Publl. Math. Debrecen
55 / 1-2 (1999), 123–140

Equilibrium distributions for the M/G/1 and related systems

By LÁSZLÓ LAKATOS (Budapest)

Dedicated to Professors I. Kátai and Z. Daróczy on their sixtieth birthday

Abstract. For the equilibrium description of the M/G/1 system one usually uses the embedded Markov chain technique and gets the Pollaczek–Khinchin formula, the generating function of ergodic distribution. From it the equilibrium probabilities may be obtained by means of differentiation leading to rather complicated expressions. In this paper we try to determine them on a simpler way. According to an ergodicity theorem for Kovalenko's piecewise-linear processes they may be found on the basis of mean values of duration of the busy period and times spent in different states for it. Here we derive a recurrence relation for these mean values by using only the transition probabilities of the embedded chain. A natural generalization leads to the queueing systems with vacation. In this case after each busy period an additional time interval called vacation follows and the sequence of busy periods and successive vacations is terminated when for a vacation no request enters. Considering a cycle starting with the first busy period and terminated by the last vacation the same reasoning can be used and we are able to obtain a similar recurrence relation for the desired mean values in this system, too.

Mathematics Subject Classification: 60K25, 90B22.
Key words and phrases: M/G/1 system, ergodic distribution, recurrence relation, system with vacation.
Research supported by OTKA 14974/95 and T022796 and FKFP-0144.
The main results of the paper were presented on the Second Scandinavian–Ukrainian Conference in Mathematical Statistics in Umeå, Sweden, June 8–13, 1997.

1. Introduction

In the queueing theory the M/G/1 system is one of the most well-known ones, for its equilibrium description one usually uses the embedded Markov chain technique leading to the Pollaczek–Khinchin formula, the generating function of ergodic distribution. From mathematical viewpoint it gives the complete solution of the problem, but in practice it can lead to complications. One had such kind of difficulties at computation of percentiles. For this purpose it is necessary to take one by one and sum up the probabilities $p_0, p_1, p_2, \ldots$ till the sum achieves a given probability level. In order to obtain these probabilities the differentiation of the Pollaczek–Khinchin formula seems to be the most natural way, but it gives very complicated expressions. In [6], [7] Brière and Chaudry consider this problem from another viewpoint in case of bulk-arrival and bulk-service systems, the inversion of generating function is realized by comparing the coefficients at the corresponding powers of z, a recursive algorithm is obtained for different concrete service time distributions. Their work includes both sample numerical results and easily implementable algorithms. Software packages realizing these algorithms are also available on such systems [9]. [8] presents a unified approach for the numerical solutions of the M/G/1 queue. It gives an overview of existing methods and discusses the possibility of their applications. There is mentioned e.g. Neuts' matrix-analytic method which by Powell and Van Hoorn is good for mathematical treatment, but it is difficult to implement computationally at least for high values of parameters. The authors present a possible solution in case the service time distribution has a rational Laplace–Stieltjes transform, under such assumptions explicit closed-form expressions can be obtained in terms of roots of associated characteristic equation (it corresponds to the denominator of generating function). We used another approach. The functioning of M/G/1 system may be characterized by the help of Kovalenko's piecewise linear processes, it gives possibility to compute the desired probabilities on the basis of a busy period and the transition probabilities of the embedded Markov chain. The arrival rate, the mean value of service time and the probability of appearence of a given number of requests for the service time are the primary information about the functioning of the system, the desired probabilities are obtained directly from them avoiding the generating functions, even we have not to know the concrete service time distribution.

In [11] Levy and Yechiali introduced the idea and investigated the M/G/1 system with vacation. It means that after each busy period an additional time interval called vacation is required (e.g. to repair the equipment), i.e. having completed the service of all present requests we cannot come at once to free state. If during the vacation no request enters we reach the usual free state, in another case a further busy period begins, it generates again a vacation, etc. This process will be terminated only in case, when for a vacation no request appears. In the future the period from the beginning of the first busy period till the end of last vacation will be called “cycle”. Introducing the special state of vacation and reasoning on the same way as in case of the simple M/G/1 system we are able to find the ergodic distribution for this model, too.

2. Kovalenko's results and some notations

The piecewise linear processes were introduced in [3], [4] and they are described in the third chapter of [1]. There is a useful theorem concerning how to determine the ergodic distribution at the same place in the fourth chapter. Since the functioning of the M/G/1 system may be described by means of these processes, we shortly point out the basic moments which are necessary for the fulfilment of conditions of these theorems. In case of investigation of the M/G/1 system by the help of embedded Markov chain its states are identified by the number of requests there at moments $t_n + 0$, i.e. they coincide with the number of requests after having serviced the n-th one. There is no restriction on the waiting room, so the number of states is countable. The ergodicity theorem from [1] (a detailed proof is given in [14]) requires a finite number of states, so we unite into one the cases when the number of present ones is equal to or greater than k. Furthermore, we assume that the mean value of service time of a request is finite, and if the service process of a request started, it is continued till the end without interruption. Under these conditions according to [1] the ergodic distribution exists and it can be computed on the basis of the mean value of duration of busy period and the mean value of sojourn on different levels for it (the expressions to have k requests in the system and to stay at level k we will use in the same sense). Introducing the special state for systems with vacation and reasoning the same way as above, one can also show the applicability of piecewise linear processes and the ergodicity theorem to it (actually one has to only consider a cycle and sojourns in different states for it). So our purpose is to find the mean values of a busy period and a cycle, the mean values of time spent in different states for them.

