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Finsler spaces with the h-curvature tensor dependent on position alone

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Abstract. In a Finsler space the components of a tensor field are usually functions of position (x^i) and direction (y^i). The main purpose of the present paper is to consider Finsler spaces having h-curvature tensor whose components are functions of position alone.

1. Introduction

When we have devoted ourselves to the theory of Douglas spaces [3], we were greatly surprised and delighted at the discovery of the following remarkable fact: For a Douglas space the components ${W_i}^h{}_{jk}$ of the projective Weyl tensor are functions of position (x^i) alone.

In a Finsler space almost all tensor fields depend on E. Cartan's supporting element (x^i, y^i), that is, they are functions not on the underlying manifold but on the tangent bundle. We have obtained the rigorous definition of such a Finslerian tensor field ([1, 2.2.3]; [4, Definition 6.2]), and it is well-known that it is a singular case for a Finsler space to have some tensor fields dependent on position alone.

The main purpose of the present paper is to consider Finsler spaces whose h-curvature tensor depends on position alone.

Let $F^n = \{M^n, L(x, y)\}$ be an n-dimensional Finsler space on a smooth n-manifold M^n, equipped with the fundamental metric func-

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tion $L(x,y)$. When considering the extremals of the length integral $\int L(x, dx/dt)dt$, we obtain the functions

$$G^i(x,y) = g^{ij}\{(\dot\partial_j\partial_r F)y^r - \partial_j F\}, \qquad F = L^2/2,$$

and $G^i{}_j(x,y) = \dot\partial_j G^i$ constitute a nonlinear connection (or spray connection, [1, p. 72]). Then we get the Berwald connection $B\Gamma = \{G^i{}_j, G_j{}^i{}_k, 0\}$, where $G_j{}^i{}_k = \dot\partial_k G^i{}_j$ and the last term $\dot\partial$ means that the v-covariant differentiation ∇^v in $B\Gamma$ is nothing but $\partial/\partial y$.

F^n is called a Berwald space if the h-connection coefficients $G_j{}^i{}_k$ of $B\Gamma$ are functions of position (x^i) alone, that is, $G^i(x,y)$ are homogeneous polynomials in (y^i) of degree two. A Berwald space is similar to a Riemannian space and has certain characteristics as follows:

(1) The hv-curvature tensor $G_i{}^h{}_{jk} = \dot\partial G_i{}^h{}_j$ of $B\Gamma$ vanishes identically.

(2) The h-connection coefficients $F_j{}^i{}_k$ of the Cartan connection $C\Gamma = \{G^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k\}$ are functions of position alone [4, Proposition 25.1].

(3) The C-tensor $(C_j{}^i{}_k)$ is h-covariant constant $(C_j{}^i{}_{k|h} = 0)$ in $C\Gamma$.

(4) The hv-curvature tensor $F_i{}^h{}_{jk} = \dot\partial_k F_i{}^h{}_j$ of the Rund connection $R\Gamma = \{G^i{}_j, F_j{}^i{}_k, 0\}$ vanishes identically.

Thus the hv-curvature tensors of a Berwald space vanish in $B\Gamma$ and $R\Gamma$. On the other hand, a Landsberg space is characterized by the vanishing of the hv-curvature tensor $P_i{}^h{}_{jk}$ of $C\Gamma$.

Let us consider the h-curvature tensors in these Finsler connections. The h-curvature tensor $H = (H_i{}^h{}_{jk})$ in $B\Gamma$ is given by

$$H: \qquad H_i{}^h{}_{jk} = \delta_k G_i{}^h{}_j + G_i{}^r{}_j G_r{}^h{}_k - (j/k),$$

where $\delta_k = \partial_k - G^r{}_k\dot\partial_r$ and the symbol (j/k) denotes the interchange of the indices j,k. The h-curvature tensor $K = (K_i{}^h{}_{jk})$ in $R\Gamma$ is given by

$$K: \qquad K_i{}^h{}_{jk} = \delta_k F_i{}^h{}_j + F_i{}^r{}_j F_r{}^h{}_k - (j/k).$$

Hence (1) and (2) as above show that both H and K are functions of position alone for a Berwald space.

