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# On homologies of Klingenberg projective spaces over special commutative local rings

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**Abstract.** In this article homologies of Kliengenberg projective spaces over communitative local rings of a special type are investigated. Especially sets of invariant points are studied.

## 1. Introduction

Klingenberg projective spaces (KPS) over a local ring were described by F. Machala [6]. Let us have in the KPS $P$ a hyperplane $\mathbf{H}$ and a point $\mathbf{C}$ non-neighbour with $\mathbf{H}$ and let there exist a homology of $P$ such that every point of $\mathbf{H}$ and the point $\mathbf{C}$ are invariant. In the case of projective spaces over fields (which is a special case of a KPS over a local ring) the set of invariant points of the considered homology is just $\mathbf{H} \cup \{\mathbf{C}\}$. However in the case of KPS over a local ring there exist certain invariant points which do not belong to $\mathbf{H} \cup \{\mathbf{C}\}$. If we consider KPS over local rings of the following special type then we may introduce the notion "*degree of neighbourhood*" of two points and by this notion we shall describe the sets of invariant points.

In this paper we shall consider the local commutative ring $\mathbf{A}$ the maximal ideal $a$ of which has the following properties:

(1) $\exists m \in \mathbb{N} : (a^m = 0) \wedge (a^{m-1} \neq 0)$,

(2) $a = \eta \mathbf{A}$[1].

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[1] The factor ring of polynomials $\mathbf{R}[x]/(x^m)$ is an example of the ring of this type.

Throughout the paper the capital $\mathbf{A}$ always denotes the ring of described type.

Clearly, for every $\tau \in \mathbf{A}$, $\tau \neq 0$, there exists a unit $\tau'$ and an integer $k$, $0 \leq k \leq m-1$, such that $\tau = \eta^k \tau'$. The number $k$ is called an *order of* $\tau$. The order of $\tau = 0$ is equal to $m$.

Let $\mathbf{M}$ be a free finite dimensional module over $\mathbf{A}$. It is well known that all bases of $\mathbf{M}$ have the same number of elements and from every system of generators of $\mathbf{M}$ we may select a basis of $\mathbf{M}$.

Moreover, in our case the module $\mathbf{M}$ has the following properties (proved in [4]):

**1.** Any linearly independent system can be completed to a basis of $\mathbf{M}$.

**2.** A submodule of $\mathbf{M}$ is a free module if and only if it is a direct summand of $\mathbf{M}$.

*Remark.* Free finite dimensional modules over a local ring $\mathbf{R}$ are called $\mathbf{R}$-*spaces* (see e.g. [3]) and their direct summands $\mathbf{R}$-*subspaces*.

We get that in our case the $\mathbf{A}$-subspaces of an $\mathbf{A}$-space $\mathbf{M}$ are just all the free submodules of $\mathbf{M}$.

For $\mathbf{A}$-subspaces of $\mathbf{M}$ we have (proved in [4]):

**3.** Let $K, L$ be $\mathbf{A}$-subspaces of an $\mathbf{A}$-space $\mathbf{M}$. Then $K + L$ is an $\mathbf{A}$-subspace if and only if $K \cap L$ is an $\mathbf{A}$-subspace. In this case the dimensions of the $\mathbf{A}$-subspaces fulfil the following relation:

$$\dim(K+L) + \dim(K \cap L) = \dim K + \dim L.$$

**Lemma 1.** *Let* $\mathbf{M}$ *be an* $\mathbf{A}$*-space and let* $\bar{\mathbf{M}}$ *be a vector space* $\mathbf{M}/a\mathbf{M}$. *Then the elements* $\mathbf{u}_1, \dots, \mathbf{u}_k$ *form a linearly independent system in* $\mathbf{M}$ *if and only if cosets* $\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_k$ *form a linearly independent system in* $\bar{\mathbf{M}}$.

Proof. It follows from the first property that a system $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subseteq \mathbf{M}$ is linearly independent iff it may be completed to a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_{n+1}\}$ of $\mathbf{M}$. This is (according to the Theorem I.2 of [3]) equivalent to the fact that the cosets $\{\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_k, \bar{\mathbf{u}}_{k+1}, \dots, \bar{\mathbf{u}}_{n+1}\}$ form a basis of the vector space $\bar{\mathbf{M}} = \mathbf{M}/a\mathbf{M}$ which means that $\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_k$ are linearly independent vectors.

