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Continuity of multivalued maps in generalized uniform spaces

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Abstract. For compact-valued maps with values in generalized uniform spaces theorems on sets of semicontinuity points are given.

The notion of uniform space, expressed in terms of covers, has been generalized by Morita [3] and studied – among others – by Rinow and Poppe [4–6]. In general, a generalized uniform space need not be uniformizable, but under assumption of regularity various results similar to those in uniform spaces can be obtained.

In this paper we investigate continuity points of multivalued maps with values in generalized uniform spaces. We adopt some ideas from [2] and we extend Fort theorems, which were formulated for the metric case.

1. Basic notions

Let $\mathcal{A}$ be a cover of a set Y. For a point $x \in Y$ and a set $M \subset Y$ we will use the following notations: $\mathrm{St}(x,\mathcal{A}) = \bigcup\{A \in \mathcal{A} : x \in A\}$,

$$\mathrm{St}(M,\mathcal{A}) = \bigcup\{\mathrm{St}(x,\mathcal{A}) : x \in M\} = \bigcup\{A \in \mathcal{A} : M \cap A \neq \emptyset\},$$

$\mathrm{St}^{n+1}(M,\mathcal{A}) = \mathrm{St}(\mathrm{St}^n(M,\mathcal{A}))$ for $n = 1, 2, \dots$

If $\mathcal{A}$, $\mathcal{B}$ are two covers of Y, then we write $\mathcal{A} \prec \mathcal{B}$ iff $\mathcal{A}$ refines $\mathcal{B}$; we mean also that $\mathcal{A} \prec \mathcal{A}$.

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A system Σ of covers of a set Y is said to be a generalized uniform structure (or generalized uniformity) on Y if for each $\mathcal{A}_1, \mathcal{A}_2 \in \Sigma$ there is $\mathcal{A} \in \Sigma$ which refines both $\mathcal{A}_1$ and $\mathcal{A}_2$, [3]. The pair (Y, Σ) is called a generalized uniform space.

A generalized uniform structure Σ determines – in general – two topologies $\overline{\tau}_\Sigma$ and τ_Σ on Y in the following way: $U \in \overline{\tau}_\Sigma$ iff for each $x \in U$ there is $\mathcal{A} \in \Sigma$ with $\mathrm{St}(x, \mathcal{A}) \subset U$, [3]; the τ_Σ is given by the subbase $\{A \subset Y : A \in \mathcal{A},\ \mathcal{A} \in \Sigma\}$, [5], [6]. Then we have $\overline{\tau}_\Sigma \subset \tau_\Sigma$ and $\mathcal{A} \subset \tau_\Sigma$ for each $\mathcal{A} \in \Sigma$. In the sequel we will use the topology τ_Σ on Y. For a set $M \subset Y$ its closure and interior we denote by $\mathrm{Cl}(M)$ and $\mathrm{Int}(M)$, respectively.

A generalized uniform structure Σ on Y is said to be:

– Hausdorff, if for each points $x, y \in Y$, $x \neq y$ there exists $\mathcal{A} \in \Sigma$ such that $\mathrm{St}(x, \mathcal{A}) \cap \mathrm{St}(y, \mathcal{A}) \neq \emptyset$, [4];

– weakly regular, if for each finite number of sets $A_1, \dots, A_n$ with $A_i \in \mathcal{A}_i$ and $\mathcal{A}_i \in \Sigma$ for $i = 1, 2, \dots, n$; and for each point $x \in \bigcap_{i=1}^n A_i$ there exists $\mathcal{B} \in \Sigma$ such that $\mathrm{St}(x, \mathcal{B}) \subset \bigcap_{i=1}^n A_i$, [5], [6];

– regular, if it is weakly regular and for each $\mathcal{A} \in \Sigma$ there exists $\mathcal{B} \in \Sigma$ with the property, that for each $B \in \mathcal{B}$ there exists $\mathcal{G} \in \Sigma$ and $A \in \mathcal{A}$ such that $\mathrm{St}(B, \mathcal{G}) \subset A$, [3].

It simply follows from the definitions that:

(A) A generalized uniform structure Σ is weakly regular if and only if $\overline{\tau}_\Sigma = \tau_\Sigma$.

Lemma 1 [1]. *Assume that a generalized uniform structure Σ on Y is regular. If $U \subset Y$ is an open set, $M \subset U$ is compact, then for each $n = 1, 2, \dots$ there exists $\mathcal{G} \in \Sigma$ such that $\mathrm{St}^n(M, \mathcal{G}) \subset U$.*

Lemma 2 [1]. *If a generalized uniform structure Σ on Y is weakly regular, then for each set $M \subset Y$ it holds:*

$$\mathrm{Cl}(M) = \bigcap \{\mathrm{St}(M, \mathcal{G}) : \mathcal{G} \in \Sigma\}.$$

Given a generalized uniform space (Y, Σ) we will write $\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$ for every $\mathcal{A}, \mathcal{B} \in \Sigma$. Then, for any sets $U, V \subset Y$ it holds $\mathrm{St}(U \times V, \mathcal{A} \times \mathcal{B}) = \mathrm{St}(U, \mathcal{A}) \times \mathrm{St}(V, \mathcal{B})$. The class $\Sigma \times \Sigma = \{\mathcal{A} \times \mathcal{B} : \mathcal{A}, \mathcal{B} \in \Sigma\}$ is a generalized uniform structure on $Y \times Y$. Furthermore, it is easy to verify the following:

(B) If Σ is Hausdorff or weakly regular or regular, then $\Sigma \times \Sigma$ has the same property.

