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On reduced n-cubic spline interpolation of Hermite type

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1. Introduction

The object of this paper is to construct a continuous spline approximational method on rectangular grid for functions in several variables. This method is a local one and it will be called reduced n-cubic spline interpolation of Hermite-type, because it is a special piecewise polynomial which is cubic polynomial in each variable defined on each rectangle by prescribing the function values and the values of the first order partial derivatives at the knots. It is called reduced because we have not as much conditions for the partial derivatives at the knots as it would be possible if we generalized the one dimensional case by tensor product (Melkes [17], Delvos, Posdorf [8], Baszenski, Schumaker [3]). In this case this polynomial is of degree at most $n+2$ instead of $3n$ on each rectangle, so we need less arithmetic operations to compute its values. We give a recursive formula to compute the spline function in higher dimensions. By this recursive formula we prove approximation theorems, which show that the order of this approximation is the best possible (in the sense of the Jackson theory for the best approximation by polynomials), depending on the smoothness of the function. We show that this spline function minimalizes a special functional which is a possible generalization of the curvature in two dimensions. We remark, that for $n=2$ this spline is the so-called ADINI element on each subrectangle (Gordon [10], Lancaster, Watkins [13], Melkes [17]).

Various multidimensional spline approximational methods have been worked out, for references see the monumental bibliography by Franke, Schumaker [9] and Chui, Schumaker, Utreras (eds) [7]. To generalize the one dimensional methods on rectangular grid the most useful methods are the tensor product methods (Baszenski, Schumaker [3]) and the blending methods (Baszenski, Delvos, Posdorf [2]). For fur-

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ther details on spline approximation see e.g. Ahlberg, Nilson, Walsh [1], de Boor [5], Korneichuk [12], Schumaker [20], Stečkin, Subbotin[21], Zavialov, Kvasov, Miroshničenko [22], etc.

A similar construction of reduced n-quadratic spline interpolation of Hermite-type can be found in [15]. As possible applications of these reduced n-cubic and n-quadratic spline interpolations we mention the multiple quadrature formulas ([16]) and the approximate solution of partial differential equations ([4], [15]).

2. Notations

In what follows $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ denote the set of reals, the set of integers and the set of the natural numbers (including zero). For any vector $\mathbf{x}$ in $\mathbb{R}^n$ we denote its j-th component by $(\mathbf{x})_j = x_j$, that is $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Addition, multiplication and inequality between vectors will be defined componentwise and elsewhere for $\mathbf{x} \in \mathbb{R}^n$ we use the Euclidean norm

$$\|\mathbf{x}\| = \Big(\sum_{j=1}^{n} x_j^2\Big)^{\frac{1}{2}},$$

and let

$$|\mathbf{x}| = \sum_{j=1}^{n} |x_j|.$$

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then let

$$[\mathbf{a}, \mathbf{b}] = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}$$

and

$$\mathbf{a}^{\mathbf{b}} = \prod_{j=1}^{n} (a_j)^{b_j} \qquad (b_j \in \mathbb{Z},\ \ j = 1, \ldots, n),$$

where $0^0 = 1$. The zero vector will be denoted by $\mathbf{0}$, further $\mathbf{e} = (1, 1, \ldots, 1)$ and $\mathbf{e}_j$ denotes the vector whose j-th coordinate equals to 1, the others being zero ($j = 1, 2, \ldots, n$). The modulus of continuity of the function $u : \mathbb{R}^n \to \mathbb{R}$ will be denoted by $\omega(d; u)$, that is

$$\omega(d; u) = \sup_{\substack{\mathbf{t}, \tilde{\mathbf{t}} \in [\mathbf{a}, \mathbf{b}] \\ \|\mathbf{t} - \tilde{\mathbf{t}}\| \leq d}} |u(\mathbf{t}) - u(\tilde{\mathbf{t}})|,$$

where d denotes the (Euclidean) diameter of the set, on which the oscillation of u is considered. The differential operators for multivariable functions will be denoted as usual by

$$\partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}.$$

or by

$$D^{\alpha} = D^{\alpha_1, \alpha_2, \ldots, \alpha_n}.$$

3. Construction of the spline function

Let $\{\mathbf{t_i}\}_{\mathbf{i}\in\mathbb{Z}^n}$ be an equidistant subdivision of $\mathbb{R}^n$ with $\mathbf{h} = (h_1, h_2, \dots, h_n)$, that is $(\mathbf{t_{i+e_\mathit{j}}} - \mathbf{t_i})_j = h_j$. Let $\{u_\mathbf{i}\}_{\mathbf{i}\in\mathbb{Z}^n}$ and $\{u_\mathbf{i}^{(\mathbf{e}_j)}\}_{\mathbf{i}\in\mathbb{Z}^n}$ $(j = 1, 2, \dots, n)$ be given systems of real numbers. Let $d = \|\mathbf{h}\|$ denote the diameter corresponding to this subdivision.

For all $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$ we define

$$(1)_n \qquad S_\mathbf{i}(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbf{K}} A_\mathbf{i}^{(\mathbf{k})} (\mathbf{t} - \mathbf{t_i})^\mathbf{k},$$

where $\mathbf{K}$ is the set of all n-dimensional multi-indices $\mathbf{k}$ $\quad(\mathbf{0} \le \mathbf{k} \le 3\mathbf{e})$ satisfying the following condition: if $k_j = 2$ or $k_j = 3$ for any j, then $k_l \le 1$ for all $l \ne j$; that is, $S_\mathbf{i}$ is a special polynomial of degree at most $n+2$, which is cubic polynomial in each variable. Further the unknown coefficients $A_\mathbf{i}^{(\mathbf{k})}$ are to be chosen satisfying the conditions

$$(2)_n \qquad \begin{aligned} S_\mathbf{i}(\mathbf{t_{i+l}}) &= u_\mathbf{i+l}, \qquad \mathbf{0} \le \mathbf{l} \le \mathbf{e}, \\ \partial_j S_\mathbf{i}(\mathbf{t_{i+l}}) &= u_\mathbf{i+l}^{(\mathbf{e}_j)}, \qquad j = 1, \dots, n. \end{aligned}$$

The number of the coefficients $A_\mathbf{i}^{(\mathbf{k})}$ is equal to the number of the conditions (equations) and it is $2^n(n+1)$.