Consequenly, if we introduce three sets of a special kind of n-dimensional Finsler spaces as follows:

$B(n)$ ... Berwald spaces,

$Hx(n)$ ... spaces with the H dependent on position alone,

$Kx(n)$ ... spaces with the K dependent on position alone, then we obtain the inclusion relations

$$(1.1) \qquad (1) \quad B(n) \subset Hx(n), \qquad (2) \quad B(n) \subset Kx(n).$$

2. $Hx(n)$ and $Kx(n)$

The three connections, the Berwald connection $B\Gamma = \{G^i{}_j, G_j{}^i{}_k, 0\}$, the Cartan connection $C\Gamma = \{G^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k\}$, and the Rund connection $R\Gamma = \{G^i{}_j, F_j{}^i{}_k, 0\}$, have the same nonlinear connection $(G^i{}_j)$, and hence their $(v)h$-torsion tensors

$$R^1 : \qquad R^h{}_{jk} = \delta_k G^i{}_j - (j/k)$$

coincide. The h-curvature tensor $R = (R_i{}^h{}_{jk)}$ in $C\Gamma$ is written as [4, (18.2)]

$$(2.1) \qquad R : \qquad R_i{}^h{}_{jk} = K_i{}^h{}_{jk} + C_i{}^h{}_r R^r{}_{jk}.$$

On the other hand, the h-curvature tensor H in $B\Gamma$ is given by [1, (18.16)]

$$(2.2) \qquad H_i{}^h{}_{jk} = K_i{}^h{}_{jk} + \{P^h{}_{ij|k} + P^r{}_{ij} P^r{}_{rk} - (j/k)\},$$

where $P^h{}_{ij} = C_i{}^h{}_{j|0}$ are components of the $(v)hv$-torsion tensor of $C\Gamma$. We have the well-known relations:

$$(2.3) \qquad y^i H_i{}^h{}_{jk} = y^i K_i{}^h{}_{jk} = y^i R_i{}^h{}_{jk} = R^h{}_{jk},$$

and the H is simply constructed by [4, (18.22)]

$$(2.4) \qquad H_i{}^h{}_{jk} = \dot{\partial}_i R^h{}_{jk}.$$

Now we consider an $F^n \in Kx(n)$. Then we have from (2.3) and (2.4)

$$R^h{}_{jk} = y^r K_r{}^h{}_{jk}(x), \qquad H_i{}^h{}_{jk} = \dot{\partial}_i (y^r K_r{}^h{}_{jk}(x)) = K_i{}^h{}_{jk}(x),$$

which implies that H of F^n depends on position alone. Therefore

Theorem 1. *We have the inclusion relation $Kx(n) \subset Hx(n)$. For $F^n \in Kx(n)$, $H = K$ holds.*

Let us define further two sets:

$L(n)$ ... Landsberg spaces,
$S(n)$ ... spaces with vanishing stretch curvature.

The inclusion relations

$$B(n) \subset L(n) \subset S(n) \tag{2.5}$$

have already been given by L. Berwald in 1926 [4], [5], but the notion of stretch curvature has faded out of memory, except for C. Shibata's work in 1978 [6].

The stretch curvature tensor $\Sigma = (\Sigma_{hijk})$, reflecting the non-metrical property of $B\Gamma$, is written in the form [5]:

$$\Sigma_{hijk} = -y_r H_h{}^r{}_{jk.i} = 2(P_{hij|k} - P_{hik|j}), \tag{2.6}$$

where $.i = \dot{\partial}/\partial y^i$ and $P_{hij} = g_{hr} P^r{}_{ij}$. The latter gives $L(n) \subset S(n)$, because $F^n \in L(n)$ has $P^h{}_{ij} = 0$ and the former gives $Hx(n) \subset S(n)$, because $F^n \in Hx(n)$ has $H_h{}^r{}_{jk.i} = 0$. Therefore