**Lemma 2.** *Let* $\mathbf{x}$ *as well as* $\mathbf{y}$ *be a linearly independent element of an* $\mathbf{A}$*-space* $\mathbf{M}$*. If* $\xi\mathbf{x}+\vartheta\mathbf{y}=\mathbf{o}$ *then* $\xi,\vartheta$ *have the same order.*

Proof. If $\xi=0$ then the linear independence of $\mathbf{y}$ implies $\vartheta=0$. Analogously $\vartheta=0$ implies $\xi=0$.

Let $\xi,\vartheta\neq 0$. Then we may write $\xi=\eta^k\xi'$, $\vartheta=\eta^h\vartheta'$ where $\xi',\vartheta'$ are units and $0\leq k,\ h\leq m-1$. If $k\neq h$ (e.g. $h=k+r$, $r\in\mathbb{N}$) then multiplying the equality $\xi\mathbf{x}+\vartheta\mathbf{y}=\mathbf{o}$ by $\eta^{m-k-r}$ we obtain $\eta^{m-r}\mathbf{x}=\mathbf{o}$, which contradicts the linear independence of $\mathbf{x}$.

## 2. Klingenberg projective spaces over local rings

According to [6] we define:

*Definition 1.* Let $\mathbf{R}$ be a local ring with the maximal ideal $r$. Let us denote $\mathbf{M}=\mathbf{R}^{n+1}$, $\bar{\mathbf{M}}=\mathbf{M}/r\mathbf{M}$, $\bar{\mathbf{R}}=\mathbf{R}/r$, and let $\mu$ be a natural homomorphism $\mathbf{M}\to\bar{\mathbf{M}}$.

Then an incidence structure $P_R$ such that

(1) the points are just all submodules $[\mathbf{x}]$ of $\mathbf{M}$ such that $\mu(\mathbf{x})$ is a non-zero element of $\bar{\mathbf{M}}$,

(2) the lines are just all submodules $[\mathbf{x},\mathbf{y}]$ of $\mathbf{M}$ such that $[\mu(\mathbf{x}),\mu(\mathbf{y})]$ is a two-dimensional subspace of $\bar{\mathbf{M}}$,

(3) the incidence relation is inclusion,

is called *an* $n$*-dimensional Klingenberg coordinate space over the ring* $\mathbf{R}$.

Two points $P=[\mathbf{p}]$, $Q=[\mathbf{q}]$ such that $[\mu(\mathbf{p})]=[\mu(\mathbf{q})]$ are called *neighbour points* or *neighbours*. In the contrary case they are called *non-neighbours*.

If $X=[\mathbf{x}]$ is a point of $P_R$, then $\mathbf{x}$ will be called an *arithmetical representative of* $X$.

A submodule $\mathbf{H}$ of $\mathbf{M}$ is called a *hyperplane of* $P_R$ if $\mathbf{H}=[\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_n]$ so that $[\mu(\mathbf{x}_1),\mu(\mathbf{x}_2),\dots,\mu(\mathbf{x}_n)]$ is an $n$-dimensional subspace of $\bar{\mathbf{M}}$.

We obtain from Definition 1 and Lemma 1 the following corollaries:

**Corollary 1.** *The points of the Klingenberg space* $P_A$ *are just all one-dimensional* $\mathbf{A}$*-subspaces of* $\mathbf{M}$*. The lines of the Klingenberg space* $P_A$ *are just all two-dimensional* $\mathbf{A}$*-subspaces of* $\mathbf{M}$*. The hyperplanes of the Klingenberg space* $P_A$ *are just all* $n$*-dimensional* $\mathbf{A}$*-subspaces.*

**Corollary 2.** *Two points of the Klingenberg space $P_A$ are neighbours if and only if their arithmetical representatives form a lienarly dependent subset of* $\mathbf{M}$.

The correctness of the following definition follows from Lemma 2.

