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Additive uniqueness sets for multiplicative functions

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Abstract. We prove that if a multiplicative function f satisfies the equation

$$f(a+b)=f(a)+f(b)$$

for all $a,b\in H_2$ or for all $a,b\in H_3$, then $f(n)=n$ for all positive integers n, where $H_k:=\left\{\frac{n(n+1)\dots(n+k-1)}{1\cdot 2\cdots k}\mid n=1,2,\dots\right\}$.

Throughout this note, let $\mathbb{N}$ denote the set of positive integers and let $\mathcal{M}$ be the set of complex valued multiplicative functions f with $f(1)=1$.

In 1992, C. Spiro [4] showed that if a function $f\in\mathcal{M}$ satisfying $f(p+q)=f(p)+f(q)$ for all primes p and q, then $f(n)=n$ for all $n\in\mathbb{N}$. Recently, in [2] the identity function was characterized as the multiplicative function f for which $f(p+n^2)=f(p)+f(n^2)$ holds for all primes p and for all $n\in\mathbb{N}$. It follows from the results of [1] that a completely multiplicative function f satisfies the equation $f(n^2+m^2)=f(n^2)+f(m^2)$ for all $n,m\in\mathbb{N}$ if and only if $f(2)=2$, $f(p)=p$ for all primes $p\equiv 1 \pmod 4$ and $f(q)=q$ or $f(q)=-q$ for all primes $p\equiv 3 \pmod 4$. In [3] the second named author proved that if a multiplicative function f satisfies the equation $f(n^2+m^2+3)=f(n^2+1)+f(m^2+2)$ for all positive integers n and m, then either $f(n)=n$ or $f(n^2+1)=f(m^2+2)=f(n^2+m^2+3)=0$ holds for all $n,m\in\mathbb{N}$.

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Following C. Spiro, we call a subset E of $\mathbb{N}$ is an additive uniqueness set for $\mathcal{M}$ if there is exactly one element f of $\mathcal{M}$ which satisfies

$$f(a+b) = f(a) + f(b) \quad \text{for all } a \in E \quad \text{and } b \in E.$$

For each $k \in \mathbb{N}$ let

$$H_k := \left\{ \frac{n(n+1)\dots(n+k-1)}{1\cdot 2\cdots k} \;\Big|\; n \in \mathbb{N} \right\}.$$

The set $H_1 = \mathbb{N}$ is clearly a uniqueness set for $\mathcal{M}$. In this note we prove the same result for the sets H_2 and H_3.

Theorem 1. *The set*

$$H_2 = \left\{ t_n := \frac{n(n+1)}{2} \;\Big|\; n \in \mathbb{N} \right\}$$

is an additive uniqueness set for $\mathcal{M}$. In other words, if $f \in \mathcal{M}$ satisfies the condition

$$f\left(t_n + t_m\right) = f\left(t_n\right) + f\left(t_m\right) \tag{1}$$

for all $n, m \in \mathbb{N}$, then $f(n)$ is the identity function, i.e. $f(n) = n$ for all $n \in \mathbb{N}$.

Theorem 2. *The set*

$$H_3 = \left\{ \mathcal{L}_n := \frac{n(n+1)(n+2)}{6} \;\Big|\; n \in \mathbb{N} \right\}$$

is an additive uniqueness set for $\mathcal{M}$. In other words, if $f \in \mathcal{M}$ satisfies the condition

$$f\left(\mathcal{L}_n + \mathcal{L}_m\right) = f\left(\mathcal{L}_n\right) + f\left(\mathcal{L}_m\right) \tag{2}$$

for all $n, m \in \mathbb{N}$, then $f(n)$ is the identity function, i.e. $f(n) = n$ for all $n \in \mathbb{N}$.

Remark. Is it true that for every $k \geq 4$ the set H_k is an additive uniqueness set for $\mathcal{M}$? We think that this question can be treated similarly as the above theorems for small fixed values of k.

