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Isometric immersion of complete Riemannian manifolds

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Dedicated to Professor Béla Gyires on his 90th birthday

Abstract. Global isometric mappings φ of a Riemannian manifold M^n into a euclidean space E^{n+m} were investigated under different additional conditions on M^n, φ and m by a number of geometers. We mention here only the works of C. Tompkins [7], S. S. Chern and C. C. Hsiung [1], S. S. Chern and N. H. Kuiper [2], H. Jacobowitz [3].

Recently S. S. Yang [8] showed that in case of a compact M^n no euclidean ball $B(r) \subset E^{n+m}$ of radius r can contain $\varphi(M^n)$, provided $\|H(\varphi(x)\| < \frac{1}{\sqrt{m}\, r}$, $\forall x \in M$, where H denotes the mean curvature vector of $\varphi(M^n)$.

In this note we prove a similar theorem, replacing compactness of M^n by the weaker condition of completeness. Our result is somewhat stronger even in the case of compactness, if $m > 1$. Also isometric immersion $\varphi : M^n \to S^{n+m-1}$ of a complete M^n into a euclidean sphere S^{n+m-1} not pinched by certain geodesic balls of S^{n+m-1} is investigated. Corollary 1 concerns the diameter of $\varphi(M^n)$, and Corollary 2 concerns minimal submanifolds of the sphere.

1. Introduction

Global isometric mappings φ of a Riemannian manifold M into a euclidean space E were investigated under different additional conditions on M, φ and $\dim E$ by a number of geometers. In 1963 S. S. Chern and C. C. Hsiung [1] showed that there exists no isometric minimal immersion of a compact Riemannian manifold into a euclidean space. In 1998 S. S. Yang [8] proved the following

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Theorem (Yang [8])**.** Let $\varphi : M^n \to E^{n+m}$ be an isometric immersion of a compact Riemannian manifold M^n into the euclidean E^{n+m}, H the mean curvature vector of $\varphi(M^n)$, and $\|H\|$ the euclidean length of H. Then in case of

$$\|H(\varphi(x)\| < \frac{1}{\sqrt{m}\, r} \qquad \forall\, x \in M$$

no euclidean ball $B(r)$ of radius r can contain $\varphi(M^n)$.

Chern and Hsiung's above mentioned result is a consequence of this theorem. Indeed, since M^n is compact, so is $\varphi(M^n) \subset E^{n+m}$. Hence $\varphi(M^n)$ is bounded, and therefore contained in a sufficiently big $B(R)$. If moreover $\varphi(M^n)$ is a minimal surface, as in Chern and Hsiung's theorem, then $\|H\| \equiv 0$, and thus by Yang's theorem no ball $B(R)$ could contain $\varphi(M^n)$. This is a contradiction, what proves Chern and Hsiung's above cited result.

Yang concluded from the behaviour of $\|H\|$ to the fact that $\varphi(M^n)$ is not pinched into (not contained in) certain balls $B(r)$. However there are results which lead to this conclusion from a restriction on the sectional curvature. H. Jacobowitz [3] proved in 1973 that if the sectional curvature of a compact Riemannian manifold M^n is everywhere less than $\frac{1}{r^2}$, then there exists no isometric immersion of M^n into a ball $B(r) \subset E^{2n-1}$. This is a generalization of a result of S. S. Chern and N. H. Kuiper [2] from 1952. This result ([2; Theorem 5]) says (in an appropriate formulation) that a compact Riemannian manifold M^n with everywhere nonpositive sectional curvature cannot be isometrically embedded in E^{2n-1}. Also this result contains as a corollary the old theorem of C. Tompkins [7] from 1939, according to which the flat n-dimensional torus cannot be embedded isometrically in E^{2n-1}.

In this note we want to prove a theorem similar to Yang's theorem, replacing compactness of M^n by the weaker condition of completeness. Also isometric immersion $\varphi : M^n \to S^{n+m-1}$ of a complete Riemannian M^n into a euclidean sphere not pinched by certain geodesic balls of S^{n+m-1}, will be investigated.

2. Immersion into E^{n+m}

Theorem 1. *Let M^n be a complete Riemannian manifold whose sectional curvature K is bounded away from $-\infty$, and $\varphi : M^n \to E^{n+m}$ an* *isometric immersion into a euclidean space* E^{n+m}, *such that* $\varphi(M^n)$ *is a submanifold of* E^{n+m}. *Let us denote by* H *the mean curvature vector of* $\varphi(M^n)$, *and by* $\|H\|$ *the euclidean length of* H.