It is well-known that the Laplace–Stieltjes transform of the busy period's distribution function $\Gamma(s)$ is the unique analytical solution of the functional equation

$$\Gamma(s) = b(s + \lambda - \lambda\Gamma(s))$$

at $\Re\, s > 0$ under condition $|\Gamma(s)| \leq 1$, where λ is the arrival rate, $b(s) = \int\limits_0^\infty e^{-sx} dB(x)$, and $B(x)$ is the distribution function of the service time for a request. Generally, from this equation one cannot obtain an explicit expression for $\Gamma(s)$, but by using it we are able to find the moments, e.g. the mean value of the busy period's duration is equal to

$$\frac{\tau}{1-\rho} \qquad \left(\tau = \int\limits_0^\infty x dB(x), \quad \rho = \lambda\tau\right),$$

from which and Wald's equality it follows that the average number of requests serviced for a busy period is equal to $(1-\rho)^{-1}$.

We introduce some further notations:

$a_k = \int\limits_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dB(x)$ – the probability of appearence of k new requests for the service time of a request;

$\zeta = \frac{\tau}{1-\rho}$ – the mean value of duration of a busy period;

ζ' – the mean value of duration of a cycle;

ζ_i – the mean value of time spent above the i-th level for a busy period;

ζ_i' – the mean value of time spent above the i-th level for a cycle;

ξ_i – the mean value of time spent on the i-th level for a busy period;

ξ_i' – the mean value of time spent on the i-th level for a cycle;

$\rho = \lambda\tau$ – the utilization factor for the system without vacation;

$D(x)$ – the distribution function of vacation, $d(s) = \int\limits_0^\infty e^{-sx} dD(x)$ is its Laplace–Stieltjes transform and $\eta = \int\limits_0^\infty x dD(x)$ its mean value;

$d_k = \int\limits_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dD(x)$ – the probability of appearence of k requests for the vacation;

$\rho_v = \lambda\eta$ – the utilization factor for the system with vacation.

In this notations the ergodic probabilities for the simple M/G/1 system obviously are $p_i = \xi_i/\zeta$ $(i = 0, 1, 2, \dots)$, and for the system with vacation $p'_i = \xi'_i/\zeta'$ $(i = v, 0, 1, 2, \dots)$. Our purpose is to find these mean values.

3. Theorems

We formulate the results of the paper in the following theorems.

Theorem 1. *In the M/G/1 system*

$$\xi_0 = \tau, \qquad \xi_1 = \frac{1 - a_0}{a_0}\tau, \qquad \xi_2 = \frac{1 - a_0 - a_1}{a_0}(\xi_0 + \xi_1)$$

and the mean values of time ξ_k $(k \geq 3)$ spent in different states for a busy period satisfy the recurrence relation

$$\xi_k = \sum_{i=1}^{k-2} \frac{1 - a_0 - a_1 - \cdots - a_i}{a_0} \xi_{k-i} + \frac{1 - a_0 - a_1 - \cdots - a_{k-1}}{a_0}(\xi_0 + \xi_1).$$

Lemma. *The mean value of duration of a cycle in the M/G/1 system with vacation is*

$$\zeta' = \frac{\tau}{1 - \rho} + \frac{\eta}{d_0(1 - \rho)}.$$

Theorem 2. *The mean values of time spent on the k-th level ξ'_k for a cycle in the M/G/1 system with vacation satisfy the relations*

$$\xi'_v = \frac{\tau}{d_0}, \qquad \xi'_0 = \eta, \qquad \xi'_1 = \frac{\tau}{a_0 d_0} - \tau + \frac{d_1}{d_0}(\eta - \tau),$$

$$\begin{aligned} \xi'_k = \xi_k &+ \sum_{i=0}^{k-2} \frac{1 - d_0 - \cdots - d_i}{d_0} \xi_{k-i} \\ &+ \frac{1 - d_0 - \cdots - d_{k-1}}{d_0}(\xi_0 + \xi_1) + \frac{d_k}{d_0}(\eta - \tau) \qquad (k \geq 2), \end{aligned}$$

where ξ_i $(i \geq 0)$ are the mean values of time spent on the i-th level for a busy period in the M/G/1 system without vacation and are determined by Theorem 1.