Proof of Theorem 1. First we prove

$$f(n) = n \quad \text{for all positive integers } \ n \le 16. \tag{3}$$

Let $t_n := \frac{n(n+1)}{2}$ and $f(3) := x$. Repeated use of (1) gives $f(2) = f(1+1) = f(t_1 + t_1) = f(1) + f(1) = 2$, $f(4) = f(1+3) = f(t_1 + t_2) = 1 + x$ and thus $x(1+x) = f(3)f(4) = f(12) = f(6+6) = f(t_3 + t_3) = 2f(6) = 4x$. Then either $x = 0$ or $x = 3$. Moreover $f(22) = f(2)f(11) = 2f(1+10) = 2f(t_1 + t_4) = 2 + 4f(5)$, while also $f(22) = f(1+21) = f(t_1 + t_6) = 1 + xf(7) = 1 + x(1+2x)$. These imply that $4f(5) = 2x^2 + x - 1$, therefore since $x = 0$ or $x = 3$, $f(5) \neq 0$. Finally, $f(4)f(5) = f(20) = f(10{+}10) = f(t_4{+}t_4) = 2f(10) = 4f(5)$, consequently $f(4) = 4$ and $f(3) = 3$. It easily follows that $f(n) = n$ successively for $n = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16$, because $7 = t_1 + t_3$, $3.8 = t_2 + t_6 = 3 + 3.7$, $9 = t_2 + t_3$, $11 = t_1 + t_4$, $13 = t_2 + t_4$, $16 = t_3 + t_5$. Thus (3) is proved.

It is clear that the theorem will follow if we can prove the following assertion: If $T$is a positive integer such that $f(n) = n$ for all $n < T$, then $f(T) = T$. Because of (3), we can assume that $T \ge 17$.

Assume that $T \ge 17$ is a positive integer satisfying the condition

$$f(n) = n \quad \text{for all positive integers } n < T. \tag{4}$$

We shall prove:

$$f(T) = T. \tag{5}$$

It is obvious that (5) holds if T is not a prime power. We must therefore have $T = q^\alpha$, where q is a prime and $\alpha \in \mathbb{N}$. We note that (5) is valid for $q = 2$. Indeed, if $q = 2$, then by using the multiplicativity of f, we have

$$\begin{aligned} f(2^\alpha(2^\alpha - 1)) &= f((2^\alpha - 1)2^{\alpha-1} + (2^\alpha - 1)2^{\alpha-1}) \\ &= 2f((2^\alpha - 1)2^{\alpha-1}) = 2^\alpha(2^\alpha - 1), \end{aligned}$$

which implies $f(2^\alpha) = 2^\alpha$.

For $T = q^\alpha$, where q is an odd prime and $\alpha \in \mathbb{N}$ we now show

$$f(m^2) = m^2 \quad \text{for all } m \le T - 2. \tag{6}$$

Indeed, (1) and the facts $m^2 = t_{m-1}+t_m = \frac{(m-1)m}{2}+\frac{m(m+1)}{2}$, $(m,m-1) = (m,m+1) = 1$ with the condition (4) lead to

$$f(m^2) = f\left(\frac{(m-1)m}{2}\right) + f\left(\frac{m(m+1)}{2}\right) = m^2$$

when $m+1 < T$. Therefore (6) follows immediately.

First we consider the case when $T-1 \neq 2^\beta$, $\beta \in \mathbb{N}$. In this case (1), (4) and $q > 2$ imply $f\left[(T-1)^2\right] = (T-1)^2$ and

$$\begin{aligned}\frac{(T-2)(T-1)}{2} + \frac{T-1}{2}f(T) &= f\left(\frac{(T-2)(T-1)}{2}\right) + f\left(\frac{(T-1)T}{2}\right)\\ &= f\left(\frac{(T-2)(T-1)}{2} + \frac{(T-1)T}{2}\right) = f\left[(T-1)^2\right] = (T-1)^2,\end{aligned}$$

which gives (5).

Next we assume $T = q^\alpha = 2^\beta + 1$, where $\alpha, \beta \in \mathbb{N}$. This together with $T \geq 17$ implies $\alpha = 1$, $\beta = 2^h$, and so $T = q = 2^{2^h} + 1$ is a Fermat-prime.