Lemma 3. *Let (Y,Σ) be a generalized uniform space. Then for induced topologies in $Y\times Y$ we have*

$$\overline{\tau}_\Sigma\times\overline{\tau}_\Sigma\subset\overline{\tau}_{\Sigma\times\Sigma}\subset\tau_{\Sigma\times\Sigma}=\tau_\Sigma\times\tau_\Sigma.$$

If Σ is weakly regular, then above topologies coincide.

Proof. Let $W\in\overline{\tau}_\Sigma\times\overline{\tau}_\Sigma$ and $(x,y)\in W$; then there exist sets $U,V\in\tau_\Sigma$ with $(x,y)\in U\times V\subset W$. Now we choose $\mathcal{G}\in\Sigma$ such that $\mathrm{St}(x,\mathcal{G})\subset U$ and $\mathrm{St}(y,\mathcal{G})\subset V$. Thus $\mathrm{St}((x,y),\mathcal{G}\times\mathcal{G})\subset U\times V\subset W$ and it means $W\in\overline{\tau}_{\Sigma\times\Sigma}$. Hence we obtain $\overline{\tau}_\Sigma\times\overline{\tau}_\Sigma\subset\overline{\tau}_{\Sigma\times\Sigma}$. Assume that $W\in\tau_\Sigma\times\tau_\Sigma$ and $(x,y)\in W$; let us take $U,V\in\tau_\Sigma$ such that $(x,y)\in U\times V\subset W$. Then we can choose $\mathcal{A}_1,\dots,\mathcal{A}_n,\mathcal{B}_1,\dots,\mathcal{B}_k\in\Sigma$ and the sets $A_i\in\mathcal{A}_i$ for $i=1,2,\dots,n$, $B_j\in\mathcal{B}_j$ for $j=1,\dots,k$, with

$$x\in\bigcap_{i=1}^n A_i\subset U\quad\text{and}\quad y\in\bigcap_{j=1}^k B_j\subset V.$$

Hence

$$(x,y)\in\bigcap_{i=1}^n\bigcap_{j=1}^k(A_i\times B_j)\subset W.$$

Since $A_i\times B_j\in\mathcal{A}_i\times\mathcal{B}_j\in\Sigma\times\Sigma$, we obtain $W\in\tau_{\Sigma\times\Sigma}$ and in the consequence $\tau_\Sigma\times\tau_\Sigma\subset\tau_{\Sigma\times\Sigma}$.

Conversely, let $W\in\tau_{\Sigma\times\Sigma}$ and $(x,y)\in W$. Then for some $\mathcal{A}_i,\mathcal{B}_i\in\Sigma$, $A_i\in\mathcal{A}_i$ and $B_i\in\mathcal{B}_i$ for $i=1,2,\dots,n$ it holds

$$(x,y)\in\bigcap_{i=1}^n(A_i\times B_i)\subset W.$$

Since $A_i,B_i\in\tau_\Sigma$ for $i=1,\dots,n$,we have $\bigcap_{i=1}^n(A_i\times B_i)\in\tau_\Sigma\times\tau_\Sigma$; so $W\in\tau_\Sigma\times\tau_\Sigma$. Hence we have shown $\tau_{\Sigma\times\Sigma}\subset\tau_\Sigma\times\tau_\Sigma$. If Σ is weakly regular, then the conclusion follows from the equality $\overline{\tau}_\Sigma=\tau_\Sigma$. □

For any sets $H_1,H_2\subset Y\times Y$ we denote $H_2\bullet H_1=\{(x,y)\in Y\times Y:$ there is $z\in Y$ with $(x,z)\in H_1$ and $(z,y)\in H_2\}$. Furthermore, for each $\mathcal{A}\in\Sigma$ we will write

$$H_\mathcal{A}=\bigcup\{A\times A:A\in\mathcal{A}\}.$$

It is easy to verify that for each set $M \subset Y \times Y$ and $\mathcal{A}, \mathcal{B} \in \Sigma$ we have

$$\mathrm{St}(M, \mathcal{A} \times \mathcal{B}) = H_{\mathcal{B}} \bullet M \bullet H_{\mathcal{A}}.$$

Thus applying Lemma 2 to the space $(Y \times Y, \Sigma \times \Sigma)$, according to (B) we obtain:

Corollary 1. *Let (Y, Σ) be a generalized uniform space with a weakly regular structure. Then for each set $M \subset Y \times Y$ it holds*

$$\mathrm{Cl}(M) = \bigcap \{H_{\mathcal{B}} \bullet M \bullet H_{\mathcal{A}} : \mathcal{A}, \mathcal{B} \in \Sigma\} = \bigcap \{H_{\mathcal{A}} \bullet M \bullet H_{\mathcal{A}} : \mathcal{A} \in \Sigma\}.$$