*Definition 2.* Let $X = [\mathbf{x}]$, $Y = [\mathbf{y}]$ be points of the Klingenberg space $P_A$ and let $k$, $k \geq 0$, be an integer fulfilling the following conditions

(1) $\eta^k \mathbf{x} \in [\mathbf{y}]$

(2) $(\eta^{k-1}\mathbf{x} \notin [\mathbf{y}]) \vee (k = 0)$.

Then the integer $r = m - k$ is called a *degree of neighbourhood of the points $X$ and $Y$*.

*Remark.* It follows from Corollary 2 that for non-neighbour points we have $r = 0$, for different neighbour points we have $0 < r < m$ and for identical points we have $r = m$.

**Lemma 3.** *Let $X$ be a point of $P_A$. Then for every integer $r$, $0 \leq r \leq m$, there exists at least one point $Y \in P_A$ such that the degree of neighbourhood of the points $X, Y$ is $r$.*

Proof. Let $X = [\mathbf{x}]$. Then (according to 1 of Section 1) there exists $\mathbf{z} \in \mathbf{M}$ such that $\mathbf{x}, \mathbf{z}$ form a linearly independent couple. If $r = 0$ then the lemma holds ($Y = [\mathbf{z}]$) as well as in the case $r = m$. Let $0 < r < m$. Considering a point $Y = [\mathbf{y}]$, $\mathbf{y} = \mathbf{x} + \eta^{m-r}\mathbf{z}$, we get $\eta^r \mathbf{y} \in [\mathbf{x}]$ and $\eta^{r-1}\mathbf{x} + \eta^{m-1}\mathbf{z} = \eta^{r-1}\mathbf{y} \notin [\mathbf{x}]$.

## 3. Homologies of Klingenberg projective spaces

*Definition 3.* An automorphism of the incidence structure $P_A$ such that there exist an hyperplane $\mathbf{H}$ of invariant points and an invariant point $C$ which is non-neighbour with $\mathbf{H}$ (it means $C$ is not neighbour with any point of $\mathbf{H}$) is called an $(\mathbf{H}, C)$-*homology of* $P_A$ and the point $C$ the *center of homology*.

*Remark.* It follows from Corollary 2 (as $\mathbf{H}$ is an $\mathbf{A}$-subspace) that $C = [\mathbf{c}]$ is non-neighbour with $\mathbf{H}$ just if $[\mathbf{c}] \cap \mathbf{H} = \{\mathbf{o}\}$.

The following theorem is proved by Bacon [2] for the case when $P_A$ is a plane and $\mathbf{A}$ is an arbitrary local ring.

**Theorem 1.** *Let* $\mathbf{H}$ *be an hyperplane of the Klingenberg projective space* $P_A$*,* $C$ *a point of* $P_A$ *non-neighbour with* $\mathbf{H}$*. If* $X, Y$ *are points of* $P_A$ *such that*

(1) $X$ *and* $Y$ *are non-neighbour with* $C$ *and with* $\mathbf{H}$

(2) $C, X, Y$ *are collinear points*

*then there exists exactly one* $(C, \mathbf{H})$*-homology of* $P_A$ *such that* $X \mapsto Y$*.*

Proof. Since $[\mathbf{c}] \cap \mathbf{H} = \{\mathbf{o}\}$ we get (according to the 3 of Section 1) $[\mathbf{c}] \oplus \mathbf{H} = \mathbf{M}$.

An arbitrary $(C, \mathbf{H})$-homology $F^*$ on $P_A$ will be induced by an automorphism $F$ of $\mathbf{M}$ in a very natural way:

$$\forall X = [\mathbf{x}] \in P_A : F^*(X) = [F(\mathbf{x})].$$

As $F^*|\mathbf{H}$ is an identity mapping we may infer that there exists a unit $\lambda \in \mathbf{A}$ such that $F|\mathbf{H} = \lambda.id$. Without loss of generality we can assume that $F|\mathbf{H} = id$.

Any $\mathbf{x} \in \mathbf{M}$ may be uniquely expressed in the form

$$\mathbf{x} = \mathbf{x}' + \xi\mathbf{c},\ \mathbf{x}' \in \mathbf{H},\ \xi \in \mathbf{A}. \tag{1}$$

The automorphism $F$ is given by

$$\mathbf{x} = \mathbf{x}' + \xi\mathbf{c} \mapsto F(\mathbf{x}) = \mathbf{x}' + \alpha\xi\mathbf{c}, \tag{2}$$

where $\alpha$ is a unit.