Then $\varphi(M^n)$ *cannot be contained in any ball* $B(r) \subset E^{n+m}$ *of radius* r *satisfying*

$$\limsup_{x\in M} \|H(\varphi(x))\| < \frac{1}{r}, \tag{1}$$

i.e. (1) $\Rightarrow$ $\varphi(M^n) \not\subset B(r)$.

Both Yang's above result ([8; Theorem 1]) and our Theorem conclude from $\|H\|$ on the upper bound of the radii r of those $B(r)$ which cannot contain $\varphi(M^n)$. However our assumption on M^n is completeness which is weaker than compactness the assumption in Yang's theorem. Moreover, since $\frac{1}{\|H\|} > \frac{1}{\sqrt{m}\|H\|}$ if $m > 1$, our result allows bigger balls not containing $\varphi(M^n)$, than the balls in Yang's theorem. So our theorem deduces from a weaker condition a stronger consequence.

We can achieve this by exploiting a result of H. Omori ([6] Theorem A.), saying: for any smooth and bounded function ϕ on a complete connected Riemannian manifold Q whose sectional curvatures K are bounded away from $-\infty$, and for any $\varepsilon > 0$ there exists a point $p \in Q$ such that $\|(\operatorname{grad}\phi)(p)\| < \varepsilon$ and at this p the Hessian $\mathcal{H}_\phi$ of ϕ for any unit vector $X \in T_pQ$ is smaller than $\varepsilon : \mathcal{H}_\phi(X,X) < \varepsilon$.

Proof of Theorem 1. Suppose that $\varphi(M^n) \subset B(r)$. We want to show that this assumption contradicts (1). – Without loss of generality we can assume that the center of this ball $B(r)$ is the origin O of the E^{n+m}. Then, denoting the position vector $\overrightarrow{O\varphi(x)}$ $x \in M$ by $F(x)$ and putting $\varphi(M^n) =: Q$,

$$\phi : \varphi(M^n) \to R, \qquad \phi(x) := \|F(x)\|^2$$

is a smooth function on Q bounded by r^2.

We want to calculate the Hessian $\mathcal{H}_\phi$ of ϕ. We know that $\mathcal{H}_\phi$ has the form

$$\mathcal{H}_\phi(X,X)\varphi(x) = XX\phi - (\nabla_X X)\phi, \qquad X \in \mathfrak{X}(Q), \tag{2}$$

where ∇ denotes the Riemannian connection of Q induced on it by the canonical euclidean connection ∇' of the ambient space E^{n+m}. The relation

$$\nabla'_X Y = \nabla_X Y + \sigma(X,Y), \qquad X,Y \in \mathfrak{X}(Q)$$

between ∇ and ∇', where σ denotes the second fundamental form belonging to the applied embedding, is well known. Applying this for $\mathcal{H}_\phi$ we obtain

$$\mathcal{H}_\phi(X,X) = XX\langle F,F\rangle - (\nabla'_X X)\langle F,F\rangle + \sigma(X,X)\langle F,F\rangle, \tag{3}$$

where $\langle\, ,\,\rangle$ denotes the euclidean scalar product. In our further calculation we want to make use of two relations. First

$$\nabla'_X F = X. \tag{4}$$

To see this we use in E^{n+m} Descartes coordinates y^A $A,B = \overline{1,n+m}$. Then $F(x) = \overrightarrow{O\varphi(x)} = y^A \frac{\partial}{\partial y^A}$ and $T_{\varphi(x)}Q \ni X = \dfrac{\partial y^B}{\partial x^i}\lambda^i(x)\frac{\partial}{\partial y^B} \equiv \xi^B \frac{\partial}{\partial y^B}$, $i = \overline{1,n}$ with arbitrary λ^i. Thus in E^{n+m} we have $\nabla'_X F = XF = \xi^B \dfrac{\partial y^A}{\partial y^B}\frac{\partial}{\partial y^A} = X$. – The second relation is well known:

$$Z\langle U,V\rangle = \langle \nabla'_Z U, V\rangle + \langle U, \nabla'_Z V\rangle\,, \qquad Z,U,V \in \mathfrak{X}(E^{n+m}). \tag{5}$$

Thus

$$X\langle F,F\rangle \overset{(5)}{=} 2\,\langle \nabla'_X F, F\rangle \overset{(4)}{=} 2\langle X,F\rangle \tag{6}$$

and

$$X\langle X,F\rangle \overset{(5)}{=} \langle \nabla'_X X, F\rangle + \langle X, \nabla'_X F\rangle\,,$$

that is

$$\langle \nabla'_X X, F\rangle \overset{(4)}{=} X\langle X,F\rangle - \langle X,X\rangle. \tag{7}$$