Let X, Z be topological spaces and let $F : X \to Z$ be a multivalued map. For each $A \subset X$, $B \subset Z$ we will denote $F(A) = \bigcup\{F(x) : x \in A\}$, $F^{+}(B) = \{x \in X : F(x) \subset B\}$ and $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. We remind that a multivalued map F is called upper (lower) semicontinuous at a point $x_0 \in X$ if for each open set $W \subset Z$ such that $F(x_0) \subset W$ (resp. $F(x_0) \cap W \neq \emptyset$) there exists a neighbourhood U of x_0 with $F(x) \subset W$ (resp. $F(x) \cap W \neq \emptyset$) for each $x \in U$. The set of all points at which F is upper or lower semicontinuous will be denoted by $C^{+}(F)$ or $C^{-}(F)$ respectively. A map F is called upper (lower) semicontinuous if $C^{+}(F) = X$ (resp. $C^{-}(F) = X$). In virtue of Lemma 1 we can formulate the following characterization of the upper semicontinuity:

(C) Let X be a topological space, (Y, Σ) a generalized uniform one with a regular structure and let $F : X \to Y$ be a multivalued map with compact values. The map F is upper semicontinuous at $x \in X$ if and only if for each $\mathcal{A} \in \Sigma$ there exists a neighbourhood U of x with $F(U) \subset \mathrm{St}(F(x), \mathcal{A})$.

Now we are going to give the similar "symmetrical" characterization of the lower semicontinuity. For a filter base $\mathcal{P}$ we will write $F(\mathcal{P}) = \{F(G) : G \in \mathcal{P}\}$. We will also apply the following:

Lemma 4. *Let X, Z be topological spaces. A multivalued map $F : X \to Z$ is lower semicontinuous at a point $x \in X$ if and only if for each filter base $\mathcal{P}$ in X that converges to x, every point $y \in F(x)$ is a cluster point of the filter base $F(\mathcal{P})$,* [7].

2. Main theorems

Theorem 1. *Let X be a topological space and (Y,Σ) a generalized uniform one with a regular Hausdorff structure. For a multivalued map $F: X \to Y$ with compact values the following conditions are equivalent:*

(a) *F is lower semicontinuous at a point $x_0 \in X$;*

(b) *for each $\mathcal{A} \in \Sigma$ there exists a neighbourhood U of x_0 such that $F(x_0) \subset \mathrm{St}(F(x),\mathcal{A})$ for each $x \in U$;*

(c) *for each $\mathcal{A} \in \Sigma$ and $n = 1,2,\dots$ there exists a neighbourhood U of x_0 such that $F(x_0) \subset \mathrm{St}^n(F(x),\mathcal{A})$ for each $x \in U$.*

Proof. Let (a) be satisfied; we assume to contrary that (b) does not hold. Then there exists $\mathcal{A} \in \Sigma$ such that each neighbourhood U of x_0 contains a point x_U for which $F(x_0) \not\subset \mathrm{St}(F(x_U),\mathcal{A})$. By S we denote the family of all open neighbourhoods of x_0 and for any $U_1, U_2 \in S$ we assume $U_1 \le U_2$ iff $U_2 \subset U_1$; so $(S,\le)$ is a directed set. For each $U \in S$ we fix a point $y_U \in F(x_0) \setminus \mathrm{St}(F(x_U),\mathcal{A})$. Then $\{y_U : U \in S\}$ is a net in the compact set $F(x_0)$; by y_0 we denote a cluster point of this net. Now, for $\mathcal{A}$ we choose $\mathcal{B} \in \Sigma$ that satisfies the condition from the definition of regular structure. We fix $B \in \mathcal{B}$, $\mathcal{G} \in \Sigma$ and $A \in \mathcal{A}$ such that $y_0 \in B$ and $\mathrm{St}(B,\mathcal{G}) \subset A$. Since B is a neighbourhood of y_0, the set $S_1 = \{U \in S : y_U \in B\}$ is cofinal in S. For $U \in S_1$ we put

$$\overline{U} = \{x_V : V \in S_1, V \subset U\} \quad \text{and} \quad \mathcal{P} = \{\overline{U} : U \in S_1\};$$

thus $\mathcal{P}$ is a filter base in X that converges to x_0. On the other hand for each $U \in S_1$ we have $y_U \in B \subset \mathrm{St}(B,\mathcal{G}) \subset A$, so $A \subset \mathrm{St}(y_U,\mathcal{A})$. Since $\mathrm{St}(y_U,\mathcal{A}) \cap F(x_U) = \emptyset$ we obtain $A \cap F(x_U) = \emptyset$ for $U \in S_1$, and in the consequence $\mathrm{St}(y_0,\mathcal{G}) \cap F(x_U) = \emptyset$ for $U \in S_1$. This implies that $\mathrm{St}(y_0,\mathcal{G}) \cap F(\overline{U}) = \emptyset$ for each $\overline{U} \in \mathcal{P}$, so y_0 is not a cluster point of the filter base $F(\mathcal{P})$. According to Lemma 4, F is not lower semicontinuous at x_0; hence the implication $(a) \Rightarrow (b)$ is proved. Evidently (b) implies (c); now we are going to show that (c) implies (a). Let n be fixed, $W \in \tau_\Sigma$ such that $F(x_0) \cap W \ne \emptyset$ and let $y_0 \in F(x_0) \cap W$. According to Lemma 1 we can choose $\mathcal{A} \in \Sigma$ with $\mathrm{St}^n(y_0,\mathcal{A}) \subset W$. Following (c) there is a neighbourhood U of x_0 such that $F(x_0) \subset \mathrm{St}^n(F(x),\mathcal{A})$ for each $x \in U$. Then $y_0 \in \mathrm{St}^n(F(x),\mathcal{A})$ for $x \in U$, so $F(x) \cap \mathrm{St}^n(y_0,\mathcal{A}) \ne \emptyset$. Thus $F(x) \cap W \ne \emptyset$ for $x \in U$ and the proof is completed. □