It is clear that $T = 2^{2^h} + 1 \equiv 2 \pmod 3$, consequently

$$3 \mid T+1 \quad \text{and} \quad T+1 = 2^{2^h} + 2 \neq 3^\gamma,$$

$$4 \mid T+3 \quad \text{and} \quad T+3 = 2^{2^h} + 4 \neq 2^\delta$$

and

$$3 \mid T+4 \quad \text{and} \quad T+4 = 2^{2^h} + 5 \neq 3^\nu,$$

where $\gamma, \delta, \nu \in \mathbb{N}$. Together with (4) and (6) we obtain

(7) $f((T+1)^2) = (T+1)^2, \quad f((T+3)^2) = (T+3)^2 \quad$ and $f(T+4) = T+4$.

Finally, let us consider the relation

$$f\left(\frac{(m-1)m}{2}\right) + f\left(\frac{m(m+1)}{2}\right) = f(m^2)$$

for $m = T+1$ and for $m = T+3$. By (1), (4) and (7) we have

$$f(T+2) = T+2 \quad \text{and} \quad f(T) = T.$$

Thus Theorem 1 is proved.

PROOF of Theorem 2. We first deduce from (2)

$$f(n) = n \quad \text{for } n \le 8. \tag{8}$$

We have $f(2) = f(1+1) = f(\mathcal{L}_1 + \mathcal{L}_1) = 2f(1) = 2$, $f(5) = f(1+4) = f(\mathcal{L}_1 + \mathcal{L}_2) = f(1) + f(4) = 1 + f(4)$, consequently $f(4) + f(4)^2 = f(4)f(5) = f(20) = f(10+10) = f(\mathcal{L}_3+\mathcal{L}_3) = 2f(10) = 4f(5) = 4+4f(4)$, which implies $f(4) = 4$ or $f(4) = -1$. On the other hand, we get from (2) that $f(8) = f(4+4) = f(\mathcal{L}_2 + \mathcal{L}_2) = 2f(4)$, $2f(3)f(4) = f(3)f(8) = f(24) = f(4+20) = f(\mathcal{L}_2 + \mathcal{L}_3) = f(4) + f(4)f(5) = 2f(4) + f(4)^2$, which together with the fact $f(4) \ne 0$ show that $2f(3) = 2 + f(4)$. Furthermore, we also get from (2) that $2f(7) = f(2)f(7) = f(14) = f(4+10) = f(\mathcal{L}_2 + \mathcal{L}_3) = f(4) + 2f(5) = 2 + 3f(4)$ and $f(3)f(7) = f(21) = f(1+20) = f(\mathcal{L}_1 + \mathcal{L}_4) = 1 + f(4)f(5) = 1 + f(4) + f(4)^2$, consequently $f(4)^2 = 4f(4)$. This, compared with $f(4) \ne 0$, shows that $f(4) = 4$. Thus, (8) is proved.

As in the proof of Theorem 1, we assume that $T \ge 9$ is a positive integer for which

$$f(n) = n \quad \text{for all positive integers } n < T. \tag{9}$$

We shall prove that

$$f(T) = T. \tag{10}$$

It is obvious that (10) holds if T is not a prime power. We must therefore have $T = q^\alpha$, where q is a prime and $\alpha \in \mathbb{N}$. We note that (10) is valid for $q = 2$. Indeed, if $q = 2$, then $\alpha \ge 4$ and by (2), (9) we have

$$\begin{aligned} 2^\alpha \frac{(2^{\alpha-1}-1)(2^{\alpha-2}-1)}{3} &= 2f\left(2^{\alpha-1}\frac{(2^{\alpha-1}-1)(2^{\alpha-1}-2)}{6}\right) \\ &= f(\mathcal{L}_{2^{\alpha-1}-2} + \mathcal{L}_{2^{\alpha-1}-2}) = f\left(2^\alpha \frac{(2^{\alpha-1}-1)(2^{\alpha-2}-1)}{3}\right) \\ &= f(2^\alpha)\frac{(2^{\alpha-1}-1)(2^{\alpha-2}-1)}{3}, \end{aligned}$$

and so $f(2^\alpha) = 2^\alpha$ as asserted.