Now $\mathbf{x}$ is linearly independent (equivalently, $\mathbf{x}$ represents a point $[\mathbf{x}]$ of $P_A$) if and only if $\mathbf{x}'$ is linearly independent or $\xi \notin a$.

Moreover $\mathbf{x}$ represents a point non-neighbour with $C$ and with $\mathbf{H}$ if and only if $\mathbf{x}'$ is linearly independent and $\xi \notin a$.

Considering $Y = [\mathbf{y}]$, $Y \in XC$, we get $\mathbf{y} = \gamma\mathbf{x} + \sigma\mathbf{c}$. As $Y, C$ are not neighbour points, $\gamma$ is a unit and the arithmetical representative $\mathbf{y}$ may be expressed by

$$\mathbf{y} = \mathbf{x} + \sigma\mathbf{c}.$$

Using the expression (1) of $\mathbf{x}$ we obtain from this

$$\mathbf{y} = \mathbf{x}' + (\xi + \sigma)\mathbf{c}. \tag{3}$$

The automorphism $F^*$ maps $X$ to $Y$ just if $F(\mathbf{x}) = \varepsilon\mathbf{y}$, where $\varepsilon$ is a unit. Using (2) and (3) we get

$$(1-\varepsilon)\mathbf{x}' + (\alpha\xi - (\xi+\sigma)\varepsilon) = \mathbf{c} = \mathbf{o}.$$

As $[\mathbf{c}] \cap \mathbf{H}$ is trivial, $\varepsilon = 1$, $\alpha\xi = -(\xi+\sigma)\varepsilon = 0$, hence $\alpha = \xi^{-1}(\xi+\sigma)$. This means that $\alpha$ as well as the homology $F^*$ is determined uniquely.

*Remark.* Every $(C, \mathbf{H})$-homology of $P_A$, $C = [\mathbf{c}]$ may be expressed by the following formula:

$$X = [\mathbf{x}' + \xi\mathbf{c}] \mapsto F^*(X) = [\mathbf{x}' + \alpha\xi\mathbf{c}], \quad \mathbf{x}' \in \mathbf{H},\ \alpha \notin a. \tag{4}$$

The element $\alpha$ will be called the *coefficient of the homology* $F^*$.

**Proposition 1.** *Let $F^*$ be a $(C, \mathbf{H})$-homology of $P_A$ and $\alpha$ the coefficient of $F^*$. If $(1-\alpha)$ has order $r$ then*

(1) *for any point $X \in P_A$, $X$ and $F^*(X)$ are neighbours of order at least $r$.*

(2) *there exists $X \in P_A$ such that $X$ and $F^*(X)$ are neighbours just of degree $r$.*

Proof. As usual let $C = [\mathbf{c}]$. Let $F$ be the automorphism of $\mathbf{M}$ inducing $F^*$ and let $F$ be given by (2). Suppose that $(1-\alpha) = \alpha_0\eta^r$, where $\alpha_0$ is a unit. Using (2) we have

$$F(\mathbf{x}) = (\mathbf{x} - \xi\mathbf{c}) + \xi\alpha\mathbf{c} = \mathbf{x} + \xi = (\alpha - 1)\mathbf{c} = \mathbf{x} - \alpha_0\eta^r\xi\mathbf{c}.$$

Therefore $\eta^{m-r}F(\mathbf{x}) = \eta^{m-r}\mathbf{x}$. This means that the degree of neighbourhood of the points $X = [\mathbf{x}]$ and $F^*(X)$ is (at least) $r$.

If $\xi$ is a unit and $[\mathbf{x}]$, $C$ are not neighbours then

$$\eta^{m-r-1}F(\mathbf{x}) = \eta^{m-r-1}\mathbf{x} - (\eta^{m-1}\xi\alpha_0)\mathbf{c} \text{ where } \eta^{m-1}\xi\alpha_0 \neq 0.$$

Thus $\eta^{m-r-1}F(\mathbf{x}) \notin [\mathbf{c}]$ which means that $X$ and $F^*(X)$ are neighbours of precisely degree $r$.