By using these relations we also get

$$\begin{aligned}(\nabla'_X X)\,\langle F,F\rangle &\overset{(5)}{=} 2\left\langle \nabla'_{\nabla'_X X} F, F\right\rangle \overset{(4)}{=} 2\,\langle \nabla'_X X, F\rangle \\ &\overset{(7)}{=} 2X\langle X,F\rangle - 2\langle X,X\rangle.\end{aligned} \tag{8}$$

Hence, by (3) (6) and (8)

$$\begin{aligned}\mathcal{H}_\phi(X,X) &= 2X\langle X,F\rangle - 2X\langle X,F\rangle + 2\langle X,X\rangle + 2\langle\sigma(X,X),F\rangle\\ &= 2\langle X,X\rangle + 2\langle\sigma(X,X),F\rangle.\end{aligned}$$

Let $E_i(x)$ be an orthonormal base of $T_{\varphi(x)}Q$. Then

$$\sum_{i=1}^{n}\mathcal{H}_\phi(E_i,E_i) = 2n\left(1+\left\langle\frac{1}{n}\sum_{i=1}^{n}\sigma(E_i,E_i),F\right\rangle\right) = 2n(1+\langle H,F\rangle).$$

Now, applying Omori's cited theorem, for any $\varepsilon>0$ there exists a $\varphi(x)=p\in Q$ such that $\mathcal{H}_\phi(X,X)<\varepsilon$ for any unit vector $X\in T_pQ$. Thus $\mathcal{H}_\phi(E_i,E_i)<2\varepsilon$, $\sum_{i=1}^{n}\mathcal{H}_\phi(E_i,E_i)<2n\varepsilon$, and

$$1+\langle H,F\rangle(p)<\varepsilon$$

$$\varepsilon-1>\langle H,F\rangle\geq-\|H\|\,\|F\|$$

$$\|H\|>\frac{1-\varepsilon}{\|F\|}.$$

According to our assumption $\varphi(M^n)=Q\subset B(r)$; i.e. $\|F\|<r$. So

$$\|H\|>\frac{1-\varepsilon}{r},\qquad\forall\,\varepsilon>0.$$

Hence $\limsup_{x\in M}\|H(\varphi(x)\|<\frac{1}{r}$. However this contradicts (1). Therefore in the case of (1) our assumption $\varphi(M^n)\subset B(r)$ cannot be true. $\square$

We can use our result for an estimate of the diameter d of $\varphi(M^n)\subset E^{n+m}$.

An old result of H. E. Jung [4] says that: Each subset of E^{n+m} of diameter not greater than d lies in a ball $B(R)\subset E^{n+m}$ of radius

(9) $$R\leq\sqrt{\frac{n+m}{2(n+m)+2}}\,d.$$

Let now $\varphi(M^n)$ of our Theorem 1 be this subset of E^{n+m}. Then $\varphi(M^n)\subset B(R)$ with an R satisfying (9). On the other hand, if M^n, φ

and r satisfy the conditions of our Theorem 1, then $\varphi(M^n)$ cannot be contained in $B(r) : \varphi(M^n) \not\subset B(r)$. Hence

$$r < R \leq \sqrt{\frac{n+m}{2(n+m)+2}}\, d,$$

i.e. we obtain

Corollary 1. *If M^n, φ and r satisfy the conditions of our Theorem 1, then the diameter d of $\varphi(M^n)$ is greater than $\sqrt{\frac{2(n+m)+2}{n+m}}\, r$:*

$$d(\varphi(M^n)) \geq \sqrt{\frac{2(n+m)+2}{n+m}}\, r.$$

3. Immersion into S^{n+m-1}

We want to prove a similar theorem in the case of immersion into a euclidean unit sphere S^{n+m-1}. We apply the notations of the previous section.

Theorem 2. *Let M^m be a complete Riemannian manifold whose sectional curvature K is bounded away from $-\infty$, and $\varphi : M^n \to S^{n+m-1}$ an isometric immersion such that $\varphi(M^n)$ is a submanifold of S^{n+m-1}.*

Then $\varphi(M^n)$ cannot be contained in any geodesic ball $\widetilde{B}(r)$ of S^{n+m-1} of radius r satisfying

$$\limsup_{x \in M} \|H\varphi(x)\| < \frac{\cos r}{2 \sin \frac{r}{2}}. \tag{10}$$

Proof. The proof runs the same way as the previous theorem with some minor differences.