Lemma 5. *Assume that a generalized uniform structure Σ on Y is regular. If $D \subset Y$ is a compact set, then for each finite open cover $\{U_1, \dots, U_m\}$ of D there is $\mathcal{G} \in \Sigma$ such that $\{\mathrm{St}(x, \mathcal{G}) : x \in D\} \prec \{U_1, \dots, U_m\}$.*

Proof. For each $i \in \{1, \dots, m\}$ and each $x \in D \cap U_i$, according to Lemma 1, we can select $\mathcal{A} = \mathcal{A}(x,i) \in \Sigma$ with $\mathrm{St}^2(x, \mathcal{A}(x,i)) \subset U_i$. The open cover $\{\mathrm{St}(x, \mathcal{A}(x,i)) : i = 1, \dots, m;\ x \in D \cap U_i\}$ contains a finite subcover; thus for each $i = 1, \dots, m$ there exist points $x_{i,1}, \dots, x_{i,n(i)} \in D \cap U_i$ and $\mathcal{A}_{i,1}, \dots, \mathcal{A}_{i,n(i)} \in \Sigma$ such that $\mathrm{St}^2(x_{i,j}, \mathcal{A}_{i,j}) \subset U_i$ for $i = 1, \dots, m$; $j = 1, \dots, n(i)$ and

$$D \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n(i)} \mathrm{St}(x_{i,j}, \mathcal{A}_{i,j}).$$

By Lemma 1, $\mathcal{G} \in \Sigma$ can be taken such that $\mathcal{G} \prec \mathcal{A}_{i,j}$ for $i = 1, \dots, m$; $j = 1, \dots, n(i)$ and

$$\mathrm{St}(D, \mathcal{G}) \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n(i)} \mathrm{St}(x_{i,j}, \mathcal{A}_{i,j}).$$

If $x \in D$, then $x \in \mathrm{St}(x_{i,j}, \mathcal{A}_{i,j})$ for some $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n(i)\}$. Thus for $y \in \mathrm{St}(x, \mathcal{G})$ we have $y \in \mathrm{St}^2(x_{i,j}, \mathcal{A}_{i,j}) \subset U_i$. This gives $\{\mathrm{St}(x, \mathcal{G}) : x \in D\} \prec \{U_1, \dots, U_m\}$ which finishes the proof. □

Theorem 2. *Let X be a topological space and (Y, Σ) a generalized uniform one with a regular Hausdorff structure. If $F : X \to Y$ is a lower semicontinuous map with compact values, then $X \setminus C^+(F) = \bigcup\{K(\mathcal{A}) : \mathcal{A} \in \Sigma\}$, where $K(\mathcal{A})$ are of the first category.*

Proof. For a compact set $D \subset Y$ and $\mathcal{A} \in \Sigma$ we denote $M_{\mathcal{A}}[D] = \sup\{m$: there exist points $y_1, \dots, y_m \in D$ with $(y_i, y_j) \notin \mathrm{Cl}(H_{\mathcal{A}})$ for $i, j \in \{1, \dots, m\},\ i \neq j\}$, if $D \times D \not\subset \mathrm{Cl}(H_{\mathcal{A}})$ and $M_{\mathcal{A}}[D] = 0$ in the other case.

We will show that $M_{\mathcal{A}}[D] < \infty$. Assume to contrary $M_{\mathcal{A}}[D] = \infty$. For each $m = 2, 3, \dots$ there exist points $y_{m1}, \dots, y_{mm} \in D$ such that $(y_{mi}, y_{mj}) \notin \mathrm{Cl}(H_{\mathcal{A}})$ for $i, j \in \{1, \dots, m\}$, $i \neq j$. Let $A_1, \dots, A_k \in \mathcal{A}$ be fixed with $D \subset A_1 \cup \ldots \cup A_k$. Following Lemma 5 we can take $\mathcal{G} \in \Sigma$,

$\mathcal{G} \prec \mathcal{A}$ such that $\{\mathrm{St}(x,\mathcal{G}) : x \in D\} \prec \{A_1,\dots,A_k\}$. Now we fix $x_1,\dots,x_n \in D$ for which it holds $D \subset \bigcup_{i=1}^n \mathrm{St}(x_i,\mathcal{G})$. Then for $m = 2n$ there are $i,j \in \{1,\dots,m\}$, $i \neq j$, $p \in \{1,\dots,n\}$ and $s \in \{1,\dots,k\}$ such that $y_{mi}, y_{mj} \in \mathrm{St}(x_p,\mathcal{G}) \subset A_s$; hence $(y_{mi}, y_{mj}) \in H_{\mathcal{A}}$ which is impossible. Thus we have shown $M_{\mathcal{A}}[D] < \infty$.

