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Nontransitive quasi-uniformities

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Abstract. We show that a topological space admits only transitive quasi-uniformities if and only if it admits a unique quasi-uniformity. Furthermore we exhibit an example of a topological space that admits exactly two quasi-proximities. Finally we show that if X is a topological space of net(work) weight $\mathrm{nw}(X)$, then any compatible quasi-uniformity of X has a base of cardinality $\leq 2^{\mathrm{nw}(X)}$ and X has $\leq 2^{2^{\mathrm{nw}(X)}}$ compatible quasi-uniformities.

1. Introduction

In the interesting article [4] Losonczi obtained results on the number of (nontransitive) quasi-uniformities that various kinds of topological spaces admit. In this note we answer two of his questions by showing that a topological space admits only transitive quasi-uniformities if and only if it admits a unique quasi-uniformity; furthermore we prove that there exists a topological space that admits exactly two quasi-proximities (equivalently, totally bounded quasi-uniformities, see e.g. [2]).

Finally we point out that a modification of a technique due to Losonczi [4] yields the following results: Let X be a topological space of network weight $\mathrm{nw}(X)$. Then any compatible quasi-uniformity on X has a base of cardinality $\leq 2^{\mathrm{nw}(X)}$ and the number of compatible quasi-uniformities on X is $\leq 2^{2^{\mathrm{nw}(X)}}$.

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We recall that a quasi-uniformity on a set X is called transitive if it possesses a base consisting of transitive entourages and it is said to be totally bounded provided that for each entourage V the cover $\{(V \cap V^{-1})(x) : x \in X\}$ has a finite subcover.

We shall need the result of [3] that a topological space admits a unique quasi-uniformity if and only if it is hereditarily compact and has no strictly decreasing sequence $(H_n)_{n\in\omega}$ of open sets such that $\bigcap_{n\in\omega} H_n$ is open.

For basic results on topological and quasi-uniform spaces we refer the reader to [1], [2]. In particular let us mention (compare [6]) that each topological space X admits the transitive and totally bounded Pervin quasi-uniformity generated by the entourages $[G\times G]\cup[(X\setminus G)\times X]$ where G is an open subset of X.

2. Construction of nontransitive quasi-uniformities

Let us establish our main result first. It solves Problem 3 of Losonczi [4].

Proposition 1. *A topological space admits only transitive quasi-uniformities if and only if it admits a unique quasi-uniformity.*

Proof. As we have mentioned above, each topological space admits the transitive Pervin quasi-uniformity. To prove the remaining statement suppose that the topological space X admits more than one quasi-uniformity. We shall show that X admits a nontransitive quasi-uniformity.

(1) Assume first that X is not hereditarily compact. By Zorn's Lemma there is a maximal open noncompact subset H of X. By noncompactness of H there is a strictly increasing sequence $(E_\alpha)_{\alpha<\lambda}$ of open sets in X such that $H = \bigcup_{\alpha<\lambda} E_\alpha$ where λ is a limit ordinal (see e.g. [7, Theorem 2.2 and Remark 2.4]). For each $x \in H$ determine the well-defined smallest $\alpha_x < \lambda$ such that $x \in E_{\alpha_x} \setminus \bigcup_{\beta<\alpha_x} E_\beta$ and write $\alpha_x = \beta_x + n_x$ where β_x is the largest limit ordinal $\le \alpha_x$ and $n_x \in \omega$. If no such β_x exists, put $\beta_x = 0$.

Consider the filter $\mathcal{V}$ on $X\times X$ that is generated by $\{[G\times G]\cup[(X\setminus G)\times X] : G$ is open in X and ($G \subseteq E_\alpha$ for some $\alpha<\lambda$ or $G\setminus H \neq \emptyset$)$\} \cup \{V_\epsilon \cup [(X\setminus H)\times X] : \epsilon > 0\}$. Here $V_\epsilon = \{(x,y) \in H\times H : \alpha_y \le \alpha_x$ or ($\beta_x = \beta_y$, $n_y > n_x$ and $\sum_{k=n_x}^{n_y-1} \frac{1}{k+1} < \epsilon$)$\}$.

Note that $\{[G\times G]\cup[(X\setminus G)\times X] : G$ is open in X and ($G\subseteq E_\alpha$ for some $\alpha<\lambda$ or $G\setminus H\neq\emptyset$)$\}$ generates a compatible quasi-uniformity on X. Furthermore $\{V_\epsilon : \epsilon > 0\}$ generates a quasi-uniformity on H whose topology is coarser than the topology of H. Observe that $\{V_\epsilon \cup [(X \setminus H) \times X] : \epsilon > 0\}$ generates a quasi-uniformity on X whose topology is coarser than the topology of X (see [2, Proposition 2.19]). Hence $\mathcal{V}$ is a compatible quasi-uniformity on X.