Theorem 2. *We have the inclusion relations*

$$B(n) \subset Kx(n) \subset Hx(n) \subset S(n).$$

Next we deal with the intersections $L(n) \cap Hx(n)$ and $L(n) \cap Kx(n)$. We have the well-known relation [4, (18.14)]

$$G_j{}^i{}_k - F_j{}^i{}_k = P^i{}_{jk}.$$

From the characteristic $P^i{}_{jk} = 0$ of $F^n \in L(n)$ it follows that $B\Gamma = \{G^i{}_j, G_j{}^i{}_k, 0\} = \{G^i{}_j, F_j{}^i{}_k, 0\} = R\Gamma$, and hence

Theorem 3. $L(n) \cap Hx(n) = L(n) \cap Kx(n)$.

3. $Hx(2)$ and $Kx(2)$

The theory of two-dimensional Finsler spaces can be treated in terms of Berwald's orthonormal frame field (l, m) ([1, 3.5]; [4, §28]; [2]). The main scalar I and the h-scalar curvature R of a space F^2 are defined as

$$LC_{hij} = Im_h m_i m_j, \qquad R_{ihjk} = \varepsilon R G_{ih} G_{jk}, \tag{3.1}$$

where ε is the signature, the angular metric tensor $h_{ij} = \varepsilon m_i m_j$, and $G_{ij} = l_i m_j - l_j m_i$. Then we have the following expressions of the H- and of the K-tensor:

$$\begin{aligned} H_{ihjk} &= \varepsilon(RG_{ih} + R_{;2} m_i m_h) G_{jk}, \\ K_{ihjk} &= (\varepsilon R G_{ih} - RI m_i m_h) G_{jk}. \end{aligned} \tag{3.2}$$

In the two-dimensional case all the Bianchi identities in $C\Gamma$ are reduced to the trivial one, except (17.15) of [4] ([1, (3.5.2.4)]):

$$\varepsilon R_{;2} + RI + I_{,1,1} = 0. \tag{3.3}$$

Now the stretch curvature tensor Σ is written as $\Sigma_{hijk} = -2I_{,1,1}m_i m_j G_{jk}$. Therefore

Proposition 1. *A Finsler space F^2 belongs to $S(n)$, if and only if the main scalar I satisfies $I_{,1,1} = 0$.*

Thus (3.3) is reduced to $\varepsilon R_{;2} + RI = 0$ for an $F^2 \in S(2)$, and hence (3.2) shows

Theorem 4. $Kx(2) = Hx(2)$.

Now we deal with $Hx(2)$ only. First we recall two-dimensional Berwald spaces. $F^2 \in B(2)$ is characterized by $C_{hij|k} = 0$, that is, $I_{,1} = I_{,2} = 0$. Then one of the Ricci identities shows $I_{,1,2} - I_{,2,1} = -RI_{;2} = 0$. Consequently $F^2 \in B(2)$ is characterized by $I_{,1} = I_{,2} = 0$ and $B(2)$ is the disjoint union

$$B(2) = B_1(2) + B_2(2) + B_3(2), \tag{3.4}$$

$$\begin{cases} B_1(2) \dots R = 0, & I_{;2} \neq 0, \\ B_2(2) \dots R = 0, & I_{;2} = 0, \\ B_3(2) \dots R \neq 0, & I_{;2} = 0. \end{cases} \tag{3.4a}$$

Thus we have

$$\begin{aligned} &B_1(2) + B_2(2) \dots \text{ locally Minkowski spaces,} \\ &B_2(2) + B_3(2) \dots \text{ spaces with constant } I. \end{aligned} \tag{3.4b}$$

Now we deal with $Hx(2)$. Applying the formulae

$$\begin{gathered} Ll_{i.j} = \varepsilon m_i m_j, \qquad Ll^i{}_{.j} = \varepsilon m^i m_j, \qquad LG_{hk.j} = \varepsilon G_{hk} m_j, \\ Lm_{i.j} = -(l_i - \varepsilon I m_i) m_j, \qquad Lm^i{}_{.j} = -(l^i + \varepsilon I m^i) m_j, \end{gathered}$$

to $H_i{}^h{}_{jk} = \varepsilon\{R(l_i m^h - l^h m_i) + R_{;2} m_i m^h\} G_{jk}$, we obtain

$$LH_i{}^h{}_{jk.l} = \varepsilon\{(R_{;2;2} + \varepsilon I R_{;2}) m^h - 2(R_{;2} + \varepsilon I R) l^h\} m_i m_j G_{jk}.$$