Now for each $n = 1,2,\dots$ and $\mathcal{A} \in \Sigma$ by $B_n(\mathcal{A})$ we denote the set of all points $x \in X$ at which the following two conditions are satisfied:

(1) $M_{\mathcal{A}}[F(x)] \le n$;

(2) for each $\mathcal{G} \in \Sigma$ with $\mathcal{G} \prec \mathcal{A}$ and for each neighbourhood U of x there is a point $x' \in U$ such that $F(x') \not\subset \mathrm{St}^3(F(x),\mathcal{G})$.

Let us take $x \in X$, $\mathcal{A} \in \Sigma$ and let $M_{\mathcal{A}}[F(x)] = k > 0$. We choose points $y_1,\dots,y_k \in F(x)$ such that $(y_i,y_j) \notin \mathrm{Cl}(H_{\mathcal{A}})$ for $i,j \in \{1,\dots,k\}$, $i \neq j$. Following Corollary 1 there is $\mathcal{B} \in \Sigma$ such that $(y_i,y_j) \notin H_{\mathcal{B}} \bullet H_{\mathcal{A}} \bullet H_{\mathcal{B}}$ for $i,j \in \{1,\dots,k\}$, $i \neq j$. We take $\mathcal{G} \in \Sigma$ for which it holds $\mathrm{St}^2(y_i,\mathcal{G}) \subset \mathrm{St}(y_i,\mathcal{B})$ for $i \in \{1,\dots,k\}$; then

$$\mathrm{St}^2(y_i,\mathcal{G}) \cap \mathrm{St}(\mathrm{St}^2(y_j,\mathcal{G}),\mathcal{A}) = \emptyset \quad \text{for } i,j \in \{1,\dots,k\},\ i \neq j.$$

Since F is lower semicontinuous, there is a neighbourhood U of x such that $F(x) \subset \mathrm{St}(F(x'),\mathcal{G})$ for $x' \in U$. Let $x' \in U$ be fixed. We choose points $y'_1,\dots,y'_k \in F(x')$ with $y'_i \in \mathrm{St}(y_i,\mathcal{G})$ for $i \in \{1,\dots,k\}$. Suppose that for some $i,j \in \{1,\dots,k\}$ we have $(y'_i,y'_j) \in \mathrm{Cl}(H_{\mathcal{A}})$. Then $(y'_i,y'_j) \in H_{\mathcal{G}} \bullet H_{\mathcal{A}} \bullet H_{\mathcal{G}}$, so $\mathrm{St}(y'_i,\mathcal{G}) \cap \mathrm{St}(\mathrm{St}(y'_j,\mathcal{G}),\mathcal{A}) \neq \emptyset$. But $\mathrm{St}(y'_i,\mathcal{G}) \subset \mathrm{St}^2(y_i,\mathcal{G})$, so we obtain a contradiction. Hence $(y'_i,y'_j) \notin \mathrm{Cl}(H_{\mathcal{A}})$ for $i,j \in \{1,\dots,k\}$, $i \neq j$, and we have shown

(3) for each $x \in X$, $\mathcal{A} \in \Sigma$ there is a neighbourhood U of x such that $M_{\mathcal{A}}[F(x)] \le M_{\mathcal{A}}[F(x')]$ for $x' \in U$.

We are going to show

(4) $M_{\mathcal{A}}[F(z)] \le n$ for each $z \in \mathrm{Cl}(B_n(\mathcal{A}))$.

Suppose, contrary to our claim that for some $z \in \mathrm{Cl}(B_n(\mathcal{A}))$ we have $M_{\mathcal{A}}[F(z)] > n$. According to (3), there is a neighbourhood U of z such that $M_{\mathcal{A}}[F(x')] \ge M_{\mathcal{A}}[F(z)]$ for $x' \in U$. Then for each point $x'_1 \in U \cap B_n(\mathcal{A})$ it holds $M_{\mathcal{A}}[F(x'_1)] > n$. This contradicts (1), so (4) is proved.

Now, let $x_0 \in \mathrm{Cl}(B_n(\mathcal{A}))$, a neighbourhood V of x_0 and let $x \in B_n(\mathcal{A}) \cap V$ be given. Then $m = M_{\mathcal{A}}[F(x)] \le n$; at first suppose $m > 0$.

We choose points $y_1, \dots, y_m \in F(x)$ such that $(y_i, y_j) \notin \mathrm{Cl}(H_{\mathcal{A}})$ for $i, j \in \{1, \dots, m\}$, $i \neq j$. There exists $\mathcal{G} \in \Sigma$, $\mathcal{G} \prec \mathcal{A}$ such that $\mathrm{St}^2(y_i, \mathcal{G}) \subset \mathrm{St}(y_i, \mathcal{A})$ for $i \in \{1, \dots, m\}$ and $\mathrm{St}^3(y_i, \mathcal{G}) \cap \mathrm{St}(\mathrm{St}^3(y_j, \mathcal{G}), \mathcal{A}) = \emptyset$ for $i, j \in \{1, \dots, m\}$, $i \neq j$.