Let us show that $\mathcal{V}$ is not transitive: Otherwise there are $P = \bigcap_{i=0}^{n}([G_i \times G_i] \cup [(X \setminus G_i) \times X])$, $\rho > 0$ and a transitive $\mathcal{V}$-entourage T of X such that $(P \cap (V_\rho \cup (X \setminus H) \times X)) \subseteq T \subseteq V_1 \cup (X \setminus H) \times X$, where $n \in \omega$ and for each $i \in \{0, \ldots, n\}$, G_i is open in X and either $G_i \subseteq E_\alpha$ for some $\alpha < \lambda$ or $G_i \setminus H \neq \emptyset$.

Note that if G is open in X and $G \setminus H \neq \emptyset$, then by compactness of $G \cup H$ there is $\beta_0 < \lambda$ such that $H \setminus E_{\beta_0} \subseteq G$.

Hence we can find $\delta_0 = \epsilon_0 + s_0 < \lambda$ where ϵ_0 is the largest limit $\leq \delta_0$ (if necessary put $\epsilon_0 = 0$) and $s_0 \in \omega$ such that for each $i \in \{0, \ldots, n\}$ either $G_i \subseteq E_{\delta_0}$ or $H \setminus E_{\delta_0} \subseteq G_i$. Thus $x \in H \setminus E_{\delta_0}$ implies that $H \setminus E_{\delta_0} \subseteq P(x)$. Let $n \in \omega$ be such that $\frac{1}{n+s_0+2} < \rho$. Then choose $m \in \omega$ such that $m - n > 1$ and $\sum_{k=n+s_0+1}^{m+s_0-1} \frac{1}{k+1} \geq 1$. Furthermore find $x_i \in X$ such that $\alpha_{x_i} = \delta_0 + n + i$ where $m - n \geq i \geq 1$. Note that $(x_i, x_{i+1}) \in P \cap V_\rho$ for any $i \in \{1, \ldots, m - n - 1\}$. Then for $x = x_1$ and $y = x_{m-n}$ we have $(x, y) \in T$, but $(x, y) \notin V_1$ – a contradiction. We have shown that $\mathcal{V}$ is not transitive.

(2) Suppose now that X is hereditarily compact. By our initial assumption and the characterization of topological spaces admitting a unique quasi-uniformity mentioned in the introduction there is an open set H of X and a strictly decreasing sequence $(H_n)_{n\in\omega}$ of open sets in X such that $H = \bigcap_{n\in\omega} H_n$. By hereditary compactness we can assume that H is maximal with these properties, because any strictly increasing sequence of open sets of X is finite. Furthermore without loss of generality we can suppose that $H_0 = X$. For each $x \in X \setminus H$, there is a well-defined $n_x \in \omega$ such that $x \in H_{n_x} \setminus H_{n_x+1}$.

Let $\mathcal{U}$ be the quasi-uniformity generated by the union of the Pervin quasi-uniformity of X and of $\{V_\epsilon \cup [X \times H] : \epsilon > 0\}$. Here $V_\epsilon = \{(x, y) \in (X \setminus H) \times (X \setminus H) : n_x \leq n_y \text{ or } (n_y < n_x \text{ and } \sum_{k=n_y}^{n_x-1} \frac{1}{k+1} < \epsilon)\}$.

We observe that $\{V_\epsilon : \epsilon > 0\}$ generates a quasi-uniformity on $X \setminus H$ whose topology is coarser than the topology of $X \setminus H$. Then $\{V_\epsilon \cup [X \times H] :$

$\epsilon > 0\}$ generates a quasi-uniformity on X whose topology is coarser than the topology of X (see [2, Proposition 2.19]). Thus $\mathcal{U}$ is a compatible quasi-uniformity on X.

Consider an open set G in X such that $G \setminus H \neq \emptyset$. Then $H \subset G \cup H$. Since $\bigcup_{n\in\omega}[(X \setminus (G \cup H_n)] = X \setminus (G \cup H)$, by the maximality of H there exists $n \in \omega$ such that $G \cup H_n = G \cup H$ and thus $(H_n \setminus H) \subseteq G$. It follows that $(H_n \setminus H) \cap (X \setminus G) = \emptyset$.