Consequently it is necessary and sufficient for an $F^2 \in Hx(2)$ that

$$(1) \quad R_{;2;2} + \varepsilon I R_{;2} = 0, \qquad (2) \quad R_{;2} + \varepsilon I R = 0.$$

The latter holds, as mentioned above, and the former reduces to $I_{;2}R = 0$ by (2). Thus, similarly to the case of $B(2)$, we have

Theorem 5. *$F^2 \in Hx(2)$ satisfies $I_{,1,1} = 0$ and $I_{;2}R = 0$. $Hx(2)$ is the disjoint union*

$$Hx(2) = H_1(2) + H_2(2) + H_3(2),$$

$$\begin{cases} H_1(2) \ldots R = 0, & I_{;2} \neq 0, \\ H_2(2) \ldots R = 0, & I_{;2} = 0, \\ H_3(2) \ldots R \neq 0, & I_{;2} = 0. \end{cases}$$

Corollary 1. *The sets $B_i(2)$, $i = 1, 2, 3$, coincide with the intersections $H_i(2) \cap B(2)$, respectively.*

The T-tensor (T_{hijk}) ([1, (3.5.3.1)]; [4, (28.20)]) of F^2 is written as $LT_{hijk} = I_{;2} m_h m_i m_j m_k$. Then we have

Corollary 2. *An $F^2 \in H_i(2)$, $i = 2, 3$, has vanishing T-tensor.*

Remark. Since $L\dot{\partial}_i I = I_{;2} m_i, T = 0$ means that I depends on position alone.

We consider an $F^2 \in L(2) \cap Hx(2)$. F^2 is a Landsberg space if and only if $LC_{hij|0} = I_{,1} m_h m_i m_j = 0$, that is, $I_{,1} = 0$. Theorem 5 shows that

$$\begin{cases} L(2) \cap H_1(2) \ldots R = 0, & I_{,1} = 0, \ I_{;2} \neq 0, \\ L(2) \cap H_2(2) \ldots R = 0, & I_{,1} = 0, \ I_{;2} = 0, \\ L(2) \cap H_3(2) \ldots R \neq 0, & I_{,1} = 0, \ I_{;2} = 0. \end{cases}$$

On the other hand, one of the Ricci formulae gives $I_{,1;2} - I_{;2,1} = I_{,2}$. Hence $I_{,1} = I_{;2} = 0$ implies $I =$ constant, and hence (3.4) shows that $L(2) \cap H_i(2)$ is equal to $B_i(2)$ for $i = 2, 3$. Therefore

Theorem 6. (1) $L(2) \cap H_1(2) = B_i(2)$, $i = 2, 3$. (2) $L(2) \cap H_1(2) \supset B_1(2)$, *and* $F^2 \in L(2) \cap H_1(2)$ *belongs to* $B_1(2)$, *if and only if* $I_{,2} = 0$.

4. $Rx(n)$

We consider the h-curvature tensor $R = (R_i{}^h{}_{jk})$ of the Cartan connection $C\Gamma = \{G^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k\}$ and define the set

$Rx(n)\ldots$ spaces with the R dependent on position alone.