*Remark.* In particular, if $1-\alpha$ has order $m$ (i.e. $\alpha = 1$) the $F^*$ is an identity and all the $X, F^*(X)$ are neighbours of degree (at least) $m$, i.e. identical.

If $1-\alpha$ is a unit then there exists $X$ such that $X, F^*(X)$ are neighbours of degree zero i.e. non-neighbours.

**Proposition 2.** *Let $F^*$ be a $(C, \mathbf{H})$-homology of $P_A$ and $\alpha$ the coefficient of $F^*$. Let $X$ be an arbitrary point non-neighbour with $C$ as well as $\mathbf{H}$ and let the degree of neighbourhood of $X$ and $F^*(X)$ be $r$. Then the following hold*

(1) $(1-\alpha)$ *has order* $r$.

(2) *If $Y$ is a point non-neighbour with $C$ as well as with $\mathbf{H}$ then $Y$ and $F^*(Y)$ are neighbours of degree $r$.*

Proof. Let $F^*$ be given by the formula (4) and let $F$ be an automorphism of $\mathbf{M}$ given by (2). If $X$ and $F^*(X)$ are neighbours of order $r$ then (according to Lemma 2)

$$\eta^{m-r}F(\mathbf{x}) = \eta^{m-r}\varepsilon\mathbf{x}, \qquad \varepsilon \notin a. \tag{5}$$

It follows from (2) that $F(\mathbf{x}) = \mathbf{x} + \xi(\alpha - 1)\mathbf{c}$. Using this and (5) we obtain

$$\eta^{m-r}(1-\varepsilon)\mathbf{x} = \eta^{m-r}\xi(1-\alpha)\mathbf{c}.$$

Because $X, C$ are non-neighbours we get

$$\eta^{m-r}(1-\varepsilon)\mathbf{x} = \eta^{m-r}\xi(1-\alpha)\mathbf{c} = \mathbf{o}. \tag{6}$$

Since $X$ is non-neighbour with $\mathbf{H}$ we have that $\xi$ is a unit. Thus $(1-\alpha)$ has order at least $r$. If the order of $(1-\alpha)$ is greater than $r$ then (according to the previous proposition) we get that $U$ and $F^*(U)$ are neighbours of degree at least $r+1$ for every point $U$ and this contradicts our assumption. Therefore the order of $(1-\alpha)$ is (precisely) $r$.

Suppose there exists $Y$ (non-neighbour with $\mathbf{H}$ and with $C$) such that $Y$, $F^*(Y)$ are neighbours of degree $r+1$. This implies (by (6)) that $(1-\alpha)$ has order (at least) $r+1$ which is not possible.

**Theorem 2.** *Let $F^*$ be a $(C, \mathbf{H})$-homology of $P_A$, $\alpha$ the coefficient of $F^*$, $r$ the order of $(1-\alpha)$ and let $X$ be a point of $P_A$. Then $X$ is $F^*$-invariant if and only if it is neighbour of degree at least $m-r$ with $C$ or some point of $\mathbf{H}$.*

Proof. Let us remark that $\mathbf{H} \oplus [\mathbf{c}] = \mathbf{M}$. Let the homology $F^*$ be given by the formula (4).

In case $(1-\alpha)$ is of order $m$ (i.e. $\alpha = 1$) we get that $F^*$ is an idnetity and the proposition holds. Now suppose $r \leq m-1$.

Let $X = [\mathbf{x}]$ be an invariant point of $F^*$ and let $F$ be an automorphism of $\mathbf{M}$ given by (2). Then $F(\mathbf{x}) = \varepsilon\mathbf{x}$, $\varepsilon \notin a$. Using this and (2) we get

$$(1-\varepsilon)\mathbf{x}' = \xi(\varepsilon - \alpha)\mathbf{c}.$$

As $C$ is not neighbour with $\mathbf{H}$ this yields

$$(1-\varepsilon)\mathbf{x}' = \xi(\varepsilon - \alpha)\mathbf{c} = \mathbf{o}. \tag{7}$$

Because $\mathbf{x}$ is linearly independent $\mathbf{x}'$ is also linearly independent or $\xi$ is a unit.

