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On the utility of power-monotone sequences

By LÁSZLÓ LEINDLER (Szeged)

Dedicated to Professor Béla Gyires on his 90th birthday

Abstract. The aim of the paper is to demonstrate the usefulness of the notion of quasi power-monotone sequences. Our theorems give examples showing how some known results on the equivalency of different norms can be simplified utilizing this notion.

1. Introduction

In [4] G. H. Hardy and J. E. Littlewood proved several very important inequalities concerning numerical series and having fundamental significance in applications. Four of them have been applied repeatedly also in the proofs of theorems concerning convergence and summability of orthogonal series. Several problems in this subject have indicated the necessity of the generalizations of these inequalities. The list of the authors having certain generalizations of the classical inequalities of Hardy and Littlewood is quite lengthy, but the majority of the authors can be found in the references of the eminent papers of G. Bennett [1], [2], [3], who also gave a unified approach of the results of Hardy–Littlewood type. Here we recall only one own generalization ([7], [8]), which in the case $\lambda_n = n^{-c}$ with an appropriate positive c reduces to that of Hardy and Littlewood.

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Theorem A. *Let $a_n \ge 0$ and $\lambda_n > 0$ $(n = 1, 2 \dots)$ be given. Then, using the notations*

$$A_{m,n} := \sum_{i=m}^{n} a_i \quad \text{and} \quad \Lambda_{m,n} := \sum_{i=m}^{n} \lambda_i \quad (1 \le m \le n \le \infty),$$

we have

$$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \le p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{n,\infty}^p a_n^p \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \le p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{1,n}^p a_n^p \tag{1.2}$$

for any $p \ge 1$; and when $0 < p \le 1$, the signs of inequalities (1.1) *and* (1.2) *are reversed.*

It is easy to show that the converses of these inequalities, in general, do not hold. But, in the particular case $\lambda_n = n^{-c} (c > 1)$ for quasi τ-power-monotone sequences $\{a_n\}$, i.e. if $n^\tau a_n \downarrow (\tau < 0)$, A. A. Konyushkov [6] proved that inequality (1.1) with a constant K instead of p^p holds for $0 < p \le 1$, too; what is the converse of the original Hardy and Littlewood inequality proved with $0 < p \le 1$. As far as we know this was the first result in this theme investigating the quasi power-monotone sequences.

In [9] we also proved that the converses of inequalities (1.1) and (1.2) for $p > 1$ (or they for $0 < p < 1$), without any additional condition on the nonnegative sequence $\{a_n\}$, hold if and only if the sequence $\{\lambda_n\}$ behaves very similar to a geometrical sequence, that is, if they are quasi geometrically monotone.

Recently it also turned out that the quasi power-monotone sequences and the quasi geometrically monotone sequences are closely interlinked; furthermore that these sequences have been appearing in the generalizations of several classical results, sometimes only implicitly. Before explaining our announcements more precisely we give the exact definitions have been used above laxly.

We shall say that a sequence $\{\gamma_n\}$ of positive terms is *quasi β-power-monotone increasing (decreasing)* if there exists a constant $K = K(\beta, \gamma) \ge 1$ such that

$$K n^\beta \gamma_n \ge m^\beta \gamma_m \quad (n^\beta \gamma_n \le K m^\beta \gamma_m) \tag{1.3}$$

holds for any $n \geq m$, $m \geq 1$.

Furthermore we shall say that a sequence $\{\gamma_n\}$ of positive terms is *quasi geometrically increasing (decreasing)* if there exist a natural number μ and a constant $K = K(\gamma) \geq 1$ such that

$$\gamma_{n+\mu} \geq 2\gamma_n \text{ and } \gamma_n \leq K\gamma_{n+1} \quad \left(\gamma_{n+\mu} \leq \tfrac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \leq K\gamma_n\right) \tag{1.4}$$

hold for all natural number n. Finally a sequence $\{\gamma_n\}$ will be called bounded by blocks if the following inequalities

$$\alpha_1 \Gamma_m^{(k)} \leq \gamma_n \leq \alpha_2 \Gamma_M^{(k)}, \quad 0 < \alpha_1 \leq \alpha_2 < \infty \tag{1.5}$$

hold for any $2^k \leq n \leq 2^{k+1}$, $k = 1, 2, \dots,$ where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

We would like to point out that if the sequence $\{\gamma_n\}$ is either quasi β-power-monotone increasing or decreasing, then condition (1.5) is always fulfilled.

In [10] jointly with J. Németh we proved:

Theorem B. *If a positive sequence $\{\gamma_n\}$ is quasi β-power-monotone increasing (decreasing) with a certain negative (positive) exponent β, then the sequence $\{\gamma_{2^n}\}$ is quasi geometrically increasing (decreasing).*

Theorem C. *If a sequence $\{\gamma_n\}$ is bounded by blocks and its partial sequence $\{\gamma_{2^n}\}$ is quasi geometrically increasing (decreasing), then the whole sequence $\{\gamma_n\}$ is quasi β-power-monotone increasing (decreasing) with a certain negative (positive) exponent β.*

Recently we have realized that the conditions appearing in the theorems of M. Kinukawa [5] have very strong relation with the quasi power-monotone sequences. Kinukawa proved, claiming four conditions, the equivalency of eight different norms, among them, the equivalency between the Beurling norm and the Littlewood–Paley norm.