By the lower semicontinuity of F and since $x \in B_n(\mathcal{A})$, there is a point $y \in V$ for which we have $F(x) \subset \mathrm{St}(F(y), \mathcal{G})$ and $F(y) \not\subset \mathrm{St}^3(F(x), \mathcal{A})$. Then we take points $z_1, \dots, z_{m+1} \in F(y)$ with $y_i \in \mathrm{St}(z_i, \mathcal{G})$ for $i \in \{1, \dots, m\}$ and $z_{m+1} \in F(y) \setminus \mathrm{St}^3(F(x), \mathcal{A})$. Suppose $(z_i, z_{m+1}) \in H_{\mathcal{G}} \bullet H_{\mathcal{A}} \bullet H_{\mathcal{G}}$ for some $i \in \{1, \dots, m\}$. This gives
$z_{m+1} \in \mathrm{St}(\mathrm{St}(\mathrm{St}(z_i, \mathcal{G}), \mathcal{A}), \mathcal{G}) \subset \mathrm{St}(\mathrm{St}(\mathrm{St}^2(y_i, \mathcal{G}), \mathcal{A}), \mathcal{G}) \subset \mathrm{St}^3(y_i, \mathcal{A}) \subset \mathrm{St}^3(F(x), \mathcal{A})$, which is a contradiction. Hence $(z_i, z_{m+1}) \notin \mathrm{Cl}(H_{\mathcal{A}})$ for $i \in \{1, \dots, m\}$, so $M_{\mathcal{A}}[F(y)] \geq m + 1$.

In the other case, if $m = 0$, we fix $y_1 \in F(x)$ and $\mathcal{G} \in \Sigma$, $\mathcal{G} \prec \mathcal{A}$. By the same arguments as above, there is a point $y \in V$ for which $F(x) \subset \mathrm{St}(F(y), \mathcal{G})$ and $F(y) \not\subset \mathrm{St}^3(F(x), \mathcal{A})$. Taking $z_1 \in F(y)$ with $y_1 \in \mathrm{St}(z_1, \mathcal{G})$ and $z_2 \in F(y) \setminus \mathrm{St}^3(F(x), \mathcal{A})$ we obtain $(z_1, z_2) \in H_{\mathcal{G}} \bullet H_{\mathcal{A}} \bullet H_{\mathcal{G}}$; this leads to the conclusion $M_{\mathcal{A}}[F(y)] \geq 2$. Thus we have actually proved that

(5) if $x_0 \in \mathrm{Cl}(B_n(\mathcal{A}))$, then each neighbourhood V of x_0 contains a point y such that $M_{\mathcal{A}}[F(y)] \geq M_{\mathcal{A}}[F(x_0)] + 1$.

As an immediate consequence of the above we have $\mathrm{Int}(\mathrm{Cl}(B_1(\mathcal{A}))) = \emptyset$. Assume that $\mathrm{Int}(\mathrm{Cl}(B_n(\mathcal{A}))) = \emptyset$ and let $x_0 \in \mathrm{Cl}(B_{n+1}(\mathcal{A})) \setminus \mathrm{Cl}(B_n(\mathcal{A}))$. Since $B_n(\mathcal{A}) \subset B_{n+1}(\mathcal{A})$, we have $M_{\mathcal{A}}[F(x_0)] = n + 1$. Thus according to (5), $V \not\subset \mathrm{Cl}(B_{n+1}(\mathcal{A}))$ for each neighbourhood V of x_0; so we have shown:

(6) $\mathrm{Int}(\mathrm{Cl}(B_n(\mathcal{A}))) = \emptyset$ for each $\mathcal{A} \in \Sigma$, $n = 1, 2, \dots$.

Finally, let $x_0 \in X \setminus \bigcup_{\mathcal{A} \in \Sigma} \bigcup_{n=1}^{\infty} B_n(\mathcal{A})$, $\mathcal{A} \in \Sigma$ and let $m = M_{\mathcal{A}}[F(x_0)]$. Because $x_0 \notin B_m(\mathcal{A})$ there is $\mathcal{G} \in \Sigma$, $\mathcal{G} \prec \mathcal{A}$ and a neighbourhood U of x_0 such that $F(x') \subset \mathrm{St}^3(F(x_0), \mathcal{G})$ for $x' \in U$. This means $x_0 \in C^+(F)$, so $X \setminus C^+(F) \subset \bigcup_{\mathcal{A} \in \Sigma} \bigcup_{n=1}^{\infty} B_n(\mathcal{A})$.

Conversely, for $x_0 \in C^+(F)$ we take $\mathcal{A}, \mathcal{G} \in \Sigma$ with $\mathcal{G} \prec \mathcal{A}$. There exists a neighbourhood U of x_0 such that$F(x) \subset \mathrm{St}^3(F(x_0), \mathcal{G})$ for $x \in U$, so $x_0 \in X \setminus B_n(\mathcal{A})$ for $n = 1, 2, \dots$. Hence $x_0 \in \bigcap_{\mathcal{A}\in\Sigma} \bigcap_{n=1}^{\infty} (X \setminus B_n(\mathcal{A}))$ and in the consequence we obtain

$$X \setminus C^+(F) = \bigcup_{\mathcal{A}\in\Sigma} \bigcup_{n=1}^{\infty} B_n(\mathcal{A}).$$