Lemma 1. *There exist unique constants $A_\mathbf{i}^{(\mathbf{k})}$ such that the functions $S_\mathbf{i}$ of form $(1)_n$ satisfy $(2)_n$.*

Proof. We show by induction with respect to the dimension n, that the coefficients $A_\mathbf{i}^{(\mathbf{k})}$ are uniquely determined by conditions $(2)_n$. If $n = 1$, then then the polynomial $(1)_1$ is the Hermite interpolational cubic polynomial satisfying the prescribed conditions. In the $n+1$-dimensional case it follows by induction, that all those coefficients $A_\mathbf{i}^{(\mathbf{k})}$ are uniquely determined, where $\mathbf{k}$ has at least one zero coordinate. For the remaining $2n+1$ coefficients

$$A_\mathbf{i}^{(\mathbf{k})} \qquad (\mathbf{k} = \mathbf{e}, \mathbf{e}+\mathbf{e}_1, \dots, \mathbf{e}+\mathbf{e}_n, \mathbf{e}+2\mathbf{e}_1, \dots \mathbf{e}+2\mathbf{e}_n)$$

we have the following system of linear equations

$$\begin{aligned} S_\mathbf{i}(\mathbf{t_{i+e}}) &= u_\mathbf{i+e}, \\ \partial_j S_\mathbf{i}(\mathbf{t_{i+e}}) &= u_\mathbf{i+e}^{(\mathbf{e}_j)} \qquad (j = 1, \dots, n), \\ \partial_j S_\mathbf{i}(\mathbf{t}_{\mathbf{i+e}-\mathbf{e}_j}) &= u_{\mathbf{i+e}-\mathbf{e}_j}^{(\mathbf{e}_j)} \qquad (j = 1, \dots, n). \end{aligned}$$

By elementary transformations the matrix of this linear system can be reduced to the following one:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 & 3 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 1 & 1 & 3 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 1 & \dots & 3 \\ 1 & 0 & 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 1 & 1 & \dots & 0 \end{pmatrix}$$

which is of order $2n+1$. It is easy to see that this matrix is regular, hence the existence and uniqueness are proved.

For instance, in the two dimensional case, the function $S_{i,j}$ satisfying the conditions $(1)_2$-$(2)_2$ can be expressed as follows: for all $x \in [x_i, x_{i+1}]$ and $y \in [y_j, y_{j+1}]$

$$\begin{aligned} &S_{i,j}(x,y) = \\ &= (1-s)\{\phi_1(t)u_{i,j} + \phi_2(t)u_{i+1,j} + \phi_3(t)hu^{(1,0)}_{i,j} + \phi_4(t)hu^{(1,0)}_{i+1,j}\} + \\ &\quad + s\{\phi_1(t)u_{i,j+1} + \phi_2(t)u_{i+1,j+1} + \phi_3(t)hu^{(1,0)}_{i,j+1} + \phi_4(t)hu^{(1,0)}_{i+1,j+1}\} + \\ &\quad + (1-s)s^2 \cdot \\ &\quad \cdot \{(u_{i,j+1} - u_{i,j} - lu^{(0,1)}_{i,j+1})(1-t) + (u_{i+1,j+1} - u_{i+1,j} - lu^{(0,1)}_{i+1,j+1})t\} + \\ &\quad + (1-s)(s-1)s \cdot \\ &\quad \cdot \{(u_{i,j+1} - u_{i,j} - lu^{(0,1)}_{i,j})(1-t) + (u_{i+1,j+1} - u_{i+1,j} - lu^{(0,1)}_{i+1,j})t\}, \end{aligned}$$

where

$$\begin{aligned} \phi_1(t) &= (1-t)^2(1+2t), & \phi_2(t) &= t^2(3-2t), \\ \phi_3(t) &= t(1-t)^2, & \phi_4(t) &= -t^2(1-t), \end{aligned}$$

and we have used the notations $h = x_{i+1} - x_i$, $l = y_{j+1} - y_j$, $t = (x - x_i)/h$ and $s = (y - y_j)/l$.

We remark that $S_{i,j}(x,y)$ is the so called ADINI element (see e.g. [8], [13], [17]).

Using this formula we need only 22 multiplicative and 34 additive operations to compute the value of $S_{i,j}(x,y)$ [14].

For comparison purposes, if we use the tensor-product interpolant which not only the first order partial derivatives but also the mixed second order partial derivatives at the knots (this is the generalization of the one dimensional case by direct product), then we need 34 multiplicative and 22 additive operations [22].

More generally, it is easy to see by the uniqueness part of Lemma 1, that the $n+1$-dimensional function $S_{\mathbf{i}}^{(n+1)}$ can be expressed with the help of the n-dimensional $S_{\mathbf{i}}^{(n+1)}$ as follows:

$$S_{\mathbf{i}}^{(n+1)}(t_1,\dots,t_n,t_{n+1}) =$$
$$= v_{n+1}S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1,\dots,t_n) + (1-v_{n+1})S^{(n)}_{\mathbf{i}}(t_1,\dots,t_n) +$$
$$+(1-v_{n+1})v_{n+1}^2 \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j\big[u_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{l}} - h_{n+1}u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}}\big] +$$
$$+(1-v_{n+1})(v_{n+1}-1)v_{n+1} \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j\big[u_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{l}} - h_{n+1}u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}}\big],$$

where

$$v_j = \frac{(\mathbf{t})_j - (\mathbf{t}_{\mathbf{i}})_j}{h_j}, \quad w_j = \begin{cases} v_j, & \text{if} \quad l_j = 1 \\ 1 - v_j, & \text{if} \quad l_j = 0 \end{cases}$$

for $j = 1, 2, \dots, n, n+1$. In what follows, we shall call this formula as the *recursive formula* of the n-cubic spline function.