His conditions are as follows:

(Ψ_1): $\psi(t)$ is a positive and monotonous function on $(0, \infty)$.

(Ψ_2): $\psi(t)$ is homogeneous, that is, there exist constants K_1 and K_2 such that

$$0 < K_1 \leq \frac{\psi(m_{k+1})}{\psi(m_k)} \leq K_2 < \infty,$$

where $\{m_k\}$ denotes an Hadamard gap sequence of integers, that is,

$$1 < K_3 \leq \frac{m_{k+1}}{m_k} \leq K_4 < \infty.$$

(Ψ_3): $\sum_{j=0}^{k} \psi(m_j)^p \leq K_5 \psi(m_k)^p$, $p > 0$.

(Ψ_4): $\sum_{j=k}^{\infty} m_j^{-\nu p} \psi(m_j)^p \leq K_6 m_k^{-\nu p} \psi(m_k)^p$, $\nu, p > 0$.

Let f be an integrable function and let

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

be its Fourier series.

Under the conditions (Ψ_i) $(i = 1,2,3,4)$ M. Kinukawa studied the equivalency of the following norms:

$$\Lambda(f : \psi, a, p, \nu) = \left[\int_0^1 \|\psi(1/t) \Delta_t^\nu f(\cdot)\|_a^p t^{-1} \, dt \right]^{1/p},$$

where $\Delta_t^\nu f(x)$ is the ν-th difference of f w.r.t. $t > 0$ at x.

$$A(c_n : \psi, a, p, \nu) = \left\{ \int_0^1 \left[\sum_{n=-\infty}^{\infty} |\psi(1/t) c_n (\sin nt/2)^\nu|^a \right]^{p/a} t^{-1} \, dt \right\}^{1/p}$$

$$B(c_n : \psi, a, p) = \left\{ \sum_{n=1}^{\infty} n^{-1} \psi(n)^p \left[\sum_{|k| \geq n} |c_k|^a \right]^{p/a} \right\}^{1/p}$$

$$B^*(c_n : \psi, a, p) = \left\{ \sum_{k=0}^{\infty} \psi(m_k)^p \left[\sum_{|n| \geq m_k} |c_n|^a \right]^{p/a} \right\}^{1/p}$$

$$C(c_n : \psi, a, p, \nu) = \left\{ \sum_{n=1}^{\infty} n^{-1-p\nu} \psi(n)^p \left[\sum_{|k| \leq n} |k|^{a\nu} |c_k|^a \right]^{p/a} \right\}^{1/p}$$

$$C^*(c_n:\psi,a,p,\nu)=\left\{\sum_{k=0}^{\infty} m_k^{-p\nu}\psi(m_k)^p\left[\sum_{|n|\le m_k}|n|^{a\nu}|c_n|^a\right]^{p/a}\right\}^{1/p}$$

$$D^*(c_n:\psi,a,p)=\left\{\sum_{k=0}^{\infty}\left[\sum_{m_k\le|n|<m_{k+1}}|\psi(|n|)c_n|^a\right]^{p/a}\right\}^{1/p}.$$

$$E(c_n:\psi,a,p)={}_a\|c_n\psi(|n|)|n|^{1/a-1/p}\|_p.$$

Here the last norm is the Beurling norm, which is defined as follows:

Let us denote

$$W=\left\{(w_n): w_n>0,\ w_n=w_{-n}\downarrow,\ \|w_n\|_1=\sum_{n=1}^{\infty}w_n<\infty\right\},$$

$${}_a\|b_n\|_{p,w}=\left\{\sum_{n=1}^{\infty}|b_n|^a(w_n)^{1-a/p}\right\}^{1/a},$$

$${}_a\|b_n\|_p=\inf_{(w_n)\in W}[\|w_n\|_1^{1/p-1/a}\,{}_a\|b_n\|_{p,w}].$$

Among others the following theorems are proved in [5].

Theorem D. *Let $0<p\le a<\infty$. Suppose that ψ satisfies (Ψ_i) $(i=1,2,3,4)$. Then the norms A, B, B^*, C, C^*, D^* and E are mutually equivalent.*

Theorem E.