First let us define the Q-tensor as

$$Q_m{}^h{}_{kij} = P_m{}^h{}_{jk|i} + P_m{}^h{}_{ir} P^r{}_{jk} - (i/j).$$

Then, rewriting $|(= \nabla^v)$ by $.(= \partial/\partial y)$, one of the Bianchi identities (17.15) of [1] is written in the form

$$R_m{}^h{}_{ij.k} + S_m{}^h{}_{kr} R^r{}_{ij} + R_m{}^r{}_{ij} C_r{}^h{}_k - R_r{}^h{}_{ij} C_m{}^r{}_k + Q_m{}^h{}_{kij} = 0. \tag{4.1}$$

Consequently we have directly

Proposition 2. *A Finsler space* F^n *belongs to* $Rx(n)$, *if and only if*

$$S_m{}^h{}_{kr} R^r{}_{ij} + R_m{}^r{}_{ij} C_r{}^h{}_k - R_r{}^h{}_{ij} C_m{}^r{}_k + Q_m{}^h{}_{kij} = 0.$$

For an $F^n \in Rx(n)$ *we have from (2.3) and (2.4)*

$$H_i{}^h{}_{jk} = R^h{}_{jk.i} = (y^r R_r{}^h{}_{jk}(x))_{.i} = R_i{}^h{}_{jk}(x).$$

Thus we have

Theorem 7. $Rx(n) \subset Hx(n)$, *and* $F^n \in Rx(n)$ *has* $H_i{}^h{}_{jk} = R_i{}^h{}_{jk}(x)$.

We consider an F^n with vanishing Q-tensor. Then Proposition 2 gives

$$S_m{}^h{}_{kr} R^r{}_{ij} + R_m{}^r{}_{ij} C_r{}^h{}_k - R_r{}^h{}_{ij} C_m{}^r{}_k = 0. \tag{4.2}$$

Transvection by y^m yields $R^r{}_{ij} C_r{}^h{}_k = 0$, and consequently $S_m{}^h{}_{kr} R^r{}_{ij} = (C_m{}^s{}_r C_s{}^h{}_k - C_m{}^s{}_k C_s{}^h{}_r) R^r{}_{ij} = 0$. Thus (4.2) is reduced to

$$R_m{}^r{}_{ij} C_r{}^h{}_k - R_r{}^h{}_{ij} C_m{}^r{}_k = 0. \tag{4.3}$$

Conversely, if F^n with $Q = 0$ satisfies (4.3), then we have (4.2). Therefore

Theorem 8. *Let a Finsler space F^n satisfy $Q_m{}^h{}_{kij} = 0$. Then F^n belongs to $Rx(n)$, if and only if* (4.3) *holds identically.*

A Landsberg space is characterized by $P_i{}^h{}_{jk} = 0$ or $P^i{}_{jk} = 0$, and hence $Q = 0$. Therefore

Corollary 3. *The intersection $L(n) \cap Rx(n)$ is characterized by* (4.3).

We are specially interested in the two-dimensional case. We have in general

$$P^h{}_{ij} = I_{,1} m^h m_i m_j, \qquad LP_i{}^h{}_{jk} = I_{,1}(l_i m^h - l^h m_i) m_j m_k,$$

which gives $LQ_m{}^h{}_{kij} = I_{,1,1}(l_m m^h - l^h m_m) G_{ij} m_k$. Thus

Lemma. *The Q-tensor of the two-dimensional case vanishes, if and only if $F^2 \in S(2)$.*

The condition (4.3) of the two-dimensional case is written as $\varepsilon RIG_{ij} m_k (l_m m_h + l_h m_m) = 0$, that is, $RI = 0$. Therefore

Theorem 9. *$F^2 \in Rx(2)$ is a Riemannian space, provided that the h-scalar curvature R does not vanish.*

5. $D(n), W(n)$ and $Wx(n)$

We have two projectively invariant tensors which play a leading role in the projective theory of paths and Finsler spaces [1, Chapter 0]. One is the *Douglas tensor* $D = (D_i{}^h{}_{jk})$:

$$D_i{}^h{}_{jk} = G_i{}^h{}_{jk} - [G_{ijk} y^h + \{G_{ij} \delta^h{}_k + (i,j,k)\}]/(n+1), \tag{5.1}$$

where $G = (G_i{}^h{}_{jk})$ is the hv-curvature tensor in $B\Gamma$, $G_{ij} = G_i{}^r{}_{jr}$ the hv-Ricci tensor, $G_{ijk} = G_{ij.k}$, and the symbol (i,j,k) denotes the cyclic permutation of the indices i, j and k.