Now we define the n-cubic spline function S (corresponding to the knots $\{\mathbf{t}_{\mathbf{i}}\}$ and the systems $\{u_{\mathbf{i}}\}$ and $\{u_{\mathbf{i}}^{(\mathbf{e}_j)}\}$) on $\mathbb{R}^n$: for all $\mathbf{t} \in [\mathbf{t}_{\mathbf{i}}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$ let

$$(3)_n \qquad S(\mathbf{t}) = S_{\mathbf{i}}(\mathbf{t}).$$

Theorem 2. *The n-cubic spline function S defined by* $(1)_n$-$(3)_n$ *is continuous.*

Proof. The statement follows by induction from the uniqueness part of Lemma 1.

In the theory of spline approximation it is well-known that spline functions of different kinds minimize special functionals. For example, the classical cubic spline has minimal curvature ([11]), it minimizes the functional

$$f \to \int_a^b |f''(t)|^2 dt$$

on the class of functions which are twice differentiable on $[a, b]$, interpolate at the knots and satisfy some boundary conditions. There are more possibilities to generalize this functional in more dimensional case. Let us consider the following functional in the two dimensional case ([22])

$$J(u)=\int_a^b\int_c^d[D^{2,2}u(x,y)]^2dxdy+$$
$$+\sum_{i=0}^N\frac{1}{\rho_i}\int_c^d[D^{0,2}u(x_i,y)]^2dy+\sum_{j=0}^M\frac{1}{\sigma_j}\int_a^b[D^{2,0}u(x,y_j)]^2dx,$$

where ρ_i $(i=0,\dots,N)$, σ_j $(j=0,\dots,M)$ are positive constants and we divided the interval $[a,b]$ into N, and the interval $[c,d]$ into M equal subintervals. Let $\mathcal{M}$ denote the class of all periodic functions u which have continuous partial derivatives ∂_1^2u, ∂_2^2u, $\partial_1\partial_2u$ on all open subrectangles and their partial derivatives up to the second order can be extended on the closed subrectangles. For the sake of the unique definition of the partial derivatives let $\partial_1^\alpha\partial_2^\beta u(x,y)$ be the value of the partial derivative of u at the point $(x,y)\in[x_i,x_{i+1})\times[y_j,y_{j+1})$. Further let us assume that all functions $u\in\mathcal{M}$ satisfy the following conditions:

$$u(x_i,y_j)=u_{i,j},$$
$$\partial_1u(x_i,y_j)=u_{i,j}^{(1,0)},$$
$$\partial_2u(x_i,y_j)=u_{i,j}^{(0,1)}$$

for $i=0,\dots,N$, $j=0,\dots,M$, and for the periodicity

$$u_{0,j}^{(\alpha,\beta)}=u_{N,j}^{(\alpha,\beta)},\qquad j=0,\dots,M,$$
$$u_{i,0}^{(\alpha,\beta)}=u_{i,M}^{(\alpha,\beta)},\qquad i=0,\dots,N.$$

It is obviously that the for the n-cubic spline function satisfies $S\in\mathcal{M}$.

Theorem 3. *The n-cubic spline function S defined by $(1)_2-(3)_2$ minimizes the functional*

$$J(u)=\int_a^b\int_c^d[D^{2,2}u(x,y)]^2dxdy+$$
$$+\sum_{i=0}^N\frac{1}{\rho_i}\int_c^d[D^{0,2}u(x_i,y)]^2dy+\sum_{j=0}^M\frac{1}{\sigma_j}\int_a^b[D^{2,0}u(x,y_j)]^2dx$$

on the class $\mathcal{M}$, that is

$$J(u)\geq J(S)\qquad \textit{for all } u\in\mathcal{M},$$

where ρ_i $(i = 0, \dots, N)$, σ_j $(j = 0, \dots, M)$ *are positive constants.*

Proof. Let $u \in \mathcal{M}$ be arbitrary and S be n-cubic spline function defined by $(1)_2 - (3)_2$. Then

$$J(u - S) = J(u) - J(S) - 2\Big(I_\Delta + \sum_{i=0}^{N} \frac{1}{\rho_i} I_i + \sum_{j=0}^{M} \frac{1}{\sigma_j} \tilde{I}_j\Big),$$

where

$$\begin{aligned}
I_\Delta &= \int_a^b \int_c^d D^{2,2}[u(x,y) - S(x,y)] D^{2,2} S(x,y) dx dy,\\
I_i &= \int_c^d D^{0,2}[u(x_i,y) - S(x_i,y)] D^{0,2} S(x_i,y) dy,\\
\tilde{I}_j &= \int_a^b D^{2,0}[u(x,y_j) - S(x,y_j)] D^{2,0} S(x,y_j) dx.
\end{aligned}$$

Hence $A_{i,j}^{(2,2)} = 0$ for all i, j, so $D^{2,2}S(x,y) = 0$ on all rectanglar, and it follows $I_\Delta = 0$.

By partial integration we have:

$$\begin{aligned}
I_i &= \sum_{j=0}^{M-1} \int_{y_j}^{y_{j+1}} D^{0,2}[u(x_i,y) - S(x_i,y)] D^{0,2} S(x_i,y) dy =\\
&= \sum_{j=0}^{M-1} \Big\{ D^{0,1}[u(x_i,y_{j+1}) - S(x_i,y_{j+1})] D^{0,2} S(x_i,y_{j+1}) -\\
&\quad - D^{0,1}[u(x_i,y_j) - S(x_i,y_j)] D^{0,2} S(x_i,y_j) -\\
&\quad - \int_{y_j}^{y_{j+1}} D^{0,1}[u(x_i,y) - S(x_i,y)] c_j(x_i) dy \Big\} =\\
&= - \sum_{j=0}^{M-1} c_j(x_i) \big[u(x_i,y) - S(x_i,y)\big]_{y_j}^{y_{j+1}} = 0,
\end{aligned}$$

because the spline function S is a piecewise cubic polinomial in each variable, that is, $D^{0,3}S(x_i,y) = c_j(x_i)$. We get similarly, that $\tilde{I}_j = 0$ for all j. Finally,

$$J(u - S) = J(u) - J(S),$$

that is

$$J(u) = J(S) + J(u - S),$$

where $J(u - S) \geq 0$, so $J(u) \geq J(S)$, which was to be proved.