(i) *Suppose (Ψ_i) $(i=1,2)$. Let $1\le a\le 2$, $1/a+1/a'=1$, $0<p<\infty$ and $0<p\le a'$. Then*

$$D^*(c_n:\psi,a',p)\le K\Lambda(f:\psi,a,p,\nu).$$

(ii) *Let $1\le a\le 2$, $1/a+1/a'=1$ and $0<p\le a$. Suppose (Ψ_i) $(i=1,2,3,4)$. Then*

$$\Lambda(f:\psi,a',p,\nu)\le KD^*(c_n:\psi,a,p).$$

(iii) *Let* $0 < p \le 2$. *Suppose* (Ψ_i) $(i = 1, 2, 3, 4)$. *Then* $\Lambda(f : \psi, 2, p, \nu)$, $B(c_n : \psi, 2, p)$, $C(c_n : \psi, 2, p, \nu)$, $D^*(c_n : \psi, 2, p)$ *and* $E(c_n : \psi, 2, p)$ *are mutually equivalent.*

There are some further similar results in the paper by Kinukawa, but we do not recall them because the aim of the present note is only to give one more example showing the usefulness of the notion of the quasi power-monotone sequences. Namely we shall demonstrate now again that this notion will simplify the conditions (Ψ_i) $(i = 2, 3, 4)$. We verify that under the condition (Ψ_1) the conditions (Ψ_i) $(i = 2, 3, 4)$ are equivalent to the following two hypotheses:

(P_1): the sequence $\{\psi(n)\}$ is quasi ε-power-monotone increasing with some negative ε.

(P_2): the sequence $\{n^{-\nu}\psi(n)\}$ is quasi β-power-monotone decreasing with some positive β.

2. Results

More precisely we prove the following theorem.

Theorem. *Let* $\psi(t)$ *be a positive and monotonous function on* $(0, \infty)$. *Then the conditions* (Ψ_i) $(i = 2, 3, 4)$ *simultaneously hold if and only if the conditions* (P_1) *and* (P_2) *are satisfied together.*

In my opinion the conditions (P_1) and (P_2) together are more natural than (Ψ_i) $(i = 2, 3, 4)$ jointly. I know, the matter of taste is very personal.

Utilizing our new Theorem it is easy to formulate the dual pairs of the cited thorems of Kinukawa. Now we establish only the twin couple of Theorem D as follows:

Theorem D*. *Let* $0 < p \le a < \infty$, *furthermore let* $\psi(t)$ *be a positive and monotone function on* $(0, \infty)$ *satisfying the conditions* (P_1) *and* (P_2). *Then the norms* A, B, B^*, C, C^*, D^*, *and* E *are mutually equivalent.*

3. Lemmas

To prove our Theorem we need the following lemmas.

Lemma 1 ([9]). *For any positive sequence* $\{\gamma_n\}$ *the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \le K\gamma_m, \quad m = 1, 2, \ldots, K \ge 1;$$

or

$$\sum_{n=1}^{m} \gamma_n \le K\gamma_m, \quad m = 1, 2 \ldots, K \ge 1$$

hold if and only if the sequence $\{\gamma_n\}$ *is quasi geometrically decreasing or increasing, respectively.*

Lemma 2. *If a positive sequence* $\{\gamma_n\}$ *is quasi geometrically decreasing or increasing, then for any* $p > 0$ *the sequence* $\{\gamma_n^p\}$ *has the same property.*

Proof. The assertion is an obvious consequence of the definitions given in (1.4). □

Lemma 3. *If the function* $\psi(t)$ *is homogeneous, that is, if condition* (Ψ_2) *holds, then the sequence* $\{\psi(n)\}$ *is bounded by blocks.*

Proof. Due to the definitions the declaration is trivial. □

4. Proof of Theorem

First we underline that the sequence $\{m_k\}$ can be chosen to $\{2^k\}$, that is, $m_k = 2^k$.

Thus, by Lemmas 1, 2, 3, and Theorem C, we see that the conditions (Ψ_2) and (Ψ_3) imply that the sequence $\{\psi(n)\}$ is quasi ε-power-monotone increasing with a certain negative exponent ε, that is, we have got the implication (Ψ_2) and $(\Psi_3) \Rightarrow (\mathrm{P}_1)$.

The same arguing with (Ψ_4) in place of (Ψ_3) conveys the implication (Ψ_2) and $(\Psi_4) \Rightarrow (\mathrm{P}_2)$.

Conversely, first an elementary consideration gives that conditions (P_1) and (P_2) together imply the condition (Ψ_2), see the definitions in (1.3).

Furthermore the conditon (P_1) by Theorem B with $\gamma_n := \psi(n)$ implies that the sequence $\{\psi(2^k)\}$ is quasi geometrically increasing, thus, by Lemmas 1 and 2, the condition (Ψ_3) holds. Briefly we have got that (P_1) $\Rightarrow$ (Ψ_3).

Similarly with (P_2) in place of (P_1) we obtain that (P_2) $\Rightarrow$ (Ψ_4). Namely the condition (P_2) via Theorem B with $\gamma_n := n^{-\nu}\psi(n)$ implies that the sequence $\{2^{-\nu k}\psi(2^k)\}$ is quasi geometrically decreasing, whence by Lemmas 1 and 2, the condition (Ψ_4) follows, as stated.

The proof is complete.

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L. LEINDLER
BOLYAI INSTITUTE
JÓZSEF ATTILA UNIVERSITY
ARADI VÉRTANÚK TERE 1
H–6720 SZEGED
HUNGARY

E-mail: leindler@math.u-szeged.hu

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