In virtue of (6) it suffices to put $K(\mathcal{A}) = \bigcup_{n=1}^{\infty} B_n(\mathcal{A})$ and the proof is completed. $\square$

Theorem 3. *Let X be a topological space and (Y, Σ) a generalized uniform one with a regular Hausdorff structure. If $F : X \to Y$ is an upper semicontinuous map with compact values, then $X \setminus C^-(F) = \bigcup\{E(\mathcal{A}) : \mathcal{A} \in \Sigma\}$, where $E(\mathcal{A})$ are nowhere dense sets.*

Proof. For a compact set $D \subset Y$ and $\mathcal{A} \in \Sigma$ we denote $N_{\mathcal{A}}[D] = \inf\{m :$ there are points $y_1, \dots, y_m \in Y$ such $D \subset \bigcup_{i=1}^{m} \mathrm{St}(y_i, \mathcal{A})\}$. By $E_n(\mathcal{A})$ we denote the set of all points $x \in X$ satisfying the following two conditions:

(1) $N_{\mathcal{A}}[F(x)] \geq n$;

(2) each neighbourhood U of x contains a point x' with $F(x) \not\subset \mathrm{St}^3(F(x'), \mathcal{A})$.

Let us remark that $E_{n+1}(\mathcal{A}) \subset E_n(\mathcal{A})$ for each $\mathcal{A} \in \Sigma$, $n = 1, 2, \dots$. We fix $x \in X$, $\mathcal{A} \in \Sigma$ and let $m = N_{\mathcal{A}}[F(x)]$; then there are points $y_1, \dots, y_m \in Y$ such that $F(x) \subset \bigcup_{i=1}^{m} \mathrm{St}(y_i, \mathcal{A})$. Since F is upper semicontinuous, there is a neighbourhood U of x such that $F(x') \subset \bigcup_{i=1}^{m} \mathrm{St}(y_i, \mathcal{A})$ for $x' \in U$. This means $N_{\mathcal{A}}[F(x')] \leq m$ for $x' \in U$, thus

(3) for each $x \in X$, $\mathcal{A} \in \Sigma$ there is a neighbourhood U of x with $N_{\mathcal{A}}[F(x')] \leq N_{\mathcal{A}}[F(x)]$ for $x' \in U$.

If for $x \in X$ we have $N_{\mathcal{A}}[F(x)] < n$, then according to (3) there is a neighbourhood U of x such that $N_{\mathcal{A}}[F(x')] \leq N_{\mathcal{A}}[F(x)] < n$ for $x' \in U$, i.e. $U \cap E_n(\mathcal{A}) = \emptyset$. Then $x \notin \mathrm{Cl}(E_n(\mathcal{A}))$, so we obtain:

(4) $N_{\mathcal{A}}[F(x)] \geq n$ for each $x \in \mathrm{Cl}(E_n(\mathcal{A}))$.

For a point $x_0 \in \mathrm{Cl}(E_m(\mathcal{A})) \setminus \mathrm{Cl}(E_{m+1}(\mathcal{A}))$ and a neighbourhood V of x_0 it holds $(V \setminus \mathrm{Cl}(E_{m+1}(\mathcal{A}))) \cap E_m(\mathcal{A}) \neq \emptyset$. Let $x \in (V \setminus \mathrm{Cl}(E_{m+1}(\mathcal{A}))) \cap E_m(\mathcal{A})$; then $N_{\mathcal{A}}[F(x)] = m$. We choose points $y_1, \dots, y_m \in Y$ for which we have $F(x) \subset \bigcup_{i=1}^{m} \mathrm{St}(y_i, \mathcal{A})$. Using the upper semicontinuity of F and (2) we can select a point $x_1 \in V$ such that $F(x_1) \subset \mathrm{St}(F(x), \mathcal{A})$ and $F(x) \not\subset \mathrm{St}^3(F(x_1), \mathcal{A})$. Let $y \in F(x) \setminus \mathrm{St}^3(F(x_1), \mathcal{A})$, then $y \in \mathrm{St}(y_j, \mathcal{A})$ for some $j \in \{1, \ldots, m\}$. Suppose $p \in F(x_1) \cap \mathrm{St}(y_j, \mathcal{A})$; then $p, y \in \mathrm{St}(y_j, \mathcal{A})$ implies $y \in \mathrm{St}^2(p, \mathcal{A}) \subset \mathrm{St}^2(F(x_1), \mathcal{A})$ which is impossible. Hence $F(x_1) \cap \mathrm{St}(y_j, \mathcal{A}) = \emptyset$ and in the consequence $F(x_1) \subset \bigcup_{i=1, i \neq j}^{m} \mathrm{St}(y_i, \mathcal{A})$. This means $N_{\mathcal{A}}[F(x_1)] \leq m - 1$, so in virtue of (4) we obtain $x_1 \notin \mathrm{Cl}(E_m(\mathcal{A}))$. Thus we have shown for each $\mathcal{A} \in \Sigma$, $m = 1, 2, \ldots$:

(5) if $x \in \mathrm{Cl}(E_m(\mathcal{A})) \setminus \mathrm{Cl}(E_{m+1}(\mathcal{A}))$, then $V \not\subset \mathrm{Cl}(E_m(\mathcal{A}))$ for each neighbourhood V of x.