4. Approximation properties of the spline function

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function having first order partial derivatives with respect to each variable. We define for all $\mathbf{i} \in \mathbb{Z}^n$ and $j = 1, 2, \ldots, n$

$$(4)_n \qquad u_{\mathbf{i}} = u(\mathbf{t}_{\mathbf{i}})$$

and

$$(5)_n \qquad u_{\mathbf{i}}^{(\mathbf{e}_j)} = \partial_j u(\mathbf{t}_{\mathbf{i}}).$$

In the following theorems we study the approximating properties of this n-cubic spline function (defined by the conditions $(1)_n - (5)_n$) depending on the smoothness of the function u. Although this n-cubic spline function S is only continuous, by defining (uniquely) its first order partial derivatives, for example

$$\partial_k S(\mathbf{t}) = \partial_k S_{\mathbf{i}}(\mathbf{t}) \qquad \mathbf{t} \in [\mathbf{t}_{\mathbf{i}}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}], \quad (\mathbf{t})_j \neq (\mathbf{t}_{\mathbf{i}})_j + h_j \quad (j = 1, \ldots, n),$$

respectively

$$\partial_k \partial_l S(\mathbf{t}) = \partial_k \partial_l S_{\mathbf{i}}(\mathbf{t}) \qquad \mathbf{t} \in [\mathbf{t}_{\mathbf{i}}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}], \quad (\mathbf{t})_j \neq (\mathbf{t}_{\mathbf{i}})_j + h_j \quad (j = 1, \ldots, n),$$

the spline function approximates not only the function u but also its partial derivatives.

Lemma 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function and for $t \in [t_i, t_{i+1}]$ let*

$$(6) \quad S_i(t) = \phi_1(v) f(t_i) + \phi_2(v) f(t_{i+1}) + \phi_3(v) h f'(t_i) + \phi_4(v) h f'(t_{i+1}),$$

where

$$\phi_1(v) = (1-v)^2(1+2v), \qquad \phi_2(v) = v^2(3-2v),$$
$$\phi_3(v) = v(1-v)^2, \qquad \phi_4(v) = -v^2(1-v),$$

and $h = t_{i+1} - t_i$, $v = (t - t_i)/h$. If $f \in C^1(\mathbb{R})$, then

$$|f(t) - S_i(t)| \leq \frac{3}{8} h \omega(h; f'),$$
$$|f'(t) - S_i'(t)| \leq \frac{3}{2} \omega(h; f'),$$

if $f \in C^2(\mathbb{R})$, then

$$|f(t) - S_i(t)| \leq \frac{1}{32} h^2 \omega(h; f''),$$
$$|f'(t) - S_i'(t)| \leq 0.12375 h \omega(h; f''),$$
$$|f''(t) - S_i''(t)| \leq \frac{4}{3} \omega(h; f'')$$

for all $t \in [t_i, t_{i+1}]$.

Proof. See e.g. [22] Theorem 5 in Chapter 2.

Theorem 5. *Let* $u : \mathbb{R}^n \to \mathbb{R}$ *be continuously differentiable. Then the* n*-cubic spline function* S *defined by the conditions* $(1)_n$-$(5)_n$ *satisfies*

$$|u(\mathbf{t}) - S(\mathbf{t})| \leq \frac{1}{2} \sum_{j=1}^{n} h_j \omega(d; \partial_j u),$$

and

$$|\partial_k u(\mathbf{t}) - \partial_k S(\mathbf{t})| \leq n\omega(d; \partial_k u) + \frac{1}{2h_k} \sum_{j=1}^{n} h_j \omega(d; \partial_j u)$$

for all $\mathbf{t} \in \mathbb{R}^n$ *and* $k = 1, 2, \dots, n$, *where* d *is the diameter of the subdivision.*

Proof. We prove this statement by induction on the dimension, the case of $n = 1$ is true by the previous lemma. Let us suppose, that the statement is true for n and now let us see the case of $\mathbb{R}^{n+1}$. Let $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$, then by induction and the Lagrange theorem we have

$$\begin{aligned}
|u(\mathbf{t}) - S(\mathbf{t})| &= |u(t_1, \dots, t_n, t_{n+1}) - S^{(n+1)}_{\mathbf{i}}(t_1, \dots, t_n, t_{n+1})| \leq \\
&\leq v_{n+1}|u(t_1, \dots, t_n, (\mathbf{t_{i+e}})_{n+1}) - S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1, \dots, t_n)| + \\
&\quad + (1 - v_{n+1})|u(t_1, \dots, t_n, (\mathbf{t_i})_{n+1}) - S^{(n)}_{\mathbf{i}}(t_1, \dots, t_n)| + \\
&\quad + |v_{n+1}[u(t_1, \dots, t_n, t_{n+1}) - u(t_1, \dots, t_n, (\mathbf{t_{i+e}})_{n+1})] + \\
&\quad + (1 - v_{n+1})[u(t_1, \dots, t_n, t_{n+1}) - u(t_1, \dots, t_n, (\mathbf{t_i})_{n+1})]| + \\
&\quad + h_{n+1}(1 - v_{n+1})v_{n+1} \sum_{\substack{\mathbf{0} \leq \mathbf{l} \leq \mathbf{e} \\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \cdot \\
&\quad \cdot \left\{ v_{n+1}|\partial_{n+1}u(t_1, \dots, t_n, \theta_{n+1}) - u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}}| + \right. \\
&\quad \left. - (1 - v_{n+1})|\partial_{n+1}u(t_1, \dots, t_n, \theta_{n+1}) - u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}}| \right\} \leq \\
&\leq \frac{1}{2} \sum_{j=1}^{n} h_j \omega(d; \partial_j u) + +(1 - v_{n+1})v_{n+1}h_{n+1} \cdot \\
&\quad \cdot |\partial_{n+1}u(t_1, \dots, t_n, \xi_{n+1}) - \partial_{n+1}u(t_1, \dots, t_n, \vartheta_{n+1})| + \\
&\quad + (1 - v_{n+1})v_{n+1}h_{n+1}\omega(d; \partial_{n+1}u) \sum_{\substack{\mathbf{0} \leq \mathbf{l} \leq \mathbf{e} \\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \leq
\end{aligned}$$