We now consider the case when $T = 3^\alpha$, $\alpha \ge 2$. The proof of (10) depends on the relation

$$\mathcal{L}_{n-1} + \mathcal{L}_n = \frac{(n-1)n(n+1)}{6} + \frac{n(n+1)(n+2)}{6} = \frac{n(n+1)(2n+1)}{6} \tag{11}$$

and on the fact:

(12) if $x, y \in \mathbb{N}$ satisfy $3^x + 1 = 2^y$, then $(x, y) = (1, 2)$.

Let $n = (3T-1)/2 = (3^{\alpha+1}-1)/2$. Then by (11), we have

$$\frac{(T-1)(3T-1)(3T+1)}{16} + \frac{(3T-1)(3T+1)(T+1)}{16} = \frac{(3T-1)(3T+1)T}{8}. \tag{13}$$

If $4 \mid 3T-1$, then $(\frac{3T-1}{4}, \frac{T-1}{2}\frac{3T+1}{2}) = 1$ and $(\frac{T-1}{2}, \frac{3T+1}{2}) = 1$, therefore it follows from (2), (9), (12) and (13) that

$$\frac{T-1}{2}\frac{3T-1}{4} f\left(\frac{3T+1}{2}\right) + \frac{3T-1}{4}\frac{T+1}{2} f\left(\frac{3T+1}{2}\right) = \frac{3T-1}{4} f\left(\frac{3T+1}{2}\right) f(T). \tag{14}$$

An application of (11) for $n = \frac{3T-1}{4}$ leads us to

$$\frac{1}{2}\frac{3T-5}{4}\frac{3T-1}{4}\frac{T+1}{4} + \frac{1}{2}\frac{3T-1}{4}\frac{T+1}{4}\frac{3T+7}{4} = \frac{1}{2}\frac{3T-1}{4}\frac{T+1}{4}\frac{3T+1}{2},$$

which, compared with (2) and (9), implies $f(\frac{3T+1}{2}) = \frac{3T+1}{2}$. This together with (14) proves (10). So, (10) holds for the case $T = 3^\alpha$ and $4 \mid 3T-1$.

Assume that $T = 3^\alpha$ and $4 \nmid 3T-1$. Then, by applying (2),(9), (12) and (13), we have

$$\frac{(T-1)(3T+1)}{8} f\left(\frac{3T-1}{2}\right) + \frac{(3T+1)(T+1)}{8} f\left(\frac{3T-1}{2}\right) = \frac{3T+1}{4} f\left(\frac{3T-1}{2}\right) f(T).$$

Finally, an application of (11) for $n = \frac{3T-1}{4}$, using (2) and (9), shows that $f(\frac{3T-1}{2}) = \frac{3T-1}{2}$. This with the last relation proves (10) for the case when $T = 3^\alpha$.

Now we complete the proof of Theorem 2 by showing (10) for $T = q^\alpha$, where $(T, 6) = 1$. Let us consider (11) for $n = \frac{T-1}{2}$. We have

$$\frac{T-3}{2}\frac{(T-1)(T+1)}{24} + \frac{(T-1)(T+1)}{24}\frac{T+3}{2} = \frac{(T-1)(T+1)}{24}T, \tag{15}$$

where $\frac{(T-1)(T+1)}{24}$ is a positive integer. It is clear by (9) that

$$f\left(\frac{T-3}{2}\frac{(T-1)(T+1)}{24}\right)=\frac{T-3}{2}\frac{(T-1)(T+1)}{24},$$
$$f\left(\frac{(T-1)(T+1)}{24}\frac{T+3}{2}\right)=\frac{(T-1)(T+1)}{24}\frac{T+3}{2}$$

and

$$f\left(\frac{(T-1)(T+1)}{24}T\right)=\frac{(T-1)(T+1)}{24}f(T),$$

which together with (2) and (15) imply $f(T)=T$.

The proof of Theorem 2 is finished.

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