We want to show that $\mathcal{U}$ is not transitive. Assume the converse and suppose that there are $p \in \omega$, $P = \bigcap_{i=0}^{p}([G_i \times G_i] \cup [(X \setminus G_i) \times X])$ where G_i are open sets in X whenever $i \in \{0, \dots, p\}$, $\delta > 0$ and a transitive $\mathcal{U}$-entourage T of X such that $(P \cap (V_\delta \cup (X \times H))) \subseteq T \subseteq V_1 \cup (X \times H)$.

By the argument just given there is $n_0 \in \omega$ such that for all $i \in \{0, \dots, p\}$, $(H_{n_0} \setminus H) \cap (X \setminus G_i) = \emptyset$ or $G_i \subseteq H$. We conclude that if $x \in H_{n_0} \setminus H$, then $H_{n_0} \setminus H \subseteq P^{-1}(x)$. Thus we find $n \in \omega$ such that $n \geq n_0$ and $\frac{1}{n+1} < \delta$. Furthermore we choose $m \in \omega$ such that $m > n$ and $\sum_{k=n}^{m-1} \frac{1}{k+1} \geq 1$. Finally we find $x_i \in H_i \setminus H_{i+1}$ for $i \in \{n, \dots, m\}$. Observe that $(x_{i+1}, x_i) \in P \cap V_\delta$ whenever $i \in \{n, \dots, m-1\}$. For $y = x_n$ and $x = x_m$ we have $(x, y) \in T$, but $(x, y) \notin V_1$, – a contradiction. We conclude that $\mathcal{U}$ is not transitive.

Remark 1. The given construction (see part 1) seems to leave open the following question: Does the Pervin quasi-proximity class of a topological space that admits more than one quasi-uniformity always contain a nontransitive quasi-uniformity?

Example 1. The following example answers Problem 2 of Losonczi [4] negatively: After showing that a topological space admitting more than one quasi-uniformity possesses at least $2^{2^{\aleph_0}}$ compatible quasi-uniformities, he wondered whether a similar result might be true for quasi-proximities.

Recall that a base $\mathcal{B}$ of a topological space X is called an *l-base* if it is closed under finite unions and finite intersections and $\emptyset, X \in \mathcal{B}$. In [5] it is observed that for an arbitrary topological space there is a one-to-one correspondence between the set of compatible transitive totally bounded quasi-uniformities and the set of l-bases.

Equip $X = \omega + 1$ with the topology $\tau = \{[0, n] : n \in \omega\} \cup \{\omega, \omega + 1, \emptyset\}$. Since each set $[0, n]$ is compact, $\{[0, n] : n \in \omega\} \cup \{\omega + 1, \emptyset\}$ belongs to any l-base of X. Hence X possesses the two l-bases $\{[0, n] : n \in \omega\} \cup \{\omega + 1, \emptyset\}$

and $\{[0,n] : n \in \omega\} \cup \{\omega, \omega+1, \emptyset\}$ and accordingly two corresponding transitive totally bounded quasi-uniformities.

We now show that X does not admit any totally bounded quasi-uniformity that is not transitive. To this end suppose that $\mathcal{V}$ is a compatible totally bounded quasi-uniformity on X. If $\mathcal{V}$ is equal to the Pervin quasi-uniformity of X, then $\mathcal{V}$ is transitive. So it suffices to consider the case where $\mathcal{V}$ is not the Pervin quasi-uniformity of X. Hence there is an open set G in X such that $V(G) \neq G$ whenever $V \in \mathcal{V}$. Since all open subsets of X except ω are compact, necessarily $G = \omega$ (see [2, Proposition 1.43]). Let $V \in \mathcal{V}$ and let $H \in \mathcal{V}$ be such that $H^3 \subseteq V$. Since $\mathcal{V}$ is totally bounded, there is a finite subset F of X such that $\bigcup_{x\in F}(H^{-1}(x) \cap H(x)) = X$. Let $f \in F$. Note that $\overline{H^{-1}(f)} \neq \{\omega\}$; otherwise $H^{-1}(\omega) = \{\omega\}$ which is impossible by the assumption made above. Therefore $X \setminus \overline{H^{-1}(f)}$ is compact. We conclude that there is $P \in \mathcal{V}$ such that $P^{-1}(\overline{H^{-1}(f)}) = \overline{H^{-1}(f)}$ whenever $f \in F$, because F is finite (see [2, Proposition 1.43]). Let $T = \bigcap_{f\in F}([\overline{H^{-1}(f)} \times X] \cup [(X \setminus \overline{H^{-1}(f)}) \times (X \setminus \overline{H^{-1}(f)})])$. Then T is transitive. We are going to show that $P \subseteq T \subseteq V$:

Let $x \in X$. If $x \in X \setminus \overline{H^{-1}(f)}$ for some $f \in F$, then $P(x) \subseteq (X \setminus \overline{H^{-1}(f)})$ by the choice of P. Thus $P(x) \subseteq \bigcap\{X \setminus \overline{H^{-1}(f)} : x \in X \setminus \overline{H^{-1}(f)}, f \in F\}$ (where we use the convention that $\bigcap \emptyset = X$). Thus $P \subseteq T$. Furthermore there is $f_0 \in F$ such that $x \in H(f_0) \cap H^{-1}(f_0)$. Consequently $T^{-1}(x) \subseteq \overline{H^{-1}(f_0)}$ and therefore $T^{-1}(x) \times \{x\} \subseteq \bigcup_{f\in F}(\overline{H^{-1}(f)} \times H(f)) \subseteq V$. Hence $T \subseteq V$.

We have shown that $\mathcal{V}$ has a base consisting of transitive entourages and deduce that X does not admit any nontransitive totally bounded quasi-uniformity.

Remark. No characterization seems to be known of the topological spaces having the property that all their compatible totally bounded quasi-uniformities are transitive.

In [4] Losonczi showed that the set of the reals equipped with its standard topology admits exactly $2^{2^{\aleph_0}}$ compatible quasi-uniformities. Let us point out that his techniques can be modified to yield the following results.

Proposition 2. *Let X be a topological space of network weight* $\mathrm{nw}(X)$. *Then any compatible quasi-uniformity on X has a base of cardinality* $\leq 2^{\mathrm{nw}(X)}$.

Proof. Note that the cardinality of the topology τ of X is $\leq 2^{\mathrm{nw}(X)}$. Choose a network $\mathcal{N}$ for X of cardinality $\mathrm{nw}(X)$. Let V be an entourage of a compatible quasi-uniformity $\mathcal{V}$ on X such that $V(x)$ is open whenever $x \in X$. Set $B(N) = \{x \in N : x \in N \subseteq V(x)\}$ whenever $N \in \mathcal{N}$. Then $B(N) \times B(N) \subseteq V$ whenever $N \in \mathcal{N}$. Furthermore $\bigcup_{N\in\mathcal{N}} B(N) = X$. Set $\widetilde{V} = \bigcup_{N\in\mathcal{N}}(\overline{B(N)} \times V(B(N)))$. Then $V \subseteq \widetilde{V} \subseteq V^3$. We conclude that $\mathrm{card}(\{\widetilde{V} : V$ belongs to $\mathcal{V}$ and $V(x)$ is open whenever $x \in X\}) \leq (\mathrm{card}(\tau \times \tau))^{\mathrm{nw}(X)} \leq 2^{\mathrm{nw}(X)}$. It follows from [2, p. 3] that $\mathcal{V}$ has a base of cardinality $\leq 2^{\mathrm{nw}(X)}$. □

Corollary 1. *Let X be a topological space of network weight* $\mathrm{nw}(X)$. *Then the number of transitive neighbornets of X is* $\leq 2^{\mathrm{nw}(X)}$.

Proof. For a transitive neighbornet V of X we have $V = V^3$ and $V(x)$ is open whenever $x \in X$. The assertion is a consequence of the argument just given. □

Corollary 2. *Let X be a topological space of network weight* $\mathrm{nw}(X)$. *Then the number of compatible quasi-uniformities on X is* $\leq 2^{2^{\mathrm{nw}(X)}}$.

Proof. We shall use the notation explained in the proof of Proposition 2. If $\mathcal{V}$ is a compatible quasi-uniformity on X, then $\{\widetilde{V} : V \in \mathcal{V}$ and $V(x)$ is open whenever $x \in X\}$ yields a base for $\mathcal{V}$. The assertion follows from the proof of Proposition 2. □

In fact the technique presented above yields the following result.

Remark. Suppose that (X, τ) is a topological space and let κ_1, κ_2 be (infinite) cardinal numbers such that $\mathrm{card}\,\tau \leq \kappa_1$ and for each entourage V belonging to the fine quasi-uniformity of X there is a cover $\{A_\gamma : \gamma \in \kappa_2\}$ of X such that $A_\gamma \times A_\gamma \subseteq V$ whenever $\gamma \in \kappa_2$.

Then X admits at most $2^{({\kappa_1}^{\kappa_2})}$ compatible quasi-uniformities.

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