$$\leq \frac{1}{2}\sum_{j=1}^{n} h_j\omega(d;\partial_j u) + \frac{1}{4}h_{n+1}\omega(d;\partial_{n+1}u) + \frac{1}{4}h_{n+1}\omega(d;\partial_{n+1}u) =$$
$$= \frac{1}{2}\sum_{j=1}^{n+1} h_j\omega(d;\partial_j u),$$

where $(\mathbf{t_i})_{n+1} < \xi_{n+1},\ \vartheta_{n+1},\theta_{n+1} < (\mathbf{t_{i+e}})_{n+1},\ \mathbf{t_i} \leq \bar{\mathbf{t}} \leq \mathbf{t_{i+e}}$. Here we used the simple identity

$$\sum_{\substack{\mathbf{0}\leq\mathbf{l}\leq\mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j = 1.$$

The respective statement for the derivatives can be proved similarly, and without loss of generality we prove only the case $k = 1$. Let $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$, then by induction and by the definition of the modulus of continuity we have

$$\begin{aligned}
&|\partial_1 u(\mathbf{t}) - \partial_1 S(\mathbf{t})| = \\
&\quad = |\partial_1 u(t_1,\dots,t_n,t_{n+1}) - \partial_1 S_{\mathbf{i}}^{(n+1)}(t_1,\dots,t_n,t_{n+1})| \leq \\
&\quad \leq v_{n+1}|\partial_1 u(t_1,\dots,t_n,(\mathbf{t_{i+e}})_{n+1}) - \partial_1 S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1,\dots,t_n)| + \\
&\qquad + (1-v_{n+1})|\partial_1 u(t_1,\dots,t_n,(\mathbf{t_i})_{n+1}) - \partial_1 S_{\mathbf{i}}^{(n)}(t_1,\dots,t_n)| + \\
&\qquad + v_{n+1}|\partial_1 u(t_1,\dots,t_{n+1}) - \partial_1 u(t_1,\dots,t_n,(\mathbf{t_{i+e}})_{n+1})| + \\
&\qquad + (1-v_{n+1})|\partial_1 u(t_1,\dots,t_n,t_{n+1}) - \partial_1 u(t_1,\dots,t_n,(\mathbf{t_i})_{n+1})| + \\
&\qquad + (1-v_{n+1})v_{n+1}h_{n+1}\sum_{\substack{\mathbf{0}\leq\mathbf{l}\leq\mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \frac{2}{h_1}\,\cdot \\
&\qquad \cdot\ \left\{v_{n+1}|\partial_{n+1}u(t_1,\dots,\theta_{n+1}) - u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}}| + \right. \\
&\qquad + \left.(1-v_{n+1})|\partial_{n+1}u(t_1,\dots,\theta_{n+1}) - u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}}|\right\} \leq \\
&\quad \leq n\omega(d;\partial_1 u) + \frac{1}{2h_1}\sum_{j=1}^{n} h_j\omega(d;\partial_j u) + \omega(d;\partial_1 u)\ + \\
&\qquad + (1-v_{n+1})v_{n+1}h_{n+1}\sum_{\substack{\mathbf{0}\leq\mathbf{l}\leq\mathbf{e}\\ l_{n+1}=0\\ l_1=0}} \prod_{j=2}^{n} w_j \frac{2}{h_1}\omega(d;\partial_{n+1}u) \leq
\end{aligned}$$

$$\leq n\omega(d;\partial_1 u) + \frac{1}{2h_1}\sum_{j=1}^{n} h_j\omega(d;\partial_j u) + \omega(d;\partial_1 u) + \frac{h_{n+1}}{2h_1}\omega(d;\partial_{n+1}u) =$$

$$= (n+1)\omega(d;\partial_1 u) + \frac{1}{2h_1}\sum_{j=1}^{n+1} h_j\omega(d;\partial_j u),$$

which is our statement.

Theorem 6. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then the n-cubic spline function S defined by the conditions $(1)_n$-$(5)_n$ satisties*

$$|u(\mathbf{t}) - S(\mathbf{t})| \leq \frac{1}{4}\sum_{j=1}^{n} h_j^2\omega(d;\partial_j^2 u),$$

$$|\partial_k u(\mathbf{t}) - \partial_k S(\mathbf{t})| \leq \frac{1}{4}\sum_{j=1}^{n} h_j\omega(d;\partial_k\partial_j u) + \frac{1}{8}\sum_{j=1}^{n}\frac{h_j^2}{h_k}\omega(d;\partial_j^2 u),$$

$$|\partial_k^2 u(\mathbf{t}) - \partial_k^2 S(\mathbf{t})| \leq (n+1)\omega(d;\partial_k^2 u),$$

and

$$|\partial_k\partial_l u(\mathbf{t}) - \partial_k\partial_l S(\mathbf{t})| \leq (n+\frac{5}{2})\omega(d;\partial_k\partial_l u) + \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{h_j}{h_l}\omega(d;\partial_k\partial_j u)$$

for all $\mathbf{t}\in\mathbb{R}^n$ and $k,l = 1,2,\ldots,n$, where d is the diameter corresponding to the subdivision.