Let ∇ and $\widetilde{\nabla}$ be the Levi–Civita connections on M^n and $S^{n+m-1}(\subset E^{n+m})$ resp., and ∇' the natural connection on E^{n+m}. Then

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \qquad \nabla'_X Y = \widetilde{\nabla}_X Y - \langle X, Y\rangle N$$

and

$$\nabla_X Y = \nabla'_X Y + \langle X, Y\rangle N - \sigma(X,Y) \qquad X, Y \in \mathfrak{X}(Q), \tag{11}$$

where N is the outward unit normal vector of S^{n+m-1} in E^{n+m}. Then for the Hessian $\mathcal{H}_\phi$ of a smooth function ϕ on Q we have (2), and by (11) we obtain

$$\mathcal{H}_\phi(X,X) = (XX - \nabla'_X X + \langle X,X\rangle N - \sigma(X,X))\,\phi.$$

Let A be the position vector of the center of a geodesic ball $\widetilde{B}(r)$ on S^{n+m-1} and $F(x)$ again the position vector $\overrightarrow{O\varphi(x)}$. Then $\phi(x) := \|F(x) - A\|^2$ is a smooth and bounded function on $\varphi(M^n) = Q \subset S^{n+m-1}$. A calculation using (3), (6), (8), the facts that $N = F$ on $Q \subset S^{n+m-1}$ and $\nabla'_X A = 0$, yields, similarly as in Section 2, that

$$\mathcal{H}_\phi(X,X) = 2(\langle X,X\rangle - \langle X,X\rangle\langle F,F-A\rangle + \langle\sigma(X,X),F-A\rangle).$$

For a unit vector X this gives

$$\mathcal{H}_\phi(X,X) = 2(1 - \langle F,F-A\rangle + \langle\sigma(X,X),F-A\rangle).$$

Let now again $E_i(p)$ $p = \varphi(x)$ be an orthonormal base of T_pQ. Then

$$\begin{aligned}\sum_{i=1}^n \mathcal{H}_\phi(E_i,E_i) &= 2n\left(1 + \left\langle \frac{1}{n}\sum_{i=1}^n \sigma(E_i,E_i), F-A\right\rangle - \langle F,F-A\rangle\right)\\ &= 2n(1 + \langle H,F-A\rangle - \langle F,F-A\rangle).\end{aligned}$$

Since ϕ is smooth and bounded, Q as the isometric immersion of M^n is connected and complete, and its sectional curvature is bounded away from $-\infty$, we may apply Omori's theorem, according to which we have a $\varphi(x) = p \in Q$ for any $\varepsilon > 0$ such that $\mathcal{H}_\phi(E_i,E_i)(p) < 2\varepsilon$. Hence

$$\langle H(p),F-A\rangle + 1 - \langle F,F-A\rangle = \langle H(p),F-A\rangle + 1 - \langle F,F\rangle + \langle F,A\rangle < \varepsilon.$$

Taking into consideration that $\|F\|^2 = 1$, and denoting $\langle F(\varphi(x),A\rangle$ by $\Theta(\varphi(x)$, we obtain

$$\varepsilon - \Theta(p) > \langle H(p),F-A\rangle \geq -\|H(p)\|\cdot\|F-A\| = -\|H(p)\| 2\sin\frac{\Theta(p)}{2}.$$

Thus to any $\varepsilon > 0$ there exists a $p \in Q$, such that

$$\|H(p)\| > \frac{\cos\Theta(p) - \varepsilon}{2\sin\frac{\Theta(p)}{2}} \qquad p \in Q.$$

Assume now that $Q \subset \widetilde{B}(r)$ $(r < \pi)$. Then $\Theta(\varphi(x)) \le r$ and thus $\sin \frac{\Theta(\varphi(x))}{2} \le \sin \frac{r}{2}$ and $\cos \Theta(\varphi(x)) \ge \cos r$. Hence

$$\|H(p)\| > \frac{\cos r - \varepsilon}{2 \sin \frac{r}{2}},$$

and thus

$$\limsup_{x \in M} \|H(\varphi(x))\| \ge \frac{\cos r}{2 \sin \frac{r}{2}}. \tag{12}$$

However in case of (10), (12) cannot be true, and thus our assumption cannot hold; i.e. in case of (10) there exists no $\widetilde{B}(r)$ containing $\varphi(M^n) = Q$. □

Finally we want still to show that an interesting result of S. B. Myers is a consequence of a corollary of our last theorem.

First we remark that we have confined r in our Theorem 2 by π, however, because of the inequality (9), it must be smaller even than $\frac{\pi}{2}$.

Let now $\varphi(M^n)$ be a complete and *minimal* $(H = 0)$ submanifold of S^{n+m}. Then by our Theorem 2 $\varphi(M^n)$ cannot be contained in a ball $\widetilde{B}(r)$, $r < \frac{\pi}{2}$. So we obtain

Corollary 2. *No complete minimal submanifold of a sphere can be contained in an open hemisphere if K is bounded away from $-\infty$.*

If M^n is compact and $m = 1$, then this is a result of S. B. Myers ([5; Theorem 4]).

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