4. Proof of Theorem 1

Let j requests be present in the system, one of them on service. After having serviced the actual one with probability a_1 the same number of requests remain there. With probability $1 - a_1$ we come to another level, more exactly, with probability $\dfrac{a_0}{1-a_1}$ to $j-1$ and with probability $\dfrac{1-a_0-a_1}{1-a_1}$ to a level above j.

We consider the structure of a busy period. First we shall investigate the periods during which there is only one request in the system and periods during which there are more than one request there. For us it is more convenient to characterize the system with the number of requests at the starting moment of service of a concrete one which does not change till completion. The such defined notions of state and number of present requests must be distinguished, the difference is clear from the following reasoning. If one considers service periods of requests when at the starting moment there is no another one, they correspond to the state 1 excluding two cases. The first case is the jump to a level above the first, then the service of last request from this period from the viewpoint of states corresponds to the new level (from the viewpoint of the number of present requests of course to the first). But the whole duration of time spent on the first level does not change because coming from the second level to the first the inverse situation takes place. The picture is analogous for the levels above the first, too. The second case is the service of the last request in the busy period, it corresponds to the zero state (after its service there will be no request in the system), so it must be excluded from the number of requests serviced on the first level.

We determine the mean value of a period during which there is only one request in the system. Since a_1 is the probability that during the service of a request a new one occurs, this state continues with probability a_1 and finishes with probability $1 - a_1$ (no request enters or more requests enter). For such a period with probability $1 - a_1$ is serviced one request, with probability $a_1(1-a_1)$ two requests, ..., with probability $a_1^{k-1}(1-a_1)$ k ones. The mean number of requests serviced for such a period is

$$\sum_{k=1}^{\infty} k a_1^{k-1}(1-a_1) = \frac{1}{1-a_1}.$$

Now let us determine the mean value of a period above the first level (in this case we will have the aforementioned deviation concerning the states and numbers of present requests, but finally we get the correct value without any exclusion). Assume that at the beginning of this period there are k requests in the system (during the service of last request on the first level with probability $1-a_0-a_1$ at least two new ones have occured, with probability $\frac{a_k}{1-a_0-a_1}$ $(k = 2, 3, \dots)$ we will have k ones). In order to reach again the first level we have to complete $k-1$ present and all other requests occuring for their service time. Since the service of each present $k-1$ requests with the entering ones has the same structure as the entire busy period and for a busy period one serves on average $(1-\rho)^{-1}$ requests, the mean value of duration of this period is

$$\sum_{k=2}^{\infty} \frac{a_k}{1-a_0-a_1}(k-1)\frac{\tau}{1-\rho}$$

$$= \frac{\tau}{(1-\rho)(1-a_0-a_1)}[\rho - a_1 - (1-a_0-a_1)] = \frac{\rho - 1 + a_0}{(1-\rho)(1-a_0-a_1)}\tau,$$

where we have used the equalities

$$\rho = \sum_{k=1}^{\infty} k a_k \quad \text{and} \quad \sum_{k=0}^{\infty} a_k = 1.$$

For the busy period we have a certain number of periods during which there is only one request, this period ends when no request enters (this is the end of the busy period), or two or more requests appear. So with probabilities $\frac{a_0}{1-a_1}$, $\frac{1-a_0-a_1}{1-a_1}\,\frac{a_0}{1-a_1}, \dots, \frac{(1-a_0-a_1)^k}{(1-a_1)^k}\,\frac{a_0}{1-a_1}, \dots$ we will have $0, 1, \dots, k, \dots$ periods during which there are more than one request in the system. Consequently, the mean number of requests serviced for the periods of two types are

$$\sum_{k=1}^{\infty} k \frac{(1-a_0-a_1)^{k-1}}{(1-a_1)^{k-1}} \frac{a_0}{1-a_1} \frac{1}{1-a_1} = \frac{1}{a_0},$$

$$\sum_{k=1}^{\infty} k \frac{(1-a_0-a_1)^{k}}{(1-a_1)^{k}} \frac{a_0}{1-a_1} \frac{\rho-1+a_0}{(1-\rho)(1-a_0-a_1)} = \frac{\rho-1+a_0}{a_0(1-\rho)}.$$

We derive a recurrence relation concerning the mean value of time spent above the k-th level for a busy period. Let us consider the case of second level. We have two possibilities:

1. from the first level we come to the second level;
2. from the first level we come at least to the third one.