Proof. Now we apply induction again with respect to the dimension. The case $n=1$ follows from the Lemma 4. In the $n+1$ dimensional case let $\mathbf{t}\in[\mathbf{t_i},\mathbf{t_{i+e}}]$, then by induction and by the second order Taylor-formula we have

$$|u(\mathbf{t}) - S(\mathbf{t})| = |u(t_1,\ldots,t_n,t_{n+1}) - S_{\mathbf{i}}^{(n+1)}(t_1,\ldots,t_n,t_{n+1})| \leq$$

$$\leq v_{n+1}|u(t_1,\ldots,t_n,(\mathbf{t_{i+e}})_{n+1}) - S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1,\ldots,t_n)| +$$

$$+\ (1-v_{n+1})|u(t_1,\ldots,t_n,(\mathbf{t_i})_{n+1}) - S_{\mathbf{i}}^{(n)}(t_1,\ldots,t_n)| +$$

$$+\ \Big|v_{n+1}[u(t_1,\ldots,t_n,t_{n+1}) - u(t_1,\ldots,t_n,(\mathbf{t_{i+e}})_{n+1})] +$$

$$+\ (1-v_{n+1})[u(t_1,\ldots,t_n,t_{n+1}) - u(t_1,\ldots,t_n,(\mathbf{t_i})_{n+1})] +$$

$$+\ (1-v_{n+1})v_{n+1}\sum_{\substack{\mathbf{0}\leq\mathbf{l}\leq\mathbf{e}\\ l_{n+1}=0}}\prod_{j=1}^{n} w_j\ \cdot$$

$$
\begin{aligned}
&\quad \cdot \Big\{v_{n+1}\big[u_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{l}} - h_{n+1}u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}}\big] \\
&\quad + (v_{n+1}-1)\big[u_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{l}} - h_{n+1}u^{(\mathbf{e}_{n+1})}_{\mathbf{i}+\mathbf{l}}\big]\Big\}\Bigg| \le \\
&\le \frac{1}{4}\sum_{j=1}^{n} h_j^2\omega(d;\partial_j^2 u) + \\
&\quad + (1-v_{n+1})v_{n+1}h_{n+1}\Bigg|\partial_{n+1}u(t_1,\dots,t_n,(\mathbf{t_i})_{n+1}) + \\
&\quad + \frac{1}{2}h_{n+1}v_{n+1}\partial^2_{n+1}u(t_1,\dots,t_n,\xi_{n+1}) - \partial_{n+1}u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1}) + \\
&\quad + \frac{1}{2}h_{n+1}(1-v_{n+1})\partial^2_{n+1}u(t_1,\dots,t_n,\vartheta_{n+1}) + \\
&\quad + \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \frac{1}{2}h_{n+1}\big\{v_{n+1}\partial^2_{n+1}u(\bar{\mathbf{t}}_{\mathbf{l}}) + (1-v_{n+1})\partial^2_{n+1}u(\bar{\bar{\mathbf{t}}}_{\mathbf{l}})\big\}\Bigg| \le \\
&\le \frac{1}{4}\sum_{j=1}^{n} h_j^2\omega(d;\partial_j^2 u) + (1-v_{n+1})v_{n+1}h^2_{n+1}\cdot \\
&\quad \cdot \Bigg|\frac{1}{2}\partial^2_{n+1}u(t_1,\dots,t_n,\theta_{n+1}) - \partial^2_{n+1}u(t_1,\dots,t_n,\bar\theta_{n+1}) + \\
&\quad + \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \frac{1}{2}\partial^2_{n+1}u(\tilde{\mathbf{t}}_{\mathbf{l}})\Bigg| \le \\
&\le \frac{1}{4}\sum_{j=1}^{n} h_j^2\omega(d;\partial_j^2 u) + \\
&\quad + (1-v_{n+1})v_{n+1}h^2_{n+1}|\partial^2_{n+1}u(\bar{\mathbf{t}}) - \partial^2_{n+1}u(t_1,\dots,t_n,\bar\theta_{n+1})| \le \\
&\le \frac{1}{4}\sum_{j=1}^{n+1} h_j^2\omega(d;\partial_j^2 u),
\end{aligned}
$$

where $\xi_{n+1}, \vartheta_{n+1}, \theta_{n+1}, \bar\theta_{n+1} \in (t_{n+1}, t_{n+1}+h_{n+1})$, $\bar{\mathbf{t}}_{\mathbf{l}}, \bar{\bar{\mathbf{t}}}_{\mathbf{l}}, \bar{\mathbf{t}}, \tilde{\mathbf{t}}_{\mathbf{l}} \in [\mathbf{t_i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$, and so we've proved the first statement. For the partial derivatives (without loss of generality we prove in order to the first variable) we get the following estimates:

$$
|\partial_1 u(\mathbf{t}) - \partial_1 S(\mathbf{t})| = |\partial_1 u(t_1,\dots,t_n,t_{n+1}) - \partial_1 S^{(n+1)}_{\mathbf{i}}(t_1,\dots,t_n,t_{n+1})| \le
$$