This implies

(6) $(\mathrm{Cl}(E_m(\mathcal{A})) \setminus \mathrm{Cl}(E_{m+1}(\mathcal{A}))) \cap \mathrm{Int}(\mathrm{Cl}(E_m(\mathcal{A}))) = \emptyset$ for each $\mathcal{A} \in \Sigma$, $m = 1, 2, \ldots$, which means $\mathrm{Int}(\mathrm{Cl}(E_m(\mathcal{A}))) \subset \mathrm{Cl}(E_{m+1}(\mathcal{A}))$. This gives $\mathrm{Int}(\mathrm{Cl}(E_m(\mathcal{A}))) = \mathrm{Int}(\mathrm{Cl}(E_{m+1}(\mathcal{A})))$ for $m = 1, 2, \ldots$; so (6) implies $(\mathrm{Cl}(E_n(\mathcal{A})) \setminus \mathrm{Cl}(E_{n+1}(\mathcal{A}))) \cap \mathrm{Int}(\mathrm{Cl}(E_1(\mathcal{A}))) = \emptyset$ for each $\mathcal{A} \in \Sigma$, $m = 1, 2, \ldots$. Since $\mathrm{Cl}(E_1(\mathcal{A}))$ can be presented in the form

$$\mathrm{Cl}(E_1(\mathcal{A})) = \bigcup_{n=1}^{\infty} (\mathrm{Cl}(E_n(\mathcal{A})) \setminus \mathrm{Cl}(E_{n+1}(\mathcal{A}))),$$

we have $\mathrm{Int}(\mathrm{Cl}(E_1(\mathcal{A}))) = \emptyset$ for $\mathcal{A} \in \Sigma$.

Finally, let $x \in X \setminus \bigcup\{E_1(\mathcal{A}) : \mathcal{A} \in \Sigma\}$ and let $\mathcal{A} \in \Sigma$, $m = N_{\mathcal{A}}[F(x)]$. Since $x \notin E_m(\mathcal{A})$, it follows from (2) that there is a neighbourhood U of x such that $F(x) \subset \mathrm{St}^3(F(x'), \mathcal{A})$ for $x' \in U$. Then $x \in C^-(F)$ and in the consequence

$$X \setminus C^-(F) \subset \bigcup\{E_1(\mathcal{A}) : \mathcal{A} \in \Sigma\}.$$

Conversely, if $x \in C^-(F)$, then for each $\mathcal{A} \in \Sigma$ the condition (2) is not satisfied, which leads to the conclusion

$$X \setminus C^-(F) = \bigcup\{E_1(\mathcal{A}) : \mathcal{A} \in \Sigma\}$$

and the proof is completed. $\square$

Given a Hausdorff uniform space $(Y, \mathcal{U})$ by $w(\mathcal{U})$ we denote the weight of $\mathcal{U}$ and $\mathcal{B}$ a base of $\mathcal{U}$ with $\mathrm{card}\, \mathcal{B} = w(\mathcal{U})$. For each $V \in \mathcal{U}$ we will write $V[y] = \{x \in Y : (y, x) \in V\}$; further $\mathcal{A}_V = \{V[y] : y \in Y\}$ and $\Sigma = \{\mathcal{A}_V : V \in \mathcal{B}\}$. Then Σ is a regular Hausdorff generalized uniformity in Y. In this case Theorems 2 and 3 can be rewritten in the form:

Corollary 2. *Let X be a topological space, $(Y,\mathcal{U})$ a Hausdorff uniform one and let $F : X \to Y$ be a multivalued map with compact values. If F is lower (upper) semicontinuous, then $X \setminus C^+(F) = \bigcup\{K(V) : V \in \mathcal{B}\}$, (resp. $X \setminus C^-(F) = \bigcup\{K(V) : V \in \mathcal{B}\}$), where $K(V)$ are of the first category (nowhere dense) sets and $\mathcal{B}$ is a base of $\mathcal{U}$ with* $\operatorname{card}\mathcal{B} = w(\mathcal{U})$.

Corollary 3 [2]**.** *Let X be a topological space and Y a metric one. If $F : X \to Y$ is upper (lower) semicontinuous map with compact values, then $X \setminus C^-(F)$, (resp. $X \setminus C^+(F)$) is of the first category.*

References

[1] J. Ewert, Generalized uniform spaces and almost uniform convergence, (*submitted*).

[2] M. K. Fort, Points of continuity of semicontinuous functions, *Publ. Math. Debrecen* **2** (1951), 100–102.

[3] K. Morita, On the simple extension of a space with respect to a uniformity, *I. Proc. Japan Acad.* **27** (1951), 65–72.

[4] H. Poppe, Compactness in function spaces with a generalized uniform structure, *II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom Phys.* **18** (1970), 567–573.

[5] W. Rinow, Über die verallgemainerten uniformen Strukturen von Morita und ihre Vervollständigung, General topology and its relations to modern analysis and algebra, II Proc. Second Prague Symp. 1966, *Praha*, 1967, 297–305.

[6] W. Rinow, Über die verallgemainerten uniformen Strukturen von Morita und ihre Vervollständigung, *Publ. Math. Debrecen* **15** (1968), 317–331.

[7] R. E. Smithson, A characterization of lower semicontinuity, *Amer. Math. Monthly* **75** (1968), 505.

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