The other is the *Weyl tensor* $W = (W_i{}^h{}_{jk})$:

$$W_i{}^h{}_{jk} = H_i{}^h{}_{jk} + \{\delta^h{}_i H_{jk} + y^h H_{jki} + \delta^h{}_j H_{k.i} - (j/k)\}/(n+1), \tag{5.2}$$

where $H_{jk} = H_j{}^r{}_{kr}$ is the h-Ricci tensor in $B\Gamma$, $H_{jki} = H_{jk.i}$ and $H_k = (nH_{rk} + H_{kr}) y^r/(n-1)$.

The notion of Douglas space, arising from the problem of the equations of the geodesics, has been proposed by the present authors [3] and yields interesting topics in Finsler geometry. A Finsler space is a Douglas space, if and only if the Douglas tensor D vanishes identically. Let us define the set

$$D(n)\ldots \text{ Douglas spaces.}$$

It has been proved that $L(n) \cap D(n) = B(n)$ [3, I]. As a consequence, from (2.5) we may say that $D(n)$ is a generalization of $B(n)$ in a completely different direction from $L(n)$.

On the other hand, according to Z. Szabó's theorem [7], a Finsler space F^n, $n > 2$, is of scalar curvature if and only if the Weyl tensor W vanishes identically. Thus, if we define the sets

$$W(n),\ \ n > 2, \ldots \text{ spaces of scalar curvature,}$$
$$W_0(n),\ n > 2, \ldots \text{ spaces of non-zero scalar curvature,}$$

then we may state one of the fundamental theorems of the projective theory [3, II] as follows: F^n, $n > 2$, is with rectilinear extremals or projectively flat, if and only if $F^n \in D(n) \cap W(n)$.

It is well-known from S. Numata's theorem [4, Theorem 30.6] that $L(n) \cap W_0(n) \ni F^n$ is nothing but a Riemannian space of non-zero constant curvature. This theorem has been generalized by C. Shibata [6], to whom we referred in §2: $S(n) \cap W_0(n)$ is still the set of Riemannian spaces of non-zero constant curvature.

Therefore we already know the following inclusion relations:

Proposition 3.

(1) $D(n) \cap L(n) = B(n)$,

(2) $n > 2$, $D(n) \cap W(n) =$ *(spaces with rectilinear extremals)*,

(3) $n > 2$, $S(n) \cap W_0(n) = L(n) \cap W_0(n) =$ *(Riemannian spaces of non-zero constant curvature)*.

Now we observe (5.2) for an $F^n \in Hx(n)$.

$$H_{jk} = H_i{}^r{}_{kr}(x), \qquad H_{jki} = 0, \qquad H_{k.i} = \{nH_{ik}(x) + H_{ki}(x)\}/(n-1).$$

Hence the tensor W depends on position alone. Thus we define

$$Wx(n)\ldots \text{ spaces with the } W \text{ dependent on position alone,}$$

and we have $Hx(n) \subset Wx(n)$.

Next we have proved $D(n) \subset Wx(n)$ [3, II] making use of the characteristics $Q_i{}^j{}_k(x)$ of $F^n \in D(n)$. Consequently we have

Proposition 4. *$Hx(n) \subset Wx(n)$ and $D(n) \subset Wx(n)$.*

Now we have an interesting problem, namely to consider the intersection $Hx(n) \cap D(n)$.

To deal with this problem, we first consider a Douglas space $F^n \in D(n)$. It follows from (5.1) that $D = 0$ gives

$$(n+1)G_i{}^h{}_{jk} = G_{ijk}y^h + \{G_{ij}\delta^h{}_k + (i,j,k)\}.$$

Transvection by the angular metric tensor $h^l{}_h = \delta^l{}_h - y^l y_h/L^2$ leads to

$$G_i{}^l{}_{jk} = G_i{}^0{}_{jk}y^l/L^2 + \{G_{ij}h^l{}_k + (i,j,k)\}/(n+1).$$

Consequently we obtain

$$\begin{aligned}(5.3)\qquad & G_i{}^l{}_{jk;h} - G_i{}^l{}_{jh;k} = \{G_i{}^0{}_{jk;h} - (k/h)\}y^l/L^2 \\ & +[\{G_{ij;h}h^l{}_k - (k/h)\} + \{G_{jk;h}h^l{}_i + G_{ik;h}h^l{}_j - (k/h)\}]/(n+1),\end{aligned}$$

where $; = \nabla^h$ in $B\Gamma$.