If from the first level we come to the second one, we are in the same situation as in the case of first level. We service a certain number of requests on the second level, after it we come either to the first one or to a level above the second. In the first case the sojourns on and above the second level change, and spending on average ζ_1 above it we return to the first one. In the second case the period above the second level begins with a jump from the first level to a level above the second, the mean value of time to return to the second one is equal to

$$\sum_{k=3}^{\infty} \frac{a_k}{1-a_0-a_1-a_2}(k-2)\frac{\tau}{1-\rho} = \frac{\rho-2+2a_0+a_1}{(1-\rho)(1-a_0-a_1-a_2)}\tau = \varepsilon_2.$$

After this moment we are in the previous situation, i.e. we spend above the second level on average ζ_1 time. The probabilities of two cases are correspondingly $\dfrac{a_2}{1-a_0-a_1}$ and $\dfrac{1-a_0-a_1-a_2}{1-a_0-a_1}$, so for a period beginning and ending on the first level we spend above the second level on average

$$\frac{a_2}{1-a_0-a_1}\zeta_1 + \frac{1-a_0-a_1-a_2}{1-a_0-a_1}(\zeta_1+\varepsilon_2) = \zeta_1 + \varepsilon_2'$$

time, where

$$\varepsilon_2' = \frac{\rho-2+2a_0+a_1}{(1-\rho)(1-a_0-a_1)}\tau.$$

We have i such periods with probability $\dfrac{(1-a_0-a_1)^i}{(1-a_1)^i}\,\dfrac{a_0}{1-a_1}$, so finally

$$\begin{aligned}\zeta_2 &= \sum_{i=1}^{\infty} i\frac{(1-a_0-a_1)^i}{(1-a_1)^i}\,\frac{a_0}{1-a_1}(\zeta_1+\varepsilon_2')\\ &= \frac{1-a_0-a_1}{a_0}\zeta_1 + \frac{1-a_0-a_1-a_2}{a_0}\varepsilon_2.\end{aligned}$$

Suppose that our reasoning is valid for the $k-1$-st level and we want to determine ζ_k. We consider again a period which begins and ends with the presence of one request in the system. From the first level we can come to the second, … , $k-1$-st, k-th and to a level above the k-th. In case of the second level we are in the same situation as considering the time spent above the $k-1$-st level from the viewpoint of first one, so the mean value is ζ_{k-1}. In case of third level at first we have a period starting with the presence of three requests and ending with the presence of two ones. The first period coincides with the case of the $k-2$-nd level above the first, the corresponding mean value is ζ_{k-2}. After this period we are in the previous situation (there are two requests) and the mean value of remaining part will be equal to ζ_{k-1}. So under condition that from the first level we come at once to the third one the desired mean value is $\zeta_{k-2}+\zeta_{k-1}$. Let us consider the last possibility. It takes place when from the first level we have a jump to a level above the k-th one. Computing the mean value of time spent above the k-th level at the beginning we obtain

$$\begin{aligned}&\sum_{i=k+1}^{\infty} \frac{a_i}{1-a_0-\cdots-a_k}(i-k)\frac{\tau}{1-\rho}\\ &= \frac{\rho-k+ka_0+(k-1)a_1+\cdots+2a_{k-2}+a_{k-1}}{(1-\rho)(1-a_0-a_1-\cdots-a_k)}\tau = \varepsilon_k.\end{aligned}$$

After this period we will be at the k-th level, and according to our reasoning spending ζ_1 time above the k-th level we come to the $k-1$-st, spending ζ_2 above the k-th we will be at the level $k-2$, … , and finally starting from the second level and spending ζ_{k-1} above the k-th one we arrive at the first, so in the last case the desired mean value is $\zeta_1+\zeta_2+\cdots+\zeta_{k-1}+\varepsilon_k$. The probability of first possibility is $\dfrac{a_2}{1-a_0-a_1}$, the probability of second possibility

$\dfrac{a_3}{1-a_0-a_1}, \dots$, the probability of last possibility $\dfrac{1-a_0-a_1-\dots-a_k}{1-a_0-a_1}$, so summing up the conditional mean values multiplied by the corresponding probabilities we obtain

$$\zeta_{k-1}+\frac{1-a_0-a_1-a_2}{1-a_0-a_1}\zeta_{k-2}+\dots+\frac{1-a_0-\dots-a_{k-1}}{1-a_0-a_1}\zeta_1 \\ +\frac{1-a_0-\dots-a_k}{1-a_0-a_1}\varepsilon_k.$$

For a busy period we have i periods of sojourn above the first level with probability $\dfrac{(1-a_0-a_1)^i}{(1-a_1)^i}\dfrac{a_0}{1-a_1}$, the desired mean value is equal to

$$\begin{aligned}\zeta_k &= \sum_{i=1}^{\infty} i\frac{(1-a_0-a_1)^i}{(1-a_1)^i}\frac{a_0}{1-a_1}\Bigg\{\zeta_{k-1}+\frac{1-a_0-a_1-a_2}{1-a_0-a_1}\zeta_{k-2}+\dots \\ &\quad +\frac{1-a_0-a_1-\dots-a_{k-1}}{1-a_0-a_1}\zeta_1+\frac{1-a_0-a_1-\dots-a_k}{1-a_0-a_1}\varepsilon_k\Bigg\} \\ &= \sum_{i=1}^{k-1}\frac{1-a_0-a_1-\dots-a_i}{a_0}\zeta_{k-i}+\frac{1-a_0-a_1-\dots-a_k}{a_0}\varepsilon_k.\end{aligned}$$

From this expression we derive a formula for the mean value of the number of requests serviced on the level k. Obviously,

$$\zeta_k = \zeta-\xi_0-\xi_1-\dots-\xi_k.$$

Consequently, the previous formula may be rewritten in the following form:

$$\begin{aligned}\zeta-\xi_0-\xi_1-\dots-\xi_k &= \frac{1-a_0-a_1}{a_0}(\zeta-\xi_0-\xi_1-\dots-\xi_{k-1}) \\ &+\frac{1-a_0-a_1-a_2}{a_0}(\zeta-\xi_0-\xi_1-\dots-\xi_{k-2}) \\ &+\dots+\frac{1-a_0-a_1-\dots-a_{k-2}}{a_0}(\zeta-\xi_0-\xi_1-\xi_2) \\ &+\frac{1-a_0-a_1-\dots-a_{k-1}}{a_0}(\zeta-\xi_0-\xi_1)+\frac{\rho-1+a_0}{a_0(1-\rho)}\tau\end{aligned}$$

$$- \frac{1-a_0-a_1}{a_0(1-\rho)}\tau - \frac{1-a_0-a_1-a_2}{a_0(1-\rho)}\tau - \cdots - \frac{1-a_0-a_1-\cdots-a_{k-2}}{a_0(1-\rho)}\tau$$
$$- \frac{1-a_0-a_1-\cdots-a_{k-1}}{a_0(1-\rho)}\tau.$$

Using the equalities

$$\frac{\rho-1+a_0}{a_0(1-\rho)}\tau = \zeta - \xi_0 - \xi_1, \qquad \zeta = \frac{\tau}{1-\rho}$$

we get

$$\begin{aligned} -\xi_2 - \cdots - \xi_k = {} & \frac{1-a_0-a_1}{a_0}(-\xi_0-\xi_1-\cdots-\xi_{k-1}) \\ & + \frac{1-a_0-a_1-a_2}{a_0}(-\xi_0-\xi_1-\cdots-\xi_{k-2}) + \dots \\ & + \frac{1-a_0-a_1-\cdots-a_{k-2}}{a_0}(-\xi_0-\xi_1-\xi_2) \\ & + \frac{1-a_0-a_1-\cdots-a_{k-1}}{a_0}(-\xi_0-\xi_1). \end{aligned}$$

Substituting here a similar expression for ξ_{k-1} we finally obtain

$$\xi_k = \sum_{i=1}^{k-2} \frac{1-a_0-a_1-\cdots-a_i}{a_0}\xi_{k-i} + \frac{1-a_0-a_1-\cdots-a_{k-1}}{a_0}(\xi_0+\xi_1)$$
$$(k \geq 2),$$

where $\sum\limits_{i=1}^{0} = 0$ and $\xi_0 + \xi_1 = \tau/a_0$. The theorem is proved.

Remark 1. At the derivation of recursive formula we take advantage of the mechanism of functioning of the system. It does not change even in case of bulk arrivals, so the same reasoning may be used, the concrete transition probabilities will differ of course.

Remark 2. Using the notation

$$\frac{1-a_0-a_1-\cdots-a_i}{a_0} = f_i,$$

all ξ_k $(k \geq 2)$ may be expressed via $\xi_0 + \xi_1$. We have

$$\begin{aligned}
\xi_0 + \xi_1 &= (\xi_0 + \xi_1) \cdot 1, \\
\xi_2 &= (\xi_0 + \xi_1) f_1, \\
\xi_3 &= (\xi_0 + \xi_1)(f_1^2 + f_2), \\
\xi_4 &= (\xi_0 + \xi_1)(f_1^3 + 2f_1 f_2 + f_3), \\
\xi_5 &= (\xi_0 + \xi_1)(f_1^4 + 3f_1^2 f_2 + 2f_1 f_3 + f_2^2 + f_4), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

One can easily show, by induction, that the coefficients at $\xi_0 + \xi_1$ in the expression for ξ_{k+1} give the different possibilities how k can be represented as the sum of natural numbers (it is the partition problem of number theory, the lower indices give the summands, the powers the multiplicities). The coefficients of summands are the polynomial coefficients, namely

$$\frac{(n_1 + n_2 + \cdots + n_j)!}{n_1! n_2! \dots n_j!},$$

where n_i are the powers of f_i.

Proof of the Lemma. We determine the desired mean value on the basis of the distribution function of a cycle and find it using the Laplace–Stieltjes transform. The cycle consists of the first busy period with Laplace–Stieltjes transform $\Gamma(s)$; it is followed by a certain number of periods including a vacation and a busy period (for the vacation at least one request obligatorily enters), the corresponding Laplace–Stieltjes transform is $d(s + \lambda - \lambda\Gamma(s)) - d(s + \lambda)$; and finally there will be the last vacation without entry of any request, its transform is $d(s + \lambda)$. So the Laplace–Stieltjes transform of the distribution function of a cycle is

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Gamma(s)[d(s + \lambda - \lambda\Gamma(s)) - d(s + \lambda)]^n d(s + \lambda) \\
&\qquad = \frac{\Gamma(s) d(s + \lambda)}{1 - d(s + \lambda - \lambda\Gamma(s)) + d(s + \lambda)},
\end{aligned}$$

from it on the usual way we obtain

$$\zeta' = \frac{\tau}{1 - \rho} + \frac{\eta}{d_0(1 - \rho)}. \qquad \square$$

5. Proof of Theorem 2

We will distinguish the free state, the vacation and the states $1, 2, \ldots$

To the free state corresponds the last vacation of the cycle, its mean value is η.

By the definition of states to the vacation will correspond the services of last requests in all the busy periods, the mean value is

$$\tau d_0 + 2\tau(1-d_0)d_0 + \cdots = \sum_{i=1}^{\infty} i\tau(1-d_0)^{i-1}d_0 = \frac{\tau}{d_0}.$$

As in the proof of Theorem 1 we will consider the time spent on and above the first level. For the first busy period according to our previous result the mean value of time spent on the first level is equal to $\dfrac{\tau}{a_0} - \tau$. For the other busy periods we have two possibilities. The busy period starts with one present request with probability $d_1/(1-d_0)$, in this case the mean value is equal to $\dfrac{\tau}{a_0} - \tau + \eta$ (the service time on the first level is τ/a_0, it is necessary to add the previous vacation and to subtract the service of last request, it already belongs to the state of vacation). In another case we have at least two requests with probability $\dfrac{1-d_0-d_1}{1-d_0}$, the mean value is $\dfrac{\tau}{a_0} - \tau + \tau = \dfrac{\tau}{a_0}$ (after having serviced a certain number of requests above the first level we come to it and we are in the previous situation, but the modification must be done both at the beginning and end with τ). They give

$$\frac{d_1}{1-d_0}\left(\frac{\tau}{a_0} + \eta - \tau\right) + \frac{1-d_0-d_1}{1-d_0}\frac{\tau}{a_0} = \frac{\tau}{a_0} + \frac{d_1}{1-d_0}(\eta - \tau).$$

So the mean value of time spent on the first level for a cycle

$$\sum_{i=0}^{\infty}\left\{\left[\frac{\tau}{a_0} - \tau\right] + i\left[\frac{\tau}{a_0} + \frac{d_1}{1-d_0}(\eta-\tau)\right]\right\}(1-d_0)^i d_0$$
$$= \frac{\tau}{a_0 d_0} - \tau + \frac{d_1}{1-d_0}(\eta - \tau) = \xi_1'.$$

Now we find the mean value of time spent above the first level for a cycle. By the Theorem 1 for the first busy period it is $\zeta_1 = \dfrac{\rho - 1 + a_0}{a_0(1-\rho)}\tau$.

For the other busy periods we have to distinguish two possibilities. The busy period starts with probability $\dfrac{d_1}{1-d_0}$ on the first level, the mean value is $\dfrac{\rho-1+a_0}{a_0(1-\rho)}\tau$; it starts with probability $\dfrac{1-d_0-d_1}{1-d_0}$ above the first level, on average for

$$\sum_{k=2}^{\infty}\frac{d_k}{1-d_0-d_1}(k-1)\frac{\tau}{1-\rho}=\frac{\rho_v-1+d_0}{(1-\rho)(1-d_0-d_1)}\tau$$

we come to the first level (by analogy we use the notation $\sum\limits_{k=1}^{\infty}kd_k=\rho_v$), and after it we are in the previous situation. So in this case the mean value, taking into account the necessary correction, is equal to

$$\frac{\rho_v-1+d_0}{(1-\rho)(1-d_0-d_1)}\tau+\eta-\tau+\frac{\rho-1+a_0}{a_0(1-\rho)}\tau,$$

and the mean value of time spent above the first level for a busy period (excluding the first one)

$$\frac{\rho-1+a_0}{a_0(1-\rho)}\tau+\frac{\rho_v-1+d_0}{(1-\rho)(1-d_0)}\tau+\frac{1-d_0-d_1}{1-d_0}(\eta-\tau).$$

For a cycle it will be

$$\begin{aligned}\zeta_1'&=\frac{\rho-1+a_0}{a_0(1-\rho)}\tau[d_0+2(1-d_0)d_0+3(1-d_0)^2d_0+\dots]\\&\quad+\left[\frac{\rho_v-1+d_0}{(1-\rho)(1-d_0)}\tau+\frac{1-d_0-d_1}{1-d_0}(\eta-\tau)\right]\\&\quad\times[(1-d_0)d_0+2(1-d_0)^2d_0+3(1-d_0)^3d_0+\dots]\\&=\frac{\rho-1+a_0}{a_0d_0(1-\rho)}\tau+\frac{\rho_v-1+d_0}{d_0(1-\rho)}\tau+\frac{1-d_0-d_1}{d_0}(\eta-\tau).\end{aligned}$$

We find a formula for the mean value of time spent above the second level for a cycle. For the first busy period it is equal to ζ_2. For the further busy periods we have three possibilities:

– for the previous vacation one new request appeared, the mean value is ζ_2;

– for the previous vacation two new requests appeared, the desired mean value is $\zeta_1+\zeta_2$ (spending on average ζ_1 time above the second level we come to the first one, after that spending ζ_2 above the second level the busy period is terminated);

– for the previous vacation at least three new requests appeared, on average for

$$\varepsilon_2=\sum_{k=3}^{\infty}\frac{d_k}{1-d_0-d_1-d_2}(k-2)\frac{\tau}{1-\rho}=\frac{\rho_v-2+2d_0+d_1}{(1-\rho)(1-d_0-d_1-d_2)}\tau$$

we come to the second level, and so we are in the previous situation. By modifying this period with η and τ, multiplying the three values with the corresponding probabilities, for the mean value of time spent above the second level we obtain

$$\begin{aligned}(*)\quad &\frac{d_1}{1-d_0}\zeta_2+\frac{d_2}{1-d_0}(\zeta_1+\zeta_2)\\ &\quad+\frac{1-d_0-d_1-d_2}{1-d_0}(\varepsilon_2+\eta-\tau+\zeta_1+\zeta_2)=\zeta_2+\frac{1-d_0-d_1}{1-d_0}\zeta_1\\ &\quad+\frac{1-d_0-d_1-d_2}{1-d_0}(\eta-\tau)+\frac{\rho_v-2+2d_0+d_1}{(1-\rho)(1-d_0)}\tau=\omega_2.\end{aligned}$$

For a cycle above the second level we stay on average

$$\begin{aligned}\zeta_2'&=\sum_{i=0}^{\infty}(\zeta_2+i\omega_2)(1-d_0)^i d_0=\zeta_2+\frac{1-d_0}{d_0}\zeta_2\\ &\quad+\frac{1-d_0-d_1}{d_0}\zeta_1+\frac{1-d_0-d_1-d_2}{d_0}(\eta-\tau)+\frac{\rho_v-2+2d_0+d_1}{d_0(1-\rho)}\tau.\end{aligned}$$

Let us now consider the mean value of time spent above the k-th level for a cycle. For the first busy period it is equal to ζ_k. For the other busy periods depending on the number of requests occuring for the previous vacation we have the following possibilities:

$$
\begin{aligned}
&\zeta_k\\
&\zeta_{k-1}+\zeta_k\\
&\cdots\cdots\cdots\\
&\zeta_{k-i+1}+\cdots+\zeta_k\\
&\cdots\cdots\cdots\cdots\cdots\\
&\zeta_1+\zeta_2+\cdots+\zeta_k\\
&\varepsilon_k+\eta-\tau+\zeta_1+\cdots+\zeta_k.
\end{aligned}
$$

Using this representation by means of the same reasoning as above one can get the expression

$$
\begin{aligned}
\zeta_k' = \sum_{i=0}^{\infty}(\zeta_k+i\omega_k)(1-d_0)^i d_0 &= \zeta_k+\frac{1-d_0}{d_0}\zeta_k\\
&+\frac{1-d_0-d_1}{d_0}\zeta_{k-1}+\cdots+\frac{1-d_0-\cdots-d_{i-1}}{d_0}\zeta_{k-i+1}+\cdots\\
&+\frac{1-d_0-\cdots-d_{k-1}}{d_0}\zeta_1+\frac{1-d_0-d_1-\cdots-d_k}{d_0}(\eta-\tau)\\
&+\frac{\rho-k+kd_0+(k-1)d_1+\cdots+2d_{k-2}+d_{k-1}}{d_0(1-\rho)}\tau,
\end{aligned}
$$

where ω_k is an analogous to $(*)$ expression. Having a similar expression for the level $k-1$ and taking their difference we finally obtain

$$
\begin{aligned}
\xi_k' = \zeta_{k-1}'-\zeta_k' &= \zeta_{k-1}-\zeta_k+\frac{1-d_0}{d_0}(\zeta_{k-1}-\zeta_k)\\
&+\frac{1-d_0-d_1}{d_0}(\zeta_{k-2}-\zeta_{k-1})\\
&+\cdots+\frac{1-d_0-\cdots-d_{i-1}}{d_0}(\zeta_{k-i}-\zeta_{k-i+1})+\cdots\\
&+\frac{1-d_0-\cdots-d_{k-2}}{d_0}(\zeta_1-\zeta_2)-\frac{1-d_0-d_1-\cdots-d_{k-1}}{d_0}\zeta_1
\end{aligned}
$$

$$+\left[\frac{1-d_0-\cdots-d_{k-1}}{d_0}-\frac{1-d_0-\cdots-d_k}{d_0}\right](\eta-\tau)$$

$$+\left[\frac{\rho_v-(k-1)+(k-1)d_0+(k-2)d_1+\cdots+2d_{k-3}+d_{k-2}}{d_0(1-\rho)}\right.$$

$$\left.-\frac{\rho_v-k+kd_0+(k-1)d_1+\cdots+2d_{k-2}+d_{k-1}}{d_0(1-\rho)}\right]\tau$$

$$=\xi_k+\frac{1-d_0}{d_0}\xi_k$$

$$+\frac{1-d_0-d_1}{d_0}\xi_{k-1}+\cdots+\frac{1-d_0-\cdots-d_{i-1}}{d_0}\xi_{k-i+1}$$

$$+\cdots+\frac{1-d_0-\cdots-d_{k-2}}{d_0}\xi_2$$

$$-\frac{1-d_0-d_1-\cdots-d_{k-1}}{d_0}\frac{\rho-1+a_0}{a_0(1-\rho)}+\frac{d_k}{d_0}(\eta-\tau)$$

$$+\frac{1-d_0-d_1-\cdots-d_{k-1}}{d_0(1-\rho)}\tau=\xi_k+\sum_{i=0}^{k-2}\frac{1-d_0-\cdots-d_i}{d_0}\xi_{k-i}$$

$$+\frac{1-d_0-d_1-\cdots-d_{k-1}}{d_0}(\xi_0+\xi_1)+\frac{d_k}{d_0}(\eta-\tau),$$

what proves the theorem. □

References

[1] B. V. Gnedenko i I. N. Kovalenko, Vvedenie v teoriyu massovogo obsluzhivaniya, *Nauka*, *Moskva*, 1966.

[2] L. Kleĭnrok, Teoriya massovogo obsluzhivaniya, *Mashinostroenie*, *Moskva*, 1979.

[3] I. N. Kovalenko, Nekotorye voprosy teorii nadezhnosti slozhnykh sistem, Kibernetiku – na sluzhbu kommunizmu, t.2., *Ènergiya*, *Moskva*, 1964, 194–205.

[4] I. N. Kovalenko, Nekotorye analiticheskie metody v teorii massovogo obsluzhivaniya, Kibernetiku – na sluzhbu kommunizmu, t.2., *Ènergiya*, *Moskva*, 1964, 325–337.

[5] L. Lakatosh i V. Cherich, O probleme kvantileĭ dlya sistemy M/G/1, sb. ¡O kooperiruemykh rabotakh NIVTs MGU i BUVTs¿, pod red. V.M. Repina i V.A. Morozova, *izd. MGU*, *Moskva*, 1990, 123–139.

[6] G. Brière and M. L. Chaudry, Computational analysis of single-server bulk-arrival queues $M^X/G/1$, *Comput. Oper. Res.* **15** (3) (1988), 283–292.

[7] G. Brière and M. L. Chaudry, Computational analysis of single-server bulk-service queues $M/G^Y/1$, *Adv. Appl. Probab.* **21** (1989), 207–225.

[8] M. L. Chaudry, U. C. Gupta and M. Agarwal, Exact and approximate numerical solutions to steady-state single-server queues: M/G/1 – a unified approach, *Queueing Systems* **10** (1992), 351–380.

[9] M. L. Chaudry and Hasham, Software package for the queueing system $M^X/G/1$, *Oper. Res. Letters* **6** (1987), 195–196.

[10] L. Lakatos and V. Čerić, Another approach to the ergodic distribution in the M/G/1 system, Probability theory and applications: Essays to the memory of József Mogyoródi (J. Galambos and I. Kátai, eds.), *Kluwer*, 1992, 289–298.

[11] L. Lakatos, Ergodic distribution for the M/G/1 system with vacation, Abst. 2nd Scandinavian–Ukrainian Conference in Math. Stat., *Umeå*, *Sweden*, June 8–13, 1997, 53.

[12] Y. Levy and U. Yechiali, Utilization of idle time in an M/G/1 queueing system, *Management Sci.* **22** (1975), 202–211.

[13] H. Takagi, Queueing analysis, A foundation of performance evaluation, vol. 1. Vacation and priority systems, *North-Holland*, 1991.

[14] J. Tomkó, Tartózkodási időproblémák Markov láncokra, *Alk. Mat. Lapok* **8** (1982), 91–108.

LÁSZLÓ LAKATOS
DEPARTMENT OF COMPUTER ALGEBRA
EÖTVÖS LORÁND UNIVERSITY
H–1518 BUDAPEST, P.O.B. 32
HUNGARY

E-mail: lakatos@compalg.inf.elte.hu

(Received November 27, 1997; revised October 22, 1998)