = f'$. Now $f(x)h_1(y) + h_2(y) = f(x)h'_1(y) + h'_2(y)$ implies that $h_1 = h'_1$ and $h_2 = h'_2$ because f is not constant.

Let (F, p, q) be a continuous solution of (2.2). Putting $y = [y_j]$ in (2.2) we get that

$$
\begin{aligned}
F(x)p[y_j] + q[y_j] &= \sum_{i \neq j} \int_{K_i} F(x_1, \dots, \phi_i(x_i, k_i, e_i), \dots, x_n)dk_i \\
&\quad + \int_{K_i} F(x_1, \dots, \phi_j(x_j, k_j, y_j), \dots, x_n)dk_j \\
&= (n-1)F(x) + \int_{K_j} F(x_1, \dots, \phi_j(x_j, k_j, y_j) \dots, x_n)dk_j
\end{aligned}
$$

so that

$$
\int_{K_j} F(x_1, \dots, \phi_j(x_j, k_j, y_j), \dots, x_n)dk_j = F(x)\{p[y_j] - (n-1)\} + q[y_j].
$$

Using that we compute

$$\int_{K_1\times\cdots\times K_n} F(\phi_1(x_1,k_1,y_1),\dots,\phi_n(x_n,k_n,y_n))dk_1\dots dk_n$$

$$= \int_{K_2\times\cdots\times K_n} \Big\{\int_{K_1} F(\phi_1(x_1,k_1,y_1),\phi_2(x_2,k_2,y_2),\dots,\phi_n(x_n,k_n,y_n))dk_1\Big\}dk_2\dots dk_n$$

$$= \int_{K_2\times\cdots\times K_n} \Big\{F(x_1,\phi_2(x_2,k_2,y_2),\dots,\phi_n(x_n,k_n,y_n))([p[y_1]-(n-1)]+q[y_1]\Big\}dk_2\dots dk_n$$

$$= \int_{K_2\times\cdots\times K_n} F(x_1,\phi_2(x_2,k_2,y_2),\dots,\phi_n(x_n,k_n,y_n))dk_2\dots dk_n[p[y_1]-(n-1)]+q[y_1]$$

$$= \cdots = F(x)\prod_{i=1}^{n}\{p[y_i]-(n-1)\} + \sum_{k=1}^{n} q[y_k]\prod_{i=1}^{k-1}\{p[y_i]-(n-1)\}.$$

We see that (F,h_1,h_2) is a continuous solution of (2.1) where

$$h_1(y) := \prod_{i=1}^{n}\{p[y_i]-(n-1)\}, \quad h_2(y) := \sum_{k=1}^{n} q[y_k]\prod_{i=1}^{k-1}\{p[y_i]-(n-1)\}.$$

To see that Φ is surjective it suffices to consider a solution (F,p,q) of (2.2) for which F is not a constant. With the notation just applied we see that (F,h_1,h_2) is a solution of (2.1). Now, $\Phi(F,h_1,h_2) = (F,P,Q)$ is a solution of (2.2), so we have two solutions with the same F. Thus $F(x)P(y)+Q(y) = F(x)p(y)+q(y)$. Since F is not a constant we get that $P = p$ and $Q = q$, so that $\Phi(F,h_1,h_2) = (F,p,q)$. This shows that Φ is surjective. It also shows that $\Phi\circ\Psi$ is the identity, thus verifying the claim about the inverse. □

Corollary 2.3. *The solutions $f \in C(X)$ of the following version of Jensen's functional equation*

$$\int_{K_1\times\cdots\times K_n} f(\phi_1(x_1,k_1,y_1),\dots,\phi_n(x_n,k_n,y_n))d\mu_1(k_1)\dots d\mu_n(k_n) = f(x),$$

$$x \in X,\ y \in Y,$$

are the same as the solutions $F \in C(X)$ of the following version of the rhombic functional equation

$$\sum_{i=1}^{n} \int_{K_i} F(x_1, \ldots, \phi_i(x_i, k_i, y_i), \ldots, x_n) d\mu_i(k_i) = F(x), \quad x \in X, \ y \in Y.$$

The corollary was noted in [1; §4] for the classical case of $X = \mathbb{R}^2$, and proved in [2] in the setup there.

Remark 2.4. In contrast to [2] we work with continuous solutions. This is because our integrals should make sense. However, if $K_1, \ldots, K_n$ are finite sets then we may dispense with all assumptions about continuity by equipping $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with the discrete topology. All maps are continuous with respect to the discrete topology, so in this case we work with solutions without assuming any regularity. The advantage of keeping the topology is that Theorem 2.2 and Corollary 2.3 have the same wording irrespective of the topologies involved. The discrete topology is just a special case that one can refer to if one is interested in solutions without prescribed regularity conditions like continuity.

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