(a) If $\mathbf{x}'$ is linearly independent then (by (7)) $\varepsilon = 1$, thus $\xi(1-\alpha) = 0$. Supposing $(1-\alpha)$ has order $r$ we have $\xi = \xi_0\eta^{m-r}$. Now we obtain $\mathbf{x} = \mathbf{x}' + \xi_0\eta^{m-r}\mathbf{c}$ and thus $\eta^r\mathbf{x} = \eta^r\mathbf{x}' \in \mathbf{H}$. This means that the degree of neighbourhood of $X$ and $\mathbf{H}$ is at least $m-r$.

(b) If $\xi$ is a unit then (by (7)) $\varepsilon = \alpha$ which implies that $(1-\varepsilon) = (1-\alpha)$ has order $r$. As $(1-\varepsilon)\mathbf{x}' = \mathbf{o}$, $\mathbf{x}'$ may be written as $\mathbf{x}' = \eta^{m-r}\mathbf{y}$, $\mathbf{y} \in \mathbf{H}$. Now we get $\mathbf{x} = \eta^{m-r}\mathbf{y} + \xi\mathbf{c}$ and therefore $\eta^r\mathbf{x} = \eta^r\xi\mathbf{c}$, $\xi \notin a$. This means that the degree of neighbourhood of $X$ and $C$ is at least $m-r$.

Now, suppose that degree of neighbourhood of $X = [\mathbf{x}]$ and $\mathbf{H}$ is at least $m-r$. This implies that $\eta^r\mathbf{x} \in \mathbf{H}$. As $\eta^r\mathbf{x}' + \eta^r\xi\mathbf{c} = \eta^r\mathbf{x} \in \mathbf{H}$ we have $\eta^r\xi = 0$ and thus $\xi = \xi_0\eta^{m-r}$. Supposing $1-\alpha = \alpha_0\eta^r$ we get

$$\begin{aligned} F(\mathbf{x}) &= \mathbf{x}' + \xi_0\alpha\eta^{m-r}\mathbf{c} = \mathbf{x}' + \xi_0\eta^{m-r}(1-\alpha_0\eta^r)\mathbf{c} \\ &= \mathbf{x}' + \xi\mathbf{c} - \xi_0\alpha_0\eta^m\mathbf{c} = \mathbf{x}, \end{aligned}$$

i.e. $X$ is an invariant point.

Let us suppose that $C$ and $X = [\mathbf{x}]$ are neighbours of degree at least $m-r$. Then $\eta^r\mathbf{x} \in [\mathbf{c}]$ which means that $\eta^r\mathbf{x}' + \eta^r\xi\mathbf{c} = \eta^r\mathbf{x} \in [\mathbf{c}]$. This gives $\eta^r\mathbf{x}' = \mathbf{o}$ i.e. $\mathbf{x}' = \eta^{m-r}\mathbf{y}$, $\mathbf{y} \in \mathbf{H}$.

Therefore $\mathbf{x} = \eta^{m-r}\mathbf{y} + \xi\mathbf{c}$ and $F(\mathbf{x}) = \eta^{m-r}\mathbf{y} + \xi\alpha\mathbf{c} = (1+\alpha_0\eta^r)\eta^{m-r}\mathbf{y} + \xi\alpha\mathbf{c} = \alpha\mathbf{x}' + \alpha\xi\mathbf{c} = \alpha\mathbf{x}$. This means that $X$ is an invariant point.

*Remark.* By this theorem (in view of Lemma 3) the set of $F^*$-*invariant* points uniquely determines the order of $1-\alpha$.

The following theorem is a consequence of Proposition 1, 2 and of Theorem 2:

**Theorem 3.** *Let $F^*$ be a $(C, \mathbf{H})$-homology of $P_A$ and let $X$ be an arbitrary point non-neighbour with $\mathbf{H}$ as well as with $C$. Then the points $X$, $F^*(X)$ are neighbours of degree $r$ if and only if the $F^*$-invariant points are just all the points which are neighbours with $C$ or with $\mathbf{H}$ of degree at least $m-r$.*

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