Secondly we shall recall one of the Bianchi identities in $B\Gamma$ ([1, (2.5.2.12]; [4, (18.21)]), corresponding to (4.1) in $C\Gamma$:

$$(5.4)\qquad H_i{}^l{}_{hk.j} + G_i{}^l{}_{jk;h} - G_i{}^l{}_{jh;k} = 0.$$

This yields directly

Proposition 5. *A Finsler space F^n belongs to $Hx(n)$, if and only if in $B\Gamma$*

$$G_i{}^l{}_{jk;h} - G_i{}^l{}_{jh;k} = 0.$$

Now we consider an $F^n \in D(n) \cap Hx(n)$. Then the above gives

$$G_{jk;h} - G_{jh;k} = 0, \quad G_i{}^0{}_{jk;h} - G_i{}^0{}_{jh;k} = 0.$$

Hence (5.3) reduces to $G_{ij;h}h^l{}_k - G_{ij;k}h^l{}_h = 0$, which implies $(n-2)G_{ij;h} + G_{ij;0}y_h/L^2 = 0$ and $G_{ij;0} = 0$. Thus we get $G_{ij;h} = 0$, provided $n > 2$.

Conversely, if $F^n \in D(n)$, $n > 2$, satisfies $G_{ij;k} = 0$ and

$$G_i{}^0{}_{jk;h} - G_i{}^0{}_{jh;k} = 0, \tag{5.5}$$

then (5.3) leads to $G_i{}^l{}_{jk;h} - G_i{}^l{}_{jh;k} = 0$, hence (5.4) shows that $F^n \in Hx(n)$. The condition (5.5) is nothing but $F^n \in S(n)$ because of (5.4) and (2.6) (1). Therefore

Theorem 10. *A Douglas space F^n, $n > 2$, belongs to $Hx(n)$, if and only if the hv-Ricci tensor G_{ij} is h-covariant constant in $B\Gamma$ and $F^n \in S(n)$, that is,* (5.5) *holds.*

We are concerned with the exceptional case $n = 2$ of Theorem 10. According to Theorem 5, $D(2) \cap Hx(2)$ is the direct sum

$$D(2) \cap Hx(2) = D(2) \cap H_1(2) + D(2) \cap H_2(2) + D(2) \cap H_3(2).$$

We have proved in our paper [3, I]: If a Douglas space F^2 has vanishing T-tensor, it is a Berwald space with constant main scalar I. Then Corollary 2 together with (3.4b) shows that $D(2) \cap H_i(2)$ coincides with $B_i(2)$ for $i = 2, 3$. On the other hand, (1) of Proposition 3 shows $B_1(2) \subset D(2)$ and (1.1) gives $B_1(2) \subset Hx(2)$. Therefore

Theorem 11.

(1) $D(2) \cap H_i(2) = B_i(2)$, $i = 2, 3$.

(2) $D(2) \cap H_1(2) \supset B_1(2)$ *and* $F^2 \in D(2) \cap H_1(2)$ *belongs to* $B_1(2)$, *if and only if* $I_{,1} = I_{,2} = 0$.

Finally we pay special attention to $F^2 \in Hx(2)$ having non-zero h-scalar curvature R. For these spaces Theorems 6 and 11 give a kind of reduction theorems to Berwald spaces as follows:

Theorem 12. *Let F^2 be a two-dimensional Finsler space having non-zero h-curvature tensor H dependent on position alone. If F^2 is a Landsberg space or a Douglas space, then F^2 is a Berwald space.*

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