$$
\begin{aligned}
&\le v_{n+1}|\partial_1 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1}) - \partial_1 S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1,\dots,t_n)| + \\
&\quad + (1-v_{n+1})|\partial_1 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}})_{n+1}) - \partial_1 S^{(n)}_{\mathbf{i}}(t_1,\dots,t_n)| + \\
&\quad + v_{n+1}(1-v_{n+1})\,\cdot \\
&\quad \cdot\, h_{n+1}|\partial_1\partial_{n+1}u(t_1,\dots,t_n,\xi_{n+1}) - \partial_1\partial_{n+1}u(t_1,\dots,t_n,\theta_{n+1})| + \\
&\quad + (1-v_{n+1})v_{n+1}\sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0\\ l_1=0}}\ \prod_{j=2}^{n} w_j \frac{h_{n+1}^2}{2h_1}\,\cdot \\
&\quad \cdot \big\{v_{n+1}|\partial^2_{n+1}u(\bar{\mathbf{t}}_{\mathbf{l}}) - \partial^2_{n+1}u(\bar{\bar{\mathbf{t}}}_{\mathbf{l}})| + (1-v_{n+1})|\partial^2_{n+1}u(\tilde{\mathbf{t}}_{\mathbf{l}}) - \partial^2_{n+1}u(\tilde{\tilde{\mathbf{t}}}_{\mathbf{l}})|\big\} \le \\
&\le \frac{1}{4}\sum_{j=1}^{n} h_j\omega(d;\partial_k\partial_j u) + \frac{1}{8}\sum_{j=1}^{n}\frac{h_j^2}{h_1}\omega(d;\partial_j^2 u) + \\
&\quad + \frac{1}{4}h_{n+1}\omega(d;\partial_1\partial_{n+1}u) + \frac{h_{n+1}^2}{8h_1}\omega(d;\partial^2_{n+1}u),
\end{aligned}
$$

for some $\xi_{n+1},\theta_{n+1},\bar{\mathbf{t}}_{\mathbf{l}},\bar{\bar{\mathbf{t}}}_{\mathbf{l}},\tilde{\mathbf{t}}_{\mathbf{l}},\tilde{\tilde{\mathbf{t}}}_{\mathbf{l}}$. For the second order partial derivatives we have the following estimates:

$$
\begin{aligned}
|\partial_1^2 u(\mathbf{t}) - \partial_1^2 S(\mathbf{t})| &= |\partial_1^2 u(t_1,\dots,t_n,t_{n+1}) - \partial_1^2 S^{(n+1)}_{\mathbf{i}}(t_1,\dots,t_n,t_{n+1})| \le \\
&\le v_{n+1}|\partial_1^2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1}) - \partial_1^2 S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1,\dots,t_n)| + \\
&\quad + (1-v_{n+1})|\partial_1^2 u(t_1,\dot{s},t_n,(\mathbf{t}_{\mathbf{i}})_{n+1}) - \partial_1^2 S^{(n)}_{\mathbf{i}}(t_1,\dots,t_n)| + \\
&\quad + v_{n+1}|\partial_1^2 u(t_1,\dots,t_n,t_{n+1}) - \partial_1^2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1})| + \\
&\quad + (1-v_{n+1})|\partial_1^2 u(t_1,\dots,t_n,t_{n+1}) - \partial_1^2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}})_{n+1})| \le \\
&\le (n+1)\omega(d;\partial_1^2 u) + \omega(d;\partial_1^2 u) = (n+2)\omega(d;\partial_1^2 u).
\end{aligned}
$$

Finally, for the second order mixed partial derivatives we prove the statement first in the case $n=2$. Substituting the following derivatives

$$
\begin{aligned}
\phi_1'(t) &= -6t(1-t), & \phi_2'(t) &= 6t(1-t), \\
\phi_3'(t) &= 3t^2-4t+1, & \phi_4'(t) &= 3t^2-2t,
\end{aligned}
$$

we have

$$
\begin{aligned}
&|\partial_1\partial_2 u(x,y) - \partial_1\partial_2 S_{i,j}(x,y)| = \\
&= \big|\partial_1\partial_2 u(x,y) + (6t^2-6t)\frac{1}{hl}\Delta^{1,1}u_{i,j} + (-1+4t-3t^2)\frac{1}{l}\big(u^{(1,0)}_{i,j+1} - u^{(1,0)}_{i,j}\big) + \\
&\quad + (2t-3t^2)\frac{1}{l}\big(u^{(1,0)}_{i+1,j+1} - u^{(1,0)}_{i+1,j}\big) +
\end{aligned}
$$

$$
\begin{aligned}
&+\ (2s-3s^2)\big[\frac{1}{hl}\Delta^{1,1}u_{i,j}-\frac{1}{h}\big(u^{(0,1)}_{i+1,j+1}-u^{(0,1)}_{i,j+1}\big)\big]+\\
&+\ (-1+4s-3s^2)\big[\frac{1}{hl}\Delta^{1,1}u_{i,j}-\frac{1}{hl}\big(u^{(0,1)}_{i+1,j}-u^{(0,1)}_{i,j}\big)\big]\big|\leq\\
\leq&\,|\partial_1\partial_2 u(x,y)-\frac{1}{hl}\Delta^{1,1}u_{i,j}|+\\
&+\ |3t^2-4t+1||\frac{1}{hl}\Delta^{1,1}u_{i,j}-\frac{1}{l}(u^{(1,0)}_{i,j+1}-u^{(1,0)}_{i,j})|+\\
&+\ |3t^2-2t||\frac{1}{hl}\Delta^{1,1}u_{i,j}-\frac{1}{l}(u^{(1,0)}_{i+1,j+1}-u^{(1,0)}_{i+1,j})|+\\
&+\ |2s-3s^2||\frac{1}{hl}\Delta^{1,1}u_{i,j}-\frac{1}{h}\big(u^{(0,1)}_{i+1,j+1}-u^{(0,1)}_{i,j+1}\big)\big|+\\
&+\ |-1+4s-3s^2||\frac{1}{hl}\Delta^{1,1}u_{i,j}-\frac{1}{hl}\big(u^{(0,1)}_{i+1,j}-u^{(0,1)}_{i,j}\big)\big|\leq\\
\leq&\,(1+1+1+1+1)\omega(d;\partial_1\partial_2 u)=5\omega(d;\partial_1\partial_2 u).
\end{aligned}
$$

Now we suppose that the statement is true f or n and by induction and by the recursive formula (without loss of generality) we prove the case $k=1, l=2$:

$$
\begin{aligned}
&|\partial_1\partial_2 u(\mathbf{t})-\partial_1\partial_2 S(\mathbf{t})|=\\
=&\,|\partial_1\partial_2 u(t_1,\dots,t_n,t_{n+1})-\partial_1\partial_2 S^{(n+1)}_{\mathbf{i}}(t_1,\dots,t_n,t_{n+1})|\leq\\
\leq&\,v_{n+1}|\partial_1\partial_2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1})-\partial_1\partial_2 S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1,\dots,t_n)|+\\
&+\ (1-v_{n+1})|\partial_1\partial_2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}})_{n+1})-\partial_1\partial_2 S^{(n)}_{\mathbf{i}}(t_1,\dots,t_n)|+\\
&+\ v_{n+1}|\partial_1\partial_2 u(t_1,\dots,t_n,t_{n+1})-\partial_1\partial_2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1})|+\\
&+\ (1-v_{n+1})|\partial_1\partial_2 u(t_1,\dots,t_n,t_{n+1})-\partial_1\partial_2 u(t_1,\dots,t_n,(\mathbf{t}_{\mathbf{i}})_{n+1})|+\\
&+\ (1-v_{n+1})v_{n+1}\frac{h_{n+1}}{h_2}\sum_{\substack{\mathbf{0}\leq\mathbf{l}\leq\mathbf{e}\\ l_1=0,l_2=0}}\prod_{j=3}^{n}w_j 2\omega(d;\partial_1\partial_{n+1}u)\leq\\
\leq&\,(n+\frac{5}{2})\omega(d;\partial_1\partial_2 u)+\frac{1}{2}\sum_{j=2}^{n}\frac{h_j}{h_2}\omega(d;\partial_1\partial_j u)+\\
&+\ \omega(d;\partial_1\partial_2 u)+\frac{1}{2}\frac{h_{n+1}}{h_2}\omega(d;\partial_1\partial_{n+1}u),
\end{aligned}
$$

and this proves our last statement.

References

[1] J.H. Ahlberg, E. N. Nilson and J. L. Walsh, The Theory of Splines and Their

Applications, *Academic Press*, *New York and London*, 1967.

[2] G. Baszenski, F. J. Delvos and H. Posdorf, Boolean methods in bivariate reduced Hermite interpolation, Multivariate Approximation Theory, Eds. W. Schempp, K. Zeller, *Birkhäuser*, *Basel*, 1979, pp. 47–58.

[3] G. Baszenski and L. L. Schumaker, Tensor products of abstract smoothing splines, Alfréd Haar Memorial Conference, Vol. I–II. (*Budapest,*1985), Colloq. Math. Soc. János Bolyai, 49, *North Holland*, *Amsterdam–New York*, 1987, pp. 181–192.

[4] G. Birkhoff,M. H. Schultz and R. S. Varga, Piecewise Hermite Interpolation in One and Two Variables with Applications to Partial Differential Equations, *Num. Math.* **11** (1968), 232–256.

[5] C. de Boor, A Practical Guide to Splines, *Springer Verlag*, *New York*, 1978.

[6] R. E. Carlson and F. N. Fritsch, Monotone piecewise bicubic interpolation, *SIAM J. Numer. Anal.* **22** (1985), 386–400.

[7] C. K. Chui, L. L. Schumaker and F. Utreras, Topics in Multivariate Approximation, *Academic Press*, *New York*, 1987.

[8] F. J. Delvos and H. Posdorf, A representation formula for reduced Hermite Interpolation, Numerische Methoden der Approximationstheorie, Eds. L. Collatz, G. Meinardus, H. Werner, *Birkhäus er*, *Basel*, 1977, pp. 124–137.

[9] R. F. Franke and L. L. Schumaker, A Bibliography of Multivariate Approximation, Proceedings of the International Workshop on Applied Multivariate Approximation, Eds. C. K. Chui, L. L. Schumaker, F. I. Utreras,, *Academic Press*, *New York*, 1987, pp. 275–335.

[10] W. J. Gordon, Distributive lattices and approximation of multivariate functions, Approximation with special emphasis on spline functions, Ed. I. J. Schoenberg, *Academic Press*, *New York, London*, 1969, pp. 223–277.

[11] J. C. Holladay, Smoothest curve approximation, *Math. Tables Aids Comput.* **11** (1957), 233–243.

[12] N. P. Korneichuk, Splines in approximation theory, *Nauka*, *Moscow*, 1984, (in Russian).

[13] P. Lancaster and D. S. Watkins, Interpolation in the plane and rectangular finite elements, *Numerische Behandlung von Differentialgleichungen, Band 2.*, Eds. J. Albrecht, L. Collatz, *ISNM* **31** (1976), 125–145.

[14] M. Lénárd, On the two dimensional spline interpolation of Hermite-type, Alfréd Haar Memorial Conference, Vol. I–II. (*Budapest,*1985), Colloq. Math. Soc. János Bolyai, 49, *North Holland*, *Amsterdam–New York*, 1987, pp. 531–541.

[15] M. Lénárd, On an n-dimensional quadratic spline approximation, *Journal of Approximation Theory* **68** No.2. (1992), 113–135.

[16] M. Lénárd, Multiple quadrature formulas by splines, *Annales Univ. Sci. Budapest, Sectio Computatorica* (*to appear*).

[17] F. Melkes, Reduced Picewise Bivariate Hermite Interpolations, *Numer. Math.* **19** (1972), 326–340.

[18] K. von Radziewski, Bivariate Hermite spline interpolation, Constructive Theory of Functions '87, *Sofia*, 1988, pp. 399–404.

[19] J. W. Schmidt and H. Späth, (eds.), Splines in Numerical Analysis (Mathematical Research 52), *Akademie Verlag*, *Berlin*, 1989.

[20] L. L. Schumaker, Spline Functions: Basic Theory, *Wiley*, *New York*, 1981.

[21] S. B. Stečkin and Ju. A. Subbotin, Splines in Numerical Mathematics, *Nauka*, *Moscow*, 1976, (in Russian).

[22] Yu. S. Zavialov, B. I. Kvasov and V. L. Miroshničenko, Methods of Spline Functions, *Nauka*, *Moscow*, 1980, (in Russian).

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