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Connections and regularity on the cotangent bundle

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Abstract. Considering some diffeomorphisms between the tangent and the cotangent bundles of a smooth manifold M obtained from regular geometric objets on TM and T^*M we establish some relations between mechanical structures on TM and T^*M. We construct a mechanical structure on T^*M and we also study the condition under wich a regular vector field defining a mechanical structure on T^*M is a semi-Hamilton or Hamilton vector field.

Introduction

There are some essential differences between the differential geometries of the tangent and cotangent bundles of a differentiable manifold M, although the bundles are isomorphic, as vector bundles.

The tangent bundle has a naturally defined integrable almost tangent structure J and one can define the notion of a semispray (second order differential equation vector field) on it. From a semispray one can derive easily a (nonlinear) connection on the tangent bundle. The cotangent bundle T^*M has a naturally defined symplectic structure ω but one cannot define naturally something similar to the notion of almost tangent structure or semispray.

In [1] it was introduced the notion of regularity and that of adapted almost tangent structure on the cotangent bundle T^*M, used for the study of some geometric properties of the tangent and cotangent bundles, considering some diffeomorphism between them. A regular vector field K on

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T^*M defines an integrable adapted almost tangent structure and a connection Γ_K on T^*M. On the other hand, this regular vector field defines naturally an M-bundle diffeomorphism Φ from T^*M to TM which transfers some results from the differential geometry of the cotangent bundle to that of the tangent bundle, and conversely. Using a regular vector field (which always exists) one defines the notion of mechanical structure on T^*M, which generalizes the notion of Hamiltonian system on $T^\star M$ and is correlated with the notion of mechanical system on TM. A regular vector field K on $T^\star M$ defining a mechanical structure is used to define another two symmetric connections Γ'_K and Γ''_K on $T^\star M$. In this paper we study the conditions for the existence of regular vector fields K' and K'' on $T^\star M$, with the same associated M-vector field (see [11]) like K, such that the associated connections to K' and K'', $\Gamma_{K'}$ and $\Gamma_{K''}$ are just Γ'_K and Γ''_K.

In [2], based on a similar idea, considering a regular 1-form on TM we have constructed a semispray which defines a connection on TM. The above regular 1-form defines naturally an M-bundle diffeomorphism Ψ from TM to T^*M, of the Legendre transformation type (see [4], [6]), which transfers some results from the differential geometry of the tangent bundle to that of the cotangent bundle, and conversely.

In [3] we established some relations between regular 1-forms and mechanical systems on TM. Considering the diffeomorphism Ψ from TM to $T^\star M$ determined by a regular 1-form defining a mechanical systems on TM, we construct in this paper a mechanical structure on $T^\star M$, and also, we study the conditions under which the regular vector field defining a mechanical structure on $T^\star M$ is a semi-Hamilton or Hamilton vector field.

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1. Preliminaries

We begin by describing the geometric structures of the tangent and cotangent bundles TM and T^*M of a smooth n-dimensional manifold M. First, we shall present, in short, some result included in [1], [2] and [3]. Denote by $\tau : TM \longrightarrow M$, $\pi : T^*M \longrightarrow M$ the natural projections and let (U, x^i), $i = 1, \dots, n$; be a local chart on M. Then the local chart $(\tau^{-1}(U); x^i, y^i)$ is induced on TM where $x^i = x^i \circ \tau$ (by abuse of notation) and y^i are the vector space coordinates with respect to the natural local frame $(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n})$ in TM defined by (U,x^i). Similarly the local chart $(\pi^{-1}(U);q^i,p_i)$ is induced on T^*M where $q^i=x^i\circ\pi$ and p_i are the vector space coordinates with respect to the natural local frame $(dx^1,\ldots,dx^n)$ in $T^\star M$ defined by (U,x^i).

Due to the special form of the change rules of the induced local charts on TM (on T^*M), it is possible to define the notion of an M-tensor field on these bundles (see [11]).

Definition. An M-tensor field of type (k,l) on TM (on $T^\star M$) is defined by sets of local coordinate components $T^{i_1\ldots i_k}_{j_i\ldots j_l}$, assigned to every induced local chart on TM (on $T^\star M$) such that the change rule, when a change of induced local charts on TM (on T^*M) is performed, is the same with the change rule of the local coordinate components of a tensor field of type (k,l) on M.

Remark that any M-tensor field on TM may be thought of as an ordinary tensor field T with the expression

$$T=T^{i_1\ldots i_k}_{j_1\ldots j_l}\,\frac{\partial}{\partial y^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial y^{i_k}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_l}.$$

However, there are many other possibilities to interpret an M-tensor field as an ordinary tensor field on TM. Remark also that any ordinary tensor field on the base manifold may be thought of as an M-tensor field on TM, having the same type and with the components in the induced local chart on TM, equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. The M-tensor field associated to a covariant tensor field on the base manifold M may be thought of as the pull back of this tensor field by the smooth submersion $\tau:TM\longrightarrow M$.

Consider *the natural integrable almost tangent structure on TM, defined by the tensor field J of type $(1,1)$ on TM having the local coordinate expression*

$$J=\frac{\partial}{\partial y^i}\otimes dx^i.$$

In the case of the cotangent bundle the notion of *adapted almost tangent structure is defined* ([1]) *by a tensor field J on $T^\star M$ of type $(1,1)$ such that*

$\operatorname{Ker} J = \operatorname{Im} J = VT^*M$, where $VT^*M = \operatorname{Ker}\pi_*$ is the integrable vertical distribution. The local coordinate expression of J must be

$$J = g_{ij}\frac{\partial}{\partial p_i} \otimes dq^j, \tag{1}$$

where the components $g_{ij}(q,p)$ define a nondegenerate M-tensor field of type $(0,2)$ on $T^\star M$. The integrability condition for the adapted almost tangent structure consists in the existence of a regular M-vector field $\xi^i(q,p)$ (an M-tensor field of type $(1,0)$) on $T^\star M$ such that $g^{ij} = \frac{\partial \xi^j}{\partial p_i}$, where $g^{ij}(q,p)$ are the entries of the inverse of the matrix $(g_{ij}(q,p))$, i.e. $g_{ij}g^{ik} = g_{ji}g^{ki} = \delta^k_j$, and the matrix $(g_{ij}(q,p))$ is nondegenerate.

The components $\xi^i(q,p)$ of an M-vector field on $T^\star M$ may be thought of as the last n components of an M-bundle (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$. So, the local coordinate expression of Φ is

$$x^i = q^i; \qquad y^i = \xi^i(q,p). \tag{2}$$

The inverse Φ^{-1} of Φ has the local coordinate expression

$$q^i = x^i; \qquad p_i = \zeta_i(x,y). \tag{3}$$

The components $\zeta_i(x,y)$ define an $M-1$-form on TM.
Let K be a vector field on $T^\star M$, with the local coordinate expression

$$K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}. \tag{4}$$

The components ξ^i define an M-vector field on $T^\star M$, called the associated M-vector field of K.

The vector field K is called regular if its associated M-vector field is regular, i.e. the matrix $(\frac{\partial \xi^j}{\partial p_i})$ is nondegenerate.
From a regular vector field K (from its associated M-vector field) one can derive an integrable adapted almost tangent structure J_ξ, with the local coordinate expression (1), and a nonlinear connection Γ_K, (thought of as an almost product structure on T^*M such that the eigendistribution corresponding to the eigenvalue -1 is the vertical distribution) given by

$$\Gamma_K = -\mathcal{L}_K J_\xi \tag{5}$$

Consider the canonical 1-form (the Liouville form) $\theta = p_i dq^i$ on $T^\star M$ and let $\omega = d\theta$ be the canonical symplectic structure on $T^\star M$.

Recall that *a vector field* $H = \xi^i \frac{\partial}{\partial q^i} + \chi_i \frac{\partial}{\partial p_i}$ *on* $T^\star M$ *is called a Hamilton vector field if it is regular and* $\mathcal{L}_H \omega = 0$. This condition is expressed locally by the relations

$$(6) \qquad (i)\ \frac{\partial \xi^i}{\partial p_j} = \frac{\partial \xi^j}{\partial p_i}; \quad (ii)\ \frac{\partial \chi_i}{\partial p_j} = -\frac{\partial \xi^j}{\partial q^i}; \quad (iii)\ \frac{\partial \chi_i}{\partial q^j} = \frac{\partial \chi_j}{\partial q^i}.$$

Definition. A vector field K on $T^\star M$ defines a mechanical structure on $T^\star M$ if it is regular and the matrix $(g^{ij}) = (\frac{\partial \xi^j}{\partial p_i})$ defined by its associated M-tensor field is symmetric (see [1], [4]).

In this case only the condition (6) (i) is fulfilled. There exists a smooth real function $h : T^* M \longrightarrow \mathbb{R}$ such that $\xi^i = \frac{\partial h}{\partial p_i}$ but, generally, $\chi_i \neq -\frac{\partial h}{\partial q^i}$. Consider the form $\eta = i_K \omega - (-dh)$. So $i_K \omega = -dh + \eta$. Therefore $\chi_i = -\frac{\partial h}{\partial q^i} + \eta_i$.

Then $\eta = \eta_i dq^i$ is a *semibasic 1-form* on $T^\star M$. ($\eta(V) = 0$ whenever V is a vertical vector field on $T^* M$.)

Definition. The vector field K on $T^* M$ is a semi-Hamilton vector field if it is regular and $\mathcal{L}_K \omega$ is a semibasic 2-form on $T^\star M$, i.e. $i_V(\mathcal{L}_K \omega) = 0$, whenever V is a vertical vector field on $T^\star M$.

This condition may be expressed locally by (6) (i) and (6) (ii). In the case of a vector field K defining a mechanical structure on $T^\star M$ one defines in [1] another two connections on $T^\star M$. First, one defines the symmetric tensor field g of type $(0,2)$ on $T^\star M$, by

$$g(Z_1, Z_2) = -\omega(Z_1, J_\xi Z_2); \quad Z_1, Z_2 \in \chi(T^* M).$$

Its local coordinate expression is $g = g_{ij} dq^i \otimes dq^j$. Then the nonlinear connection Γ'_K on $T^\star M$ is defined by the condition

$$\omega(Z_1, \Gamma'_K Z_2) = (\mathcal{L}_K g)(Z_1, Z_2); \quad Z_1, Z_2 \in \chi(TM).$$

Another connection Γ''_K on $T^\star M$ defined by a vector field K defining a mechanical structure is obtained in a similar way from the tensor field $\widetilde{g} = g_{ij} \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}$. Both connections Γ'_K and Γ''_K are symmetric.

Recall that *a semispray (a second order differential vector field) on the tangent bundle* TM (see [4], [6], [7], [10]) *is a vector field* S *on* TM

such that $JS = C$*, where* C *is the Liouville vector field on* TM *(the local coordinate expression is* $C = y^i \frac{\partial}{\partial y^i}$*).*

The local coordinate expression of a semispray is $S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x,y)\frac{\partial}{\partial y^i}$. There is a connection Γ (thought of as an almost product structure) on TM (see [4], [6], [7], [10]) derived from the semispray S by

$$\Gamma = -\mathcal{L}_S J. \tag{7}$$

The connection Γ is called *the canonical connection associated to the semispray* S.

In [1] it is proved the following result: *Let be* K *a regular vector field on* $T^\star M$ *and assume that its associated* M*-vector field defines a global* M*-bundle diffeomorphism* $\Phi : T^*M \longrightarrow TM$*. Then the vector field* $\Phi_\star K$ *is a semispray on* TM *whose derived connection* Γ *is just the image by* Φ *of the connection* Γ_K *defined by* (5).

In [2] it is considered a regular 1-form on TM from which one derives, by using the integrable almost tangent structure and the Liouville vector field on TM, a semispray on TM. This semispray has its associated connection. Then, the above regular 1-form defines naturally an M-bundle (local) diffeomorphism Ψ from TM to $T^\star M$ by which we can establish some correspondences between the differential geometry of these bundles.

Let α be a 1-form on TM with local coordinate expression

$$\alpha = \alpha_j(x,y)dy^j + \beta_j(x,y)dx^j. \tag{8}$$

The components α_j define an $M-1$-form on TM, called *the associated* $M-1$*-form of* α*.* The form α is called *regular* if the matrix with the entries $(g_{ij} = \frac{\partial \alpha_i}{\partial y^j})$ is nondegenerate. Assuming that the 1-form α is regular, we consider the 1-form θ on TM defined by

$$\theta = \alpha \circ J = i_J \alpha, \tag{9}$$

where i_J is the vertical derivation on TM. The 1-form θ corresponds to the Cartan 1-form defined by a regular Lagrangian on TM (see [7]) and has the local coordinate expression

$$\theta = \alpha_i(x,y)dx^i.$$

The 2-form $d\theta = di_J\alpha$ is a symplectic form. Consider the 1-form β on TM defined by

$$\beta = di_C\alpha - \alpha, \tag{10}$$

and define the vector field S on TM by

$$i_S d\theta = -\beta. \tag{11}$$

We find that S is a semispray on TM. Its associated connection is given by (7). Just like in the case of a regular M-vector field, the components $\alpha_i(x,y)$ of α may be thought as the last n components of an M-bundle map $\Psi : TM \longrightarrow T^\star M$. This is the (local) diffeomorphism mentioned above. The local coordinate expression of Ψ is

$$q^i = x^i; \qquad p_i = \alpha_i(x,y), \tag{12}$$

where $(\pi^{-1}(U); q^i, p_i)$ is the induced local chart on $T^\star M$. The inverse Ψ^{-1} of Ψ has the local coordinate expression

$$x^i = q^i; \qquad y^i = u^i(q,p). \tag{13}$$

The central result of [2] is the following:

If α is a regular 1-form on TM, there is a semispray S on TM derived from it by (9), (10), (11). *Assuming that its associated $M-1$-form defines a global M-bundle diffeomorphism $\Psi : TM \longrightarrow T^\star M$, the vector field $\Psi_\star S$ is a regular vector field on $T^\star M$ whose derived connection Γ_K is the image by Ψ of the connection Γ associated to the semispray S.*

2. Regular 1-forms and mechanical structures on TM and T^*M

In [3], some relations between regular 1-forms and regular mechanical systems on TM were studied. In this section, considering the diffeomorphism $\Psi : TM \longrightarrow T^*M$ defined by a regular 1- form α, defining a mechanical system on TM, we shall construct a mechanical structure on $T^\star M$. Then, we shall study the condition under which the regular vector field defining a mechanical structure on $T^\star M$ is a semi-Hamilton vector field.

Recall that a *mechanical system on* TM *(see* [4], [9]*) is a triple* $m = (M, F, \rho)$*, where* $F : TM \longrightarrow \mathbb{R}$ *is a smooth real valued function and* ρ *is a semibasic 1-form on* TM *(i.e.* $\rho(V) = 0$*, for any vertical vector field* V *on* TM*).*

The *vertical differentiation* d_J on TM (see [4], [7]) is a derivation of $\Lambda(M)$ of degree one, defined by

$$d_J = [i_j, d] = i_J d - d i_J.$$

Consider the form $\omega_F = dd_J F$. If ω_F is a symplectic form (an equivalent condition is for the Hessian $(\frac{\partial^2 F}{\partial y^i \partial y^j})$ of F to be nondegenerate) the mechanical system m is said to be *regular*. In this case, the vector field S defined by

$$i_S \omega_F = -dE_F + \rho, \tag{14}$$

where $E_F = CF - F$ is *the energy function associated to* F, is a semispray (see [4], [7], [8], [9]).

m is said to be *conservative* if ρ is a closed form. In this case the mechanical system is Lagrangian. In [3] it is considered a regular 1-form $\alpha = \alpha_i(x, y)dy^j + \beta_i(x, y)dx^j$ such that its associated $M-1$-form defined by α_i satisfies the conditions

$$\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}; \quad i, j = 1, \ldots, n. \text{ (i.e. } g_{ij} = g_{ij}\text{)}. \tag{15}$$

Then (15) may be expressed by

$$\alpha_i = \frac{\partial F}{\partial y^i},$$

where F is a smooth real valued function on TM.

Considering $\rho_i = \beta_i - \frac{\partial F}{\partial x^i}$, the form $\rho = \rho_i dx^i$ is a semibasic 1-form and $\alpha = dF + \rho$. The triple $m = (M, F, \rho)$ is proved to be a regular mechanical system and conversely, if $m = (M, F, \rho)$ is a regular mechanical system, then $\alpha = dF + \rho$ is a regular 1-form and its associated $M-1$-form satisfies (15).

Theorem 1. *If the differential form α on TM is regular and $\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}$; $i = 1, \dots, n$, then the (local) diffeomorphism $\Psi : TM \longrightarrow T^*M$ given by* (12) *defines a mechanical structure $K = \Psi_* S$ on T^*M.*

Proof. By a straightforward computation we get

$$K = \Psi_* S = u^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i};$$

From the condition for Ψ^{-1} to be the inverse of Ψ, we get $\frac{\partial u^j}{\partial p_i} \circ \Psi = g^{ij}$. It follows

$$\frac{\partial u^j}{\partial p_i} \circ \Psi = g^{ij} = g^{ji} = \frac{\partial u^i}{\partial p_j} \circ \Psi;$$

Therefore K defines a mechanical structure on $T^\star M$. □

Proposition 2. *If the differential form α on TM is regular and $\frac{\partial \alpha_i}{\partial y^j} = = \frac{\partial \alpha_j}{\partial y^i}$, $i, j = 1, \dots, n$, then $\Psi_\star S$ is a semi-Hamilton vector field on $T^\star M$ if and only if the form ρ is a basic 1-form on TM.*

In other words, η is a basic 1-form on $T^\star M$ if and only if ρ is a basic 1-form on TM.

Proof. From the condition for Ψ^{-1} to be the inverse of Ψ, we also get

$$\frac{\partial u^j}{\partial q^k} \circ \Psi = -g^{ij}\frac{\partial \alpha_j}{\partial x^k}.$$

$\Psi_\star S$ is a semi-Hamiltonian vector field if and only if the condition (6) (ii) is fulfilled. In this case $\xi^i = u^i$ and $\chi_i = \beta_i \circ \Psi^{-1}$. Then the condition (6) (ii) becomes

$$\frac{\partial u^j}{\partial q^i} + \frac{\partial \beta_j}{\partial y^k}\frac{\partial u^k}{\partial p_j} = 0 \quad \Leftrightarrow \quad -g^{kj}\frac{\partial \alpha_k}{\partial x^i} + \frac{\partial \beta i}{\partial y^k} g^{jk} = 0 \quad \Leftrightarrow$$

$$g^{jk}\left(\frac{\partial \alpha_k}{\partial x^i} - \frac{\partial \beta_i}{\partial y^k}\right) = 0 \quad \Leftrightarrow \quad \frac{\partial \alpha_k}{\partial x^i} = \frac{\partial \beta_i}{\partial y^k} \quad \Leftrightarrow \quad \frac{\partial}{\partial y^k}(\frac{\partial F}{\partial x^i} - \beta_i) = 0$$

$$\Leftrightarrow \quad \frac{\partial \rho_i}{\partial y^k} = 0.$$

The last conditions means that the form ρ is a basic 1-form on TM.

Proposition 3. *Under the hypothese of Proposition 1, the mechanical system defined by α on TM is conservative (Lagrangian) if and only if the mechanical system defined by $K = \Psi_\star S$ is conservative (Hamiltonian).*

In other words, ρ is a closed form on TM if and only if η is a closed form on $T^\star M$.

Proof. The mechanical system defined by $K = \Psi_\star S$ is conservative if and only if $d\eta = 0$, so $\mathcal{L}_K \omega = 0$. Taking into account the previous result, we have that (6) (iii) is equivalent with $\frac{\partial \beta_i}{\partial x^j} = \frac{\partial \beta j}{\partial x^i}$. But $\beta_i = \frac{\partial F}{\partial x^i} + \rho_i$. Therefore $\frac{\partial \beta_i}{\partial x^j} = \frac{\partial \beta j}{\partial x^i}$ if and only if $\frac{\partial \rho_i}{\partial x^j} = \frac{\partial \rho_j}{\partial x^i}$ and, since ρ is already a basic form this means that ρ is a closed form on TM. □

3. Conditions for existence of regular geometric objects on T^*M

Consider now another problem related to the above correspondences between the tangent and the cotangent bundles. Assume that K is a regular vector field on $T^\star M$ and let $\beta_i(x, y)$; $i = 1, \beta, n$; be n-smooth real valued function on the domain $\tau^{-1}(U)$ of an induced local chart on TM. We ask for the local existence of a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ (of a regular $M - 1$-form α_i) such that $\Phi_* K$ is just the semispray derived from α.

Theorem 4. *Consider a regular vector field K on $T^\star M$ with the local coordinate expression* (4) *and let $\beta_i(x, y)$; $i = 1, \dots, n$ be n-smooth real valued functions on the domain $\tau^{-1}(U)$ of an induced local chart on TM. If the ordinary differential system on $\tau^{-1}(U)$*

$$
\begin{aligned}
&\frac{dx^1}{y^1} = \cdots = \frac{dx^n}{y^n} \\
&\quad = \frac{dy^1}{\frac{\partial \xi^1}{\partial q^k}(x, \zeta(x, y))y^k + \frac{\partial \xi^1}{\partial p_k}(x, \zeta(x, y))\chi_k(x, \zeta(x, y))} = \dots \\
&\dots = \frac{dy^n}{\frac{\partial \xi^n}{\partial q^k}(x, \zeta(x, y))y^k + \frac{\partial \xi^n}{\partial p_k}(x, \zeta(x, y))\chi_k(x, \zeta(x, y))} = \dots \\
&\dots = \frac{d\alpha_1}{\beta_1} = \cdots = \frac{d\alpha_n}{\beta n}
\end{aligned}
\tag{16}
$$

has n first integrals $C^1, \dots, C^n(x, y, \alpha)$ *such that*

$$\frac{D(C_1, \dots, C_n)}{D(y^1, \dots, y^n)} \neq 0; \quad \frac{D(C_1, \dots, C_n)}{D(\alpha_1, \dots, \alpha_n)} \neq 0; \tag{17}$$

then there exists a regular 1-form $\alpha = \alpha_j dy^j + \beta_j dx^j$ *(so, with the last* n *components* β_i*) such that the semispray* $\Phi_* K$ *derives from* α*.*

Proof. Consider the (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$ and its inverse $\Phi^{-1} : TM \longrightarrow T^*M$ having the local coordinate expression (2) and (3), respectively. The semispray S derived from α has the local coordinate expression

$$S = y^i \frac{\partial}{\partial x^i} + g^{ki}(\beta_k - \frac{\partial \alpha_k}{\partial x^l} y^l) \frac{\partial}{\partial y^i}.$$

The relations $\Phi_* K = S$ are equivalent with

$$g^{ji}(\beta_j - \frac{\partial \alpha_j}{\partial x^k} y^k) = y^j (\frac{\partial \xi^i}{\partial q^j} \circ \Phi^{-1}) + (\chi_j g^{ji}) \circ \Phi^{-1}.$$

From these relations transvected by $g_{ki} = \frac{\partial \alpha_k}{\partial y^i}$ we get the quasilinear partial differential system

$$\begin{aligned} \frac{\partial \alpha_i}{\partial x^j} y^j + \frac{\partial \alpha_i}{\partial y^j} \Big[y^k \frac{\partial \xi^j}{\partial q^k}(x, \zeta(x, y)) \\ + \frac{\partial \xi^j}{\partial p_k}(x, \zeta(x, y)) \chi_k(x, \zeta(x, y)) \Big] = \beta_i \end{aligned} \tag{18}$$

whose characteristic system is just (16). The first integrals $C_1, \dots, C_n$ assure, using the implicit function theorem, the obtaining of the components $\alpha_i(x, y)$. □

Remark. Under the hypothesis of the existence of this regular $M-1$-form α_i (of a regular form α) let us consider the (local) diffeomorphism Ψ: $TM \longrightarrow T^*M$, defined by the $M-1$-form α_i. Then $\Psi = \Phi^{-1}$ if and only if

$$\alpha_i(x, y) = \zeta_i(x, y), \quad i = 1, \dots, n.$$

Using the relations (18) we obtain the following result:

Proposition 5. *Let $K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}$ be a regular vector field on T^*M and let $\beta_i(x,y)$; $i = 1,\dots,n$; be n smooth real valued functions on the domain $\tau^{-1}(U)$ of an induced local chart on TM. Consider the (local) diffeomorphism $\Phi : T^*M \longrightarrow TM$ defined by K. There exists a regular 1-form $\alpha = \alpha_j dy^j + +\beta_j dx^j$ (so, with the last n components β_i) such that the semispray $\Phi_* K$ is derived from α and Ψ is just Φ^{-1} (in this case $\Psi_* S = K$) if and only if the components $\zeta_i(x,y)$ satisfy the conditions:*

$$\frac{\partial \zeta_i}{\partial x^j} y^j + \frac{\partial \zeta_i}{\partial y^j}\left[y^k \frac{\partial \xi^j}{\partial q^k}(x,\zeta(x,y)) + \frac{\partial \xi^j}{\partial p_k}(x,\zeta(x,y))\chi_k(x,\zeta(x,y))\right] = \beta_i. \tag{19}$$

Let us consider a regular vector field $K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}$ defining a mechanical structure on $T^\star M$. From K one can derive the connection Γ_K (its associated connection) and, also, the connections Γ'_K and Γ''_K, as above. We ask for the existence of a regular vector field K' on $T^\star M$, with the same associated M-vector field (ξ^i) such that $\Gamma'_K = \Gamma_{K'}$, that is Γ'_K is just the connection associated to K'. Then, we shall consider the same problem for Γ''_K. For the study of these problems we shall use the following result of [1]:

If the components $\xi^i(q,p)$; $i = 1,\dots,n$ define a regular vector field on T^*M and Γ^* is a connection on T^*M then there exists a regular vector field K on $T^\star M$ with its associated M-vector field ξ^i and such that $\Gamma^* = \Gamma_K$ if and only if the following integrability conditions are fulfilled

$$\frac{\delta g_{kj}}{\delta q^i} - \frac{\delta g_{ki}}{\delta q^j} = g_{hi}\frac{\partial \Gamma_{kj}}{\partial p_h} - g_{hj}\frac{\partial \Gamma_{ki}}{\partial p_h}, \tag{20}$$

where $\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \Gamma_{ji}\frac{\partial}{\partial p_j}$, $i = 1,\dots,n$ are local vector field defining a local frame in the horizontal distribution $HT^\star M$, defined by Γ^*.

Theorem 6. *Let $K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}$ be a regular vector field defining a mechanical structure on T^*M. There exists a regular vector field K' with the same associated M-vector field (ξ^i) such that $\Gamma'_K = \Gamma_{K'}$ if and only if the following integrability condition are fulfilled:*

$$\left(\frac{\partial g_{ki}}{\partial p_h} g_{lj} - \frac{\partial g_{kj}}{\partial p_h} g_{li}\right)\left(\frac{\partial \xi^l}{\partial q^h} + \frac{\partial \chi_h}{\partial p_l}\right) = 0. \tag{21}$$

PROOF. The coefficients Γ'_{ij} of the connection Γ'_K are given by

$$\Gamma'_{ij} = \frac{1}{2}\left\{ K g_{ij} + g_{ik}\frac{\partial \xi^k}{\partial q^j} + g_{kj}\frac{\partial \xi^k}{\partial q^i} \right\}.$$

The result is then proved by a straightforward computation by using the relations (20). □

Remark. If K is a semi-Hamilton vector field, the expression in the second parenthesis vanishes. Therefore, the relations (21) are fulfilled in this case.

Theorem 7. *Let $K(q,p) = \xi^i(q,p)\frac{\partial}{\partial q^i} + \chi_i(q,p)\frac{\partial}{\partial p_i}$ be a regular vector field defining a mechanical structure on T^*M. There exists a regular vector field K'' with the same associated M-vector field (ξ^i), such that $\Gamma''_K = \Gamma_{K''}$ if and only if the following integrability condition are fulfilled:*

$$\frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ki}}{\partial q^j} = g_{kl}\left(g_{hj}\frac{\partial^2 \chi_i}{\partial p_h \partial p_l} - g_{hi}\frac{\partial^2 \chi_j}{\partial p_h \partial p_l} \right). \tag{22}$$

PROOF. One uses the relations (20) where the coefficients Γ''_{ij} of the connection Γ''_K are given by

$$\Gamma''_{ij} = \frac{1}{2}\left\{ K g_{ij} - g_{ik}\frac{\partial \chi_j}{\partial p_k} - g_{kj}\frac{\partial \chi_i}{\partial p_k} \